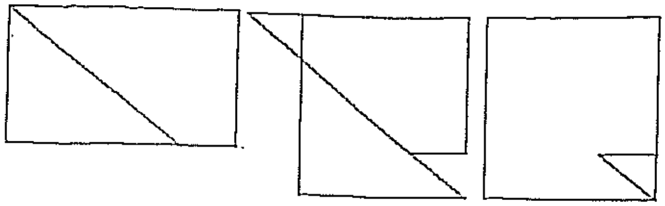
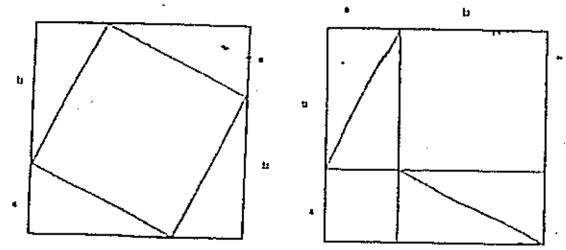
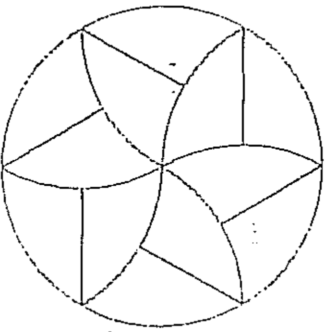


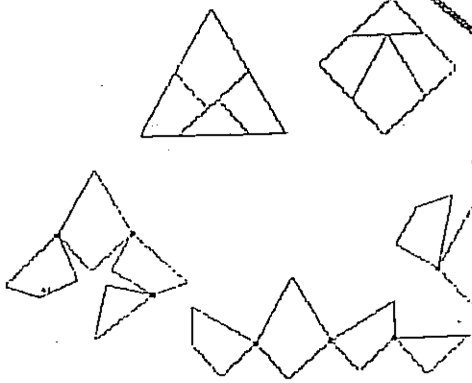
Rectangle-to-square



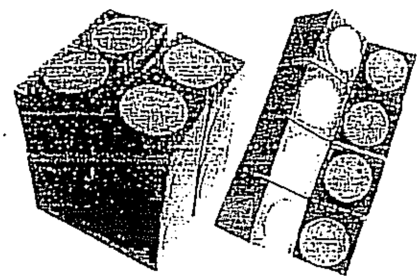
Pythagorean theorem dissection



Tessellate a circle



Hinged triangle-to-square



Magic folding cube

Geometry Gems:
Dissections That Promote Mathematical Thinking and Spatial Reasoning

Thursday, April 18, 2013, 1 to 2:15 pm
Hyatt-Regency, Mineral Hall A-C
NCTM Annual Meeting and Exposition, Denver

Patricia Baggett
Dept of Math Sci.
MSC 3MB, PO Box 30001
New Mexico State University
Las Cruces, NM 88003-8001
baggett@nmsu.edu

Andrzej Ehrenfeucht
Computer Science Dept
Box 430
Univ of Colorado at Boulder
Boulder, CO 80309-0430
andrzej.ehrenfeucht@colorado.edu

http://www.math.nmsu.edu/~breakingaway/

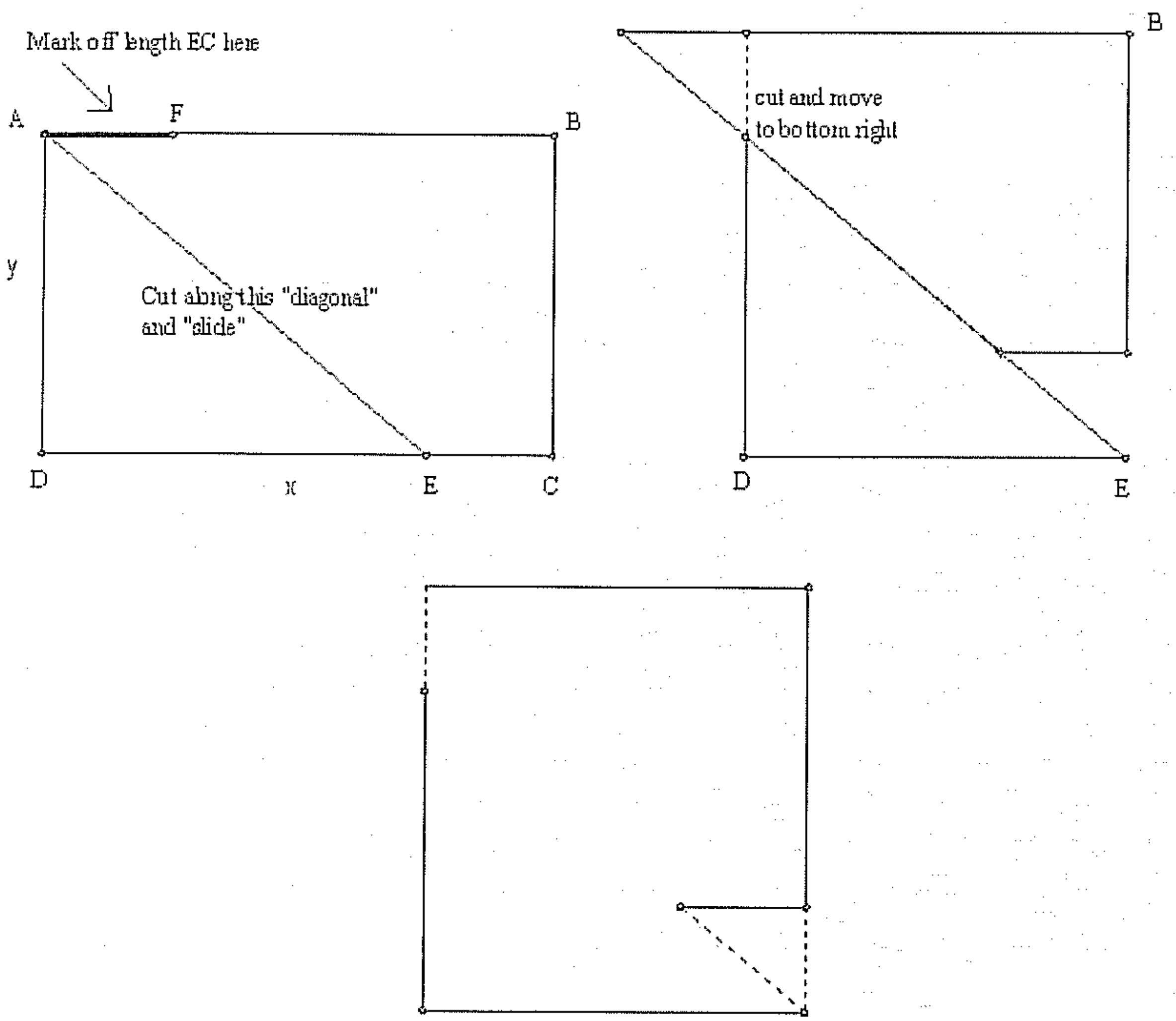
Geometric dissections involve cutting a figure into parts that are rearranged to form another figure. The mathematics is based on the theorem that any polygon can be transformed into any other of the same area by cut and paste. We will design and construct 2- and 3-D dissection puzzles and explore their mathematics and history.

*Activities 1, 3, 4, and 5 are on our website, http://www.math.nmsu.edu/~breakingaway/. There you will find detailed, step-by-step instructions for them. Here, instructions for 3, 4, and 5 are shortened to save paper.

How to transform a rectangle to a different rectangle (with the same area but a different perimeter)

Below is a method to construct a rectangle with a given area and a given perimeter. Just start with a rectangle with the area that you want, and keep transforming it! Its area will remain the same, but its perimeter will change.

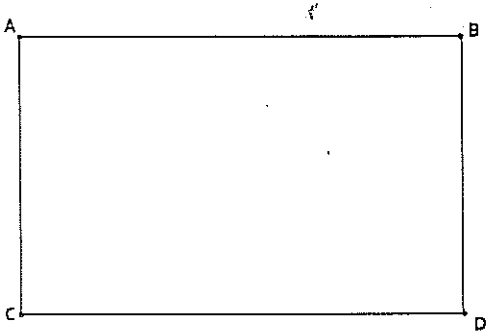
The old rectangle has dimensions x and y. The new rectangle has the same area but new dimensions. Do you see how to make the new rectangle a square?

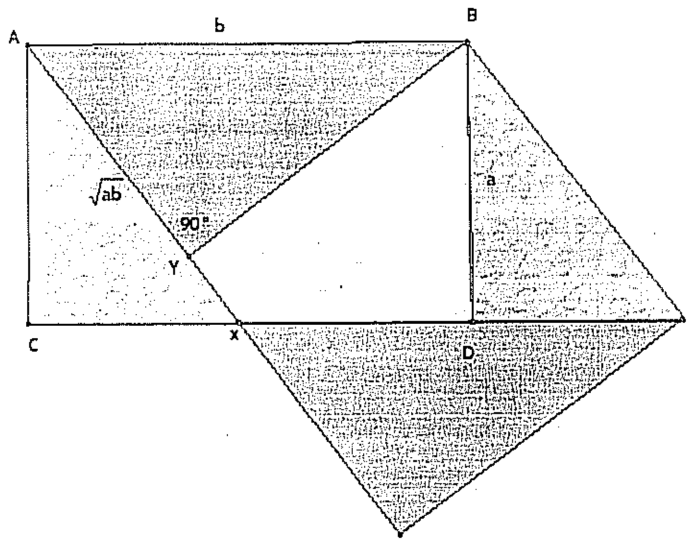


Webpage Developed by Aous Manshad
Last Modified: October 16, 2006

Turning a rectangle into a square by dissection

Consider a rectangle with sides a and b, such that, $a < b < 2*a$. We show here one way to cut it into three pieces and rearrange them into a square.



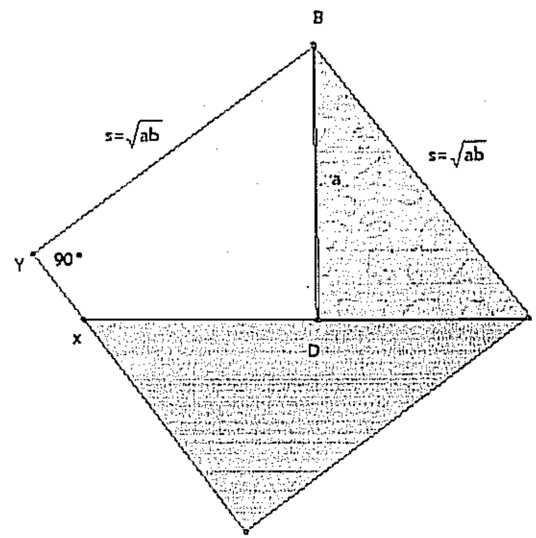


Compute, $s = \sqrt{(a*b)}$. Open a compass to length s, put a pin in corner A of the rectangle, and mark point X on side CD. The distance AX is s. Draw the line AX. Draw another line (using an index card) BY,

perpendicular to AX.

Cut out the rectangle ABDC, cut the line AX, and cut the line YB.

In order to get a square in slanted position, move the left triangle to the right, and move the top triangle to the bottom right.



Remark.
Notice that the length BY is also s. (Why?)

Task.
Convert a rectangle 2 inches by 3 inches into a square.
Click on the Flash animation below to view step-by-step how to construct a rectangle to a square:

PYTHAGOREAN PUZZLE

This lesson can be embedded in a story about selling land, planting grass, cutting cloth, or many others. It can also be presented in higher grades (e.g., middle school) as a proof of the Pythagorean theorem, because it contains such a proof. Here the lesson is provided without any embelishments.

Preparation

Children work in groups of four. Each group will make four identical right triangles from construction paper. (Different groups may use paper of different colors, and make triangles of different shapes and sizes.) The legs of the triangles should be measured and written down, For example:

Legs: $a = 6.5$ cm, $b = 12.3$ cm

Then children should draw one square whose sides are $a + b = 18.8$ cm long.

Problem

(1) Put the four triangles on the square (in the four corners) as shown in Illustration 106. What is left uncovered is one square whose sides have a length (measure it) of $c = 13.9$ cm. (The side is the hypotenuse of the right triangle.)
(2) Now put all four triangles in two corners of the square as shown in Illustration 107. What is left uncovered are two squares whose sides have lengths a and b.

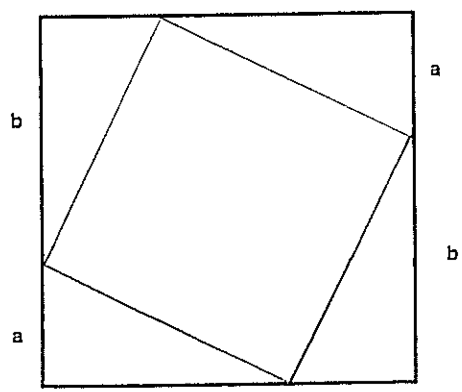


ILLUSTRATION 106.

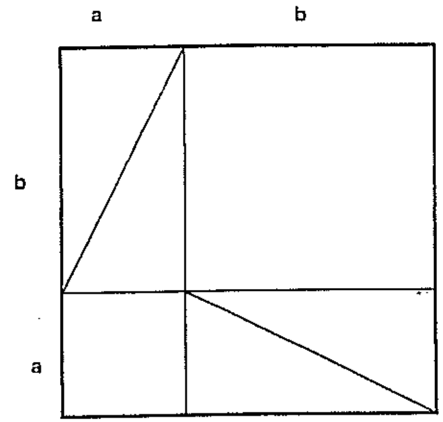


ILLUSTRATION 107.

In which of the two instances was a bigger area left uncovered? Let children make both coverings as many times as they want, and do not press them too soon for an answer. The answer is, the area uncovered is *the same*, because each time we covered the same area of a big square using the same four triangles.

Remark: It is likely that some children will not "get it." This is not necessarily surprising, because the principle involved is: If, from a region of area x, some parts of total area y are removed, what remains has an area of $x - y$, independent of its shape. This is not so obvious because "how big it looks" depends very much on its shape.

A demonstration of the Pythagorean theorem by dissection

Start with a square with side length a and a square with side length b, and build from them a square with side length $c = a^2 + b^2$, as shown below.

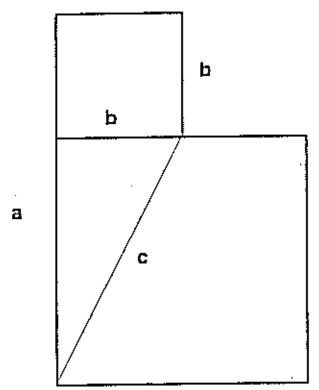


ILLUSTRATION 102.

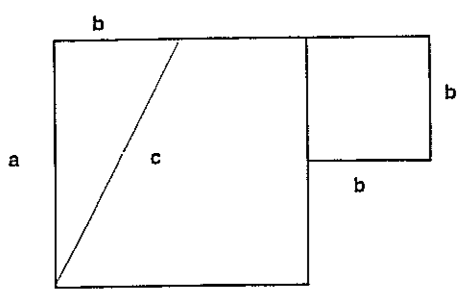


ILLUSTRATION 103.

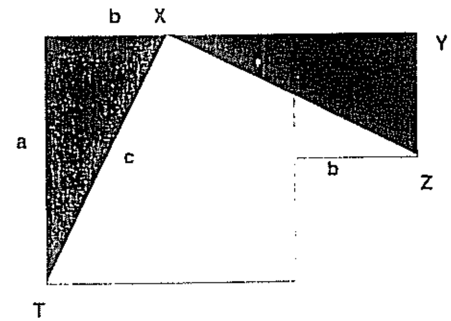


ILLUSTRATION 104.

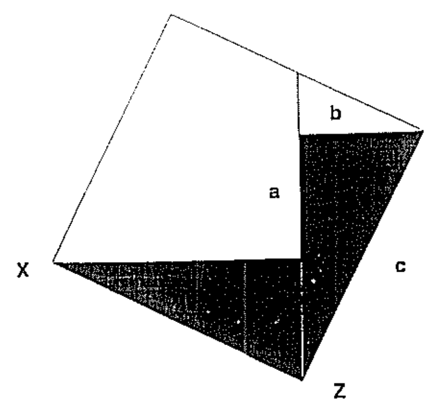
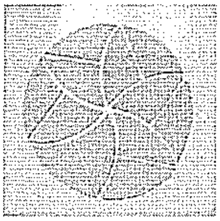
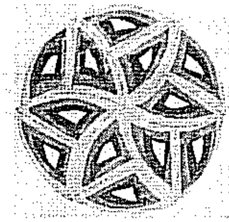


ILLUSTRATION 105.



Tessellating a Circle



Here are some definitions of to **tessellate** found on the Web:

(1) to cover the plane with a pattern in such a way as to leave no region uncovered.

(2) to decompose a curve or surface into polygonal faces.

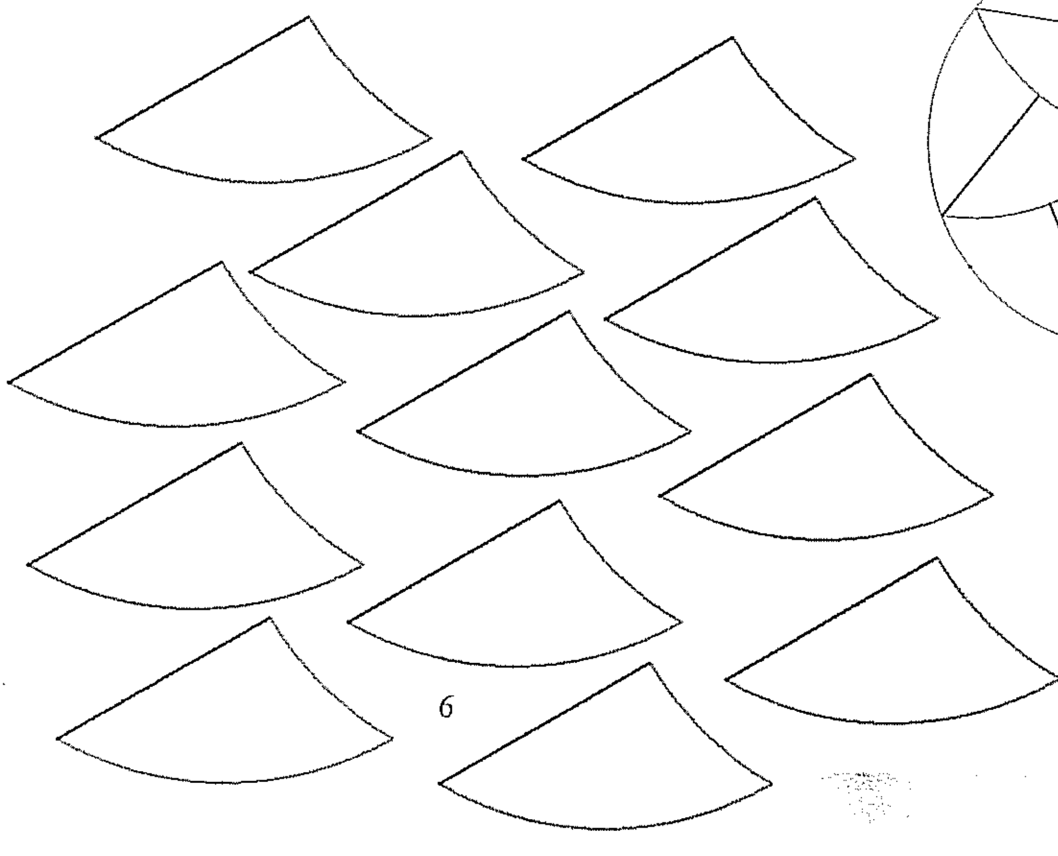
(3) to break an image into small, square regions for processing or output.

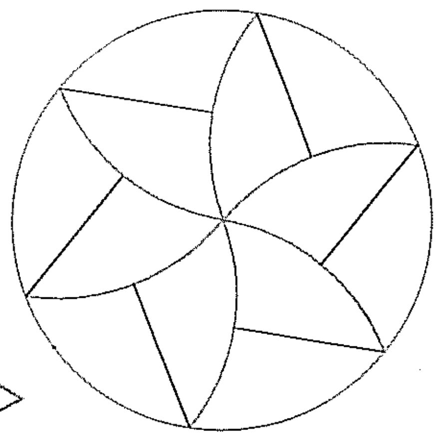
(4) to tile with tesserae (small pieces used in mosaic work); "tessellate the kitchen floor".

In school mathematics we generally think of tessellating a plane with congruent shapes. But what about tessellating a circle? We could cover it with congruent sectors, which is not very interesting. Below we give a non-trivial way to tessellate a circle with congruent shapes that are not sectors. When cut out, the pieces make a nice puzzle.

It is fun to start by having students use the twelve congruent shapes to try to make a circle. Click here for a sheet of the pieces that can be used for this task. We Xerox them on cardstock so they are sturdier.

pcs4TAC.PNG 988×732 pixels





Triangle to square: A hinged dissection

Background:

Greg Frederickson's book *Hinged Dissections: Swinging & Twisting* , was published in 2002 by Cambridge University Press. We thank him for this dissection, originally credited to Dudeney in 1907. We have adapted it for use with children.

From his website, http://www.cs.purdue.edu/homes/gnf/book2.html :

"A geometric dissection is a cutting of a geometric figure into pieces that we can rearrange to form another figure. As visual demonstrations of relationships such as the Pythagorean theorem, dissections have had a surprisingly rich history, reaching back to Arabian mathematicians a millennium ago and Greek mathematicians more than two millennia ago. As mathematical puzzles they enjoyed great popularity a century ago, in newspaper and magazine columns written by the American Sam Loyd and the Englishman Henry Ernest Dudeney. Loyd and Dudeney set as a goal the minimization of the number of pieces. Their puzzles charmed and challenged readers, especially when Dudeney introduced an intriguing variation in his 1907 book *The Canterbury Puzzles* . After presenting the remarkable 4-piece solution for the dissection of an equilateral triangle to a square, Dudeney wrote: 'I add an illustration showing the puzzle in a rather curious practical form, as it was made in polished mahogany with brass hinges for use by certain audiences. It will be seen that the four pieces form a sort of chain, and that when they are closed up in one direction they form a triangle, and when closed in the other direction they form a square."

The lesson:

For this lesson we used 9 by 12 inch rectangles of foam rubber, 6 mm thick. The brand was Darice Super Thick Foamies, available at craft shops for about $1.50 per sheet. You may use the 3 mm thick sheets, but they are more difficult to work with. If you are careful (see the diagram below), you can get four triangles with edge length 6 inches from each sheet. The foam rubber can be cut with ordinary scissors.

Supplies:

A foam rubber equilateral triangle, 6 inches on a side, scissors, ruler, calculator, compass, index card (to make right angles). Wide strapping tape (optional), to repair your triangle in case you accidentally cut it apart.

Steps one through nine are shown in the animation at the very bottom of this unit!

1. Find the area of your equilateral triangle. It has side length s = 6 inches, and of course base b = 6 inches. So 1/2 of the base is 3 inches. By the Pythagorean theorem,
(1/2*base)^2 + height^2 = 6^2
Solving for height,
height = √(36 - 9) display: 5.1961524 inches
area of triangle = 1/2 * base * height = 3*5.1961524 = 15.588457 square inches

2. Find the length of the edge of a square with area 15.588457 square inches:
√(15.588457) = 3.948222 inches
Changing this number to 3 inches and some number of 16ths of an inch:
[3.948222][-][3][=] .948222
[*][16][=] 15.171552
So the length of the edge of the square is about 3 15/16 inches.

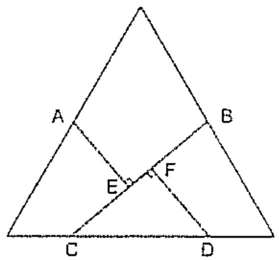
3. Mark the midpoints of two edges of your triangle, say the left and right edges. Call the midpoints A and B. (See illustration.)

4. Set your compass radius at 3 15/16 inches. Put the point of your compass on one of the midpoints that you just marked, say B. Swing an arc and mark a point one radius away on the bottom (unmarked) edge of your triangle. Call this point C. Connect points B and C with a straight line.

5. Set your compass to a radius which is the length from A to B. Put the point of your compass on C, and swing an arc that cuts the bottom edge of the triangle. Call this point D.

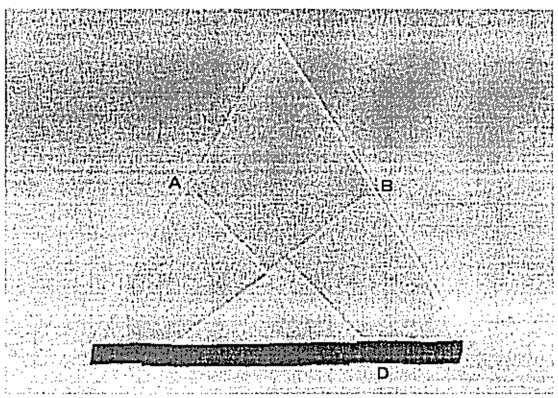
6. Now using an index card, construct a segment that goes through point A and is perpendicular to line BC, ending on line BC. Call the point E where the line intersects BC.

7. Construct a second line segment that goes through point D and is perpendicular to line BC. Name the point F where the line intersects line BC. Your triangle should look like this:



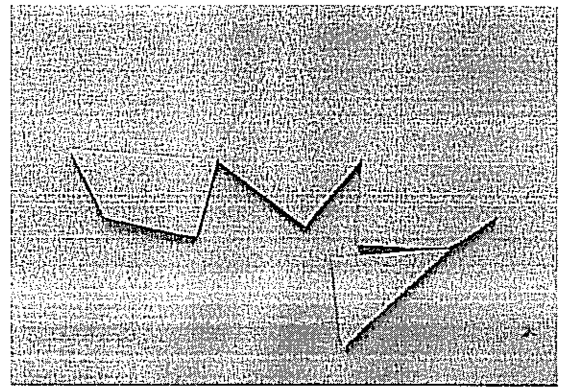
8. Be sure that all the lines above are drawn on your triangle. You are now ready to cut. This needs to be done very carefully.
Beginning at C, cut along line CB ALMOST to B. Do NOT cut through to B!
Beginning at E, cut along EA ALMOST to A. Do NOT cut through to A!
Beginning at F, Cut along line FD ALMOST to D. Do NOT cut through to D!

You have created hinges at A, B, and D:

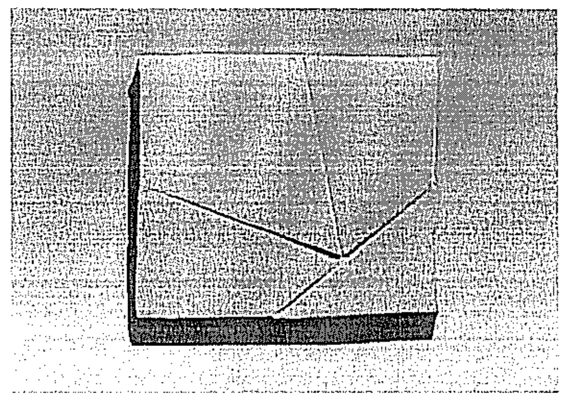


9. Now swing the pieces around to form a square!

8

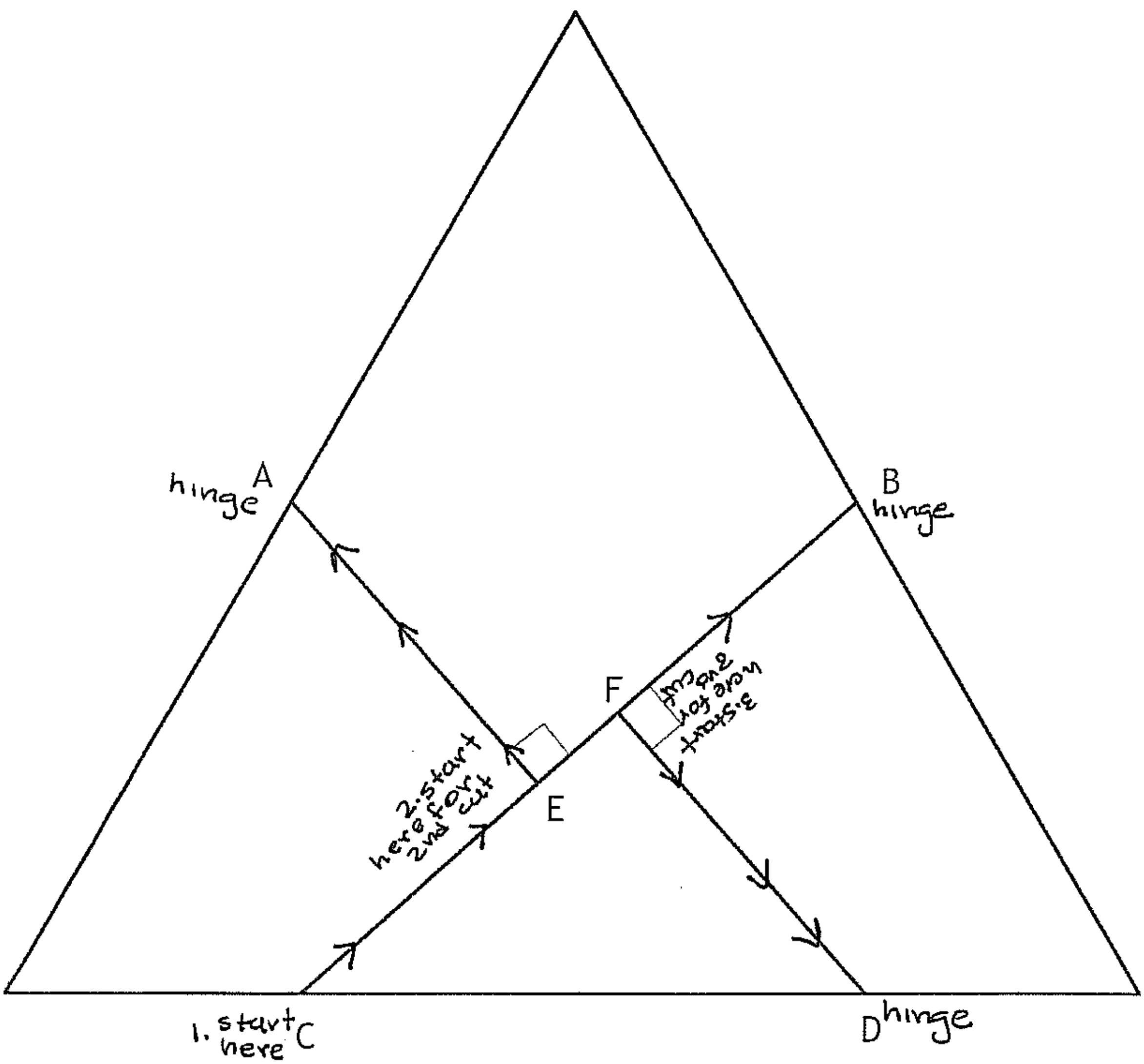


Now we have a triangle to square!



This is how your pieces should look:

Triangle to square: A hinged dissection



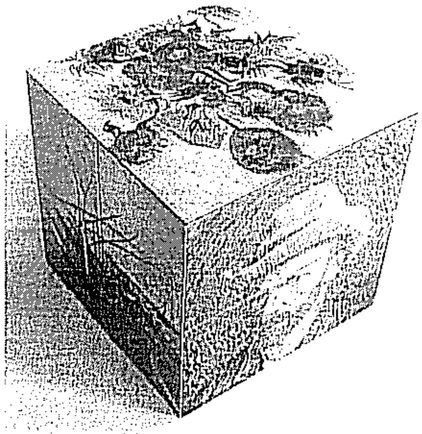
1. Follow instructions to draw the above pattern on <u>paper</u> first. Then copy it onto your thin sponge rubber.
2. Don't cut out the sponge rubber triangle yet! First, make 3 cuts as follows:
   a. Begin at C and cut ALMOST to B.
   b. Begin at E and cut ALMOST to A.
   c. Begin at F and cut ALMOST to D.

   You now have hinges at A, B, and D.
3. Carefully cut out triangle, but leave a little extra border at A, B, and D. Now swing it around to make a square!

Magic Folding Cube

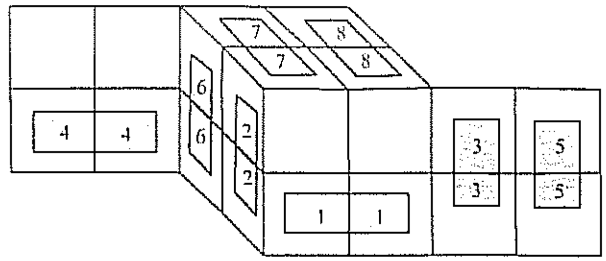
Note: Information about this cube is well known. Our contribution is to give a step-by-step procedure for building it, which has been taught to young children. The supplies needed and the instructions are given below.

The "magic" folding cube is made up of 8 single cubes. When you buy a folding cube from a store, it is usually a picture cube. The faces show nine different pictures, six outside and three inside. Here is a store-bought model which has pictures of works of art by Van Gogh.



A description of the cube

The magic cube consists of eight smaller individual cubes. Each one has six faces, so altogether there are 48 small faces. When the cube is folded, 24 small faces can be seen, and 24 are hidden. You can design your cube so that 24 small faces are their natural wood color, 8 are pink, 8 are green, and 8 are yellow (see below). You can fold the cube so that the colored faces cannot be seen because they are inside! You can also produce three rectangles (of three different colors) consisting of eight small faces each, as in the pictures below, by folding. The cube has eight hinges that are arranged as in the diagram below.

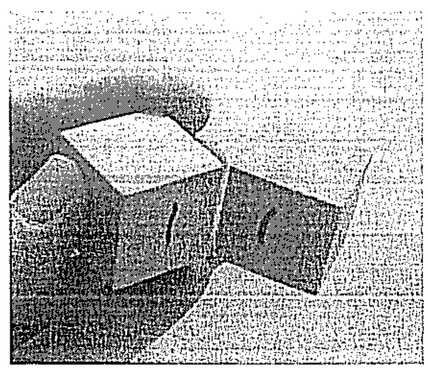


Before building the folding cube, first study the positions of the eight hinges. You join eight equal cubes (cubic inch blocks) with eight pieces of packaging tape (shown here in orange), exactly as shown in the diagram. Do you see that there will be eight hinges?

Each hinge needs to be reinforced. You do this by using two pieces of tape for each hinge, placing the second piece on the side opposite to the first piece. So you actually use 16 pieces of tape. (Click here to see how the first three reinforced hinges are made.) This makes a folding cube that is quite durable and fun to play with.

We have numbered the hinges one through eight in the picture above, and the ones below.

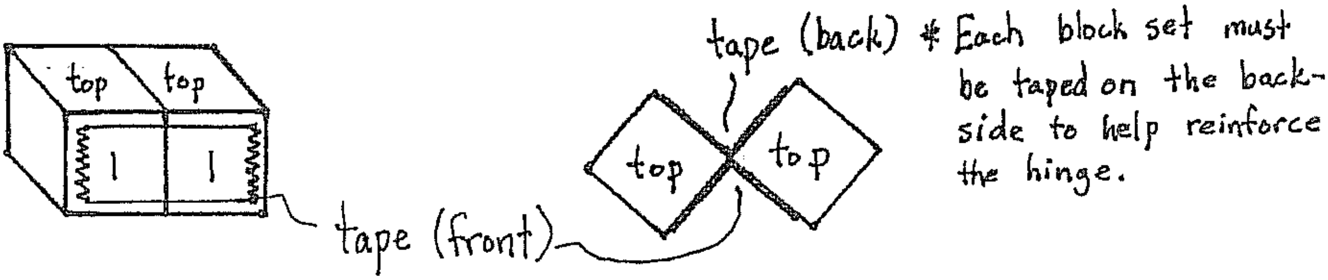
Step 1. Join two blocks with a piece of tape. Reinforce the hinge by using tape on both sides. Label the hinge 1.



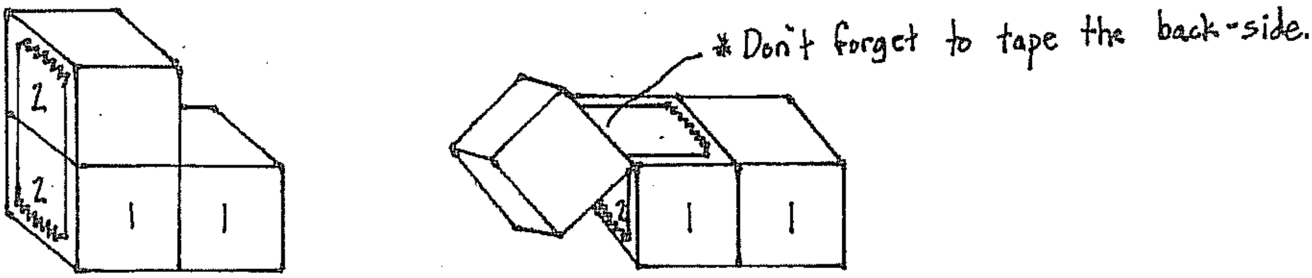
Step 2. Place a block on the left top of your construction, and make hinge 2. Be sure to reinforce the hinge on two sides. Label the hinge.

9

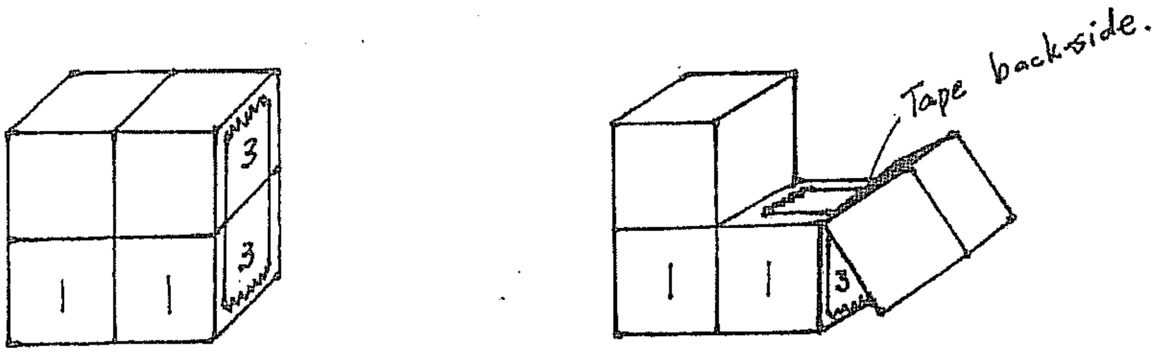
1


top
top
1
1
tape (front)
tape (back)
top
top
* Each block set must be taped on the back-side to help reinforce the hinge.

2


2
2
1
1
* Don't forget to tape the back-side.
2
1
1

3


3
3
1
1
Tape back-side.
3
1
1