Presentation 214

Shapes in Shapes

Building Number Sense and Formula Sense

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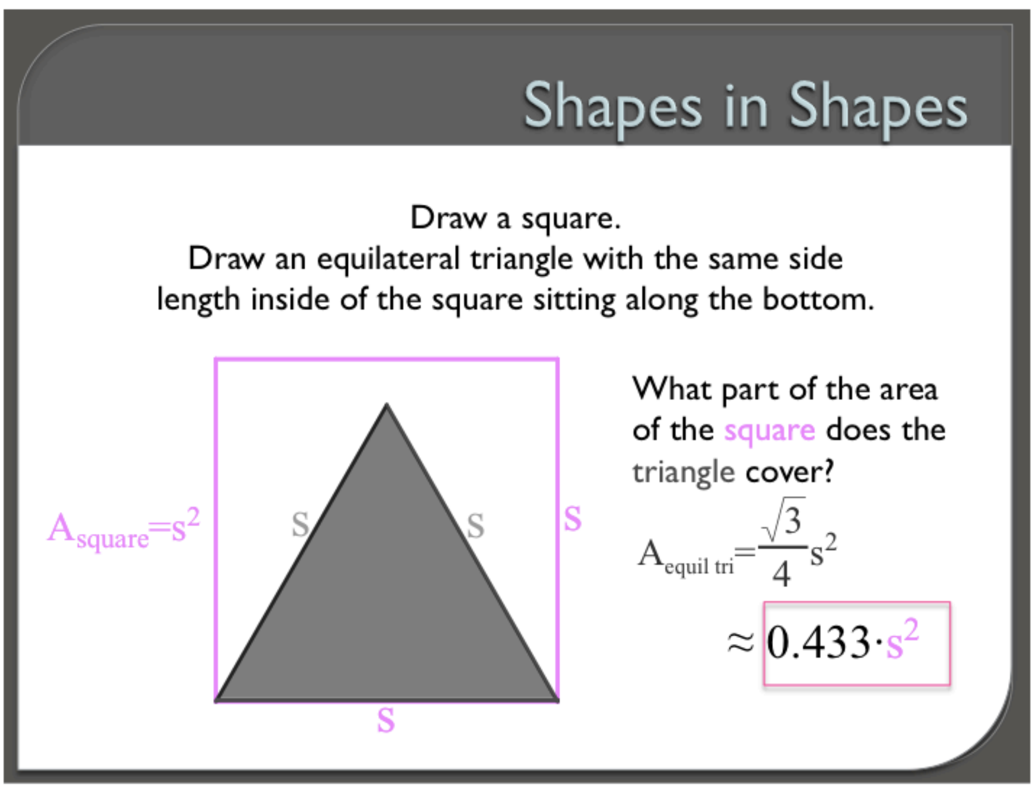
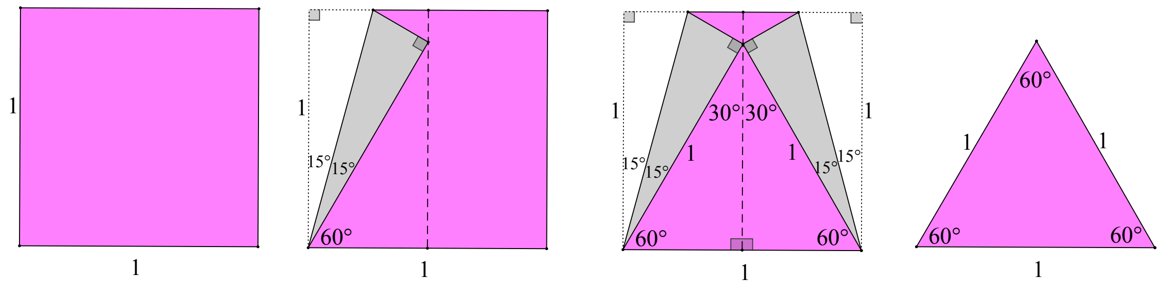
- Starting with Triangles, Squares, and Circles, fit shapes inside other shapes and compare relative Side Lengths and Areas
- Build Unit Tetrahedron out of envelopes
- Each table will collaborate to build a Stella Octangula
- Discuss further properties of these basic 3D shapes and how they fit within each other using the readily available Stella Octangula model

Question:

What part of the area of a Square does an Equilateral Triangle cover?



To fold an equilateral triangle from a square (patty paper works great):

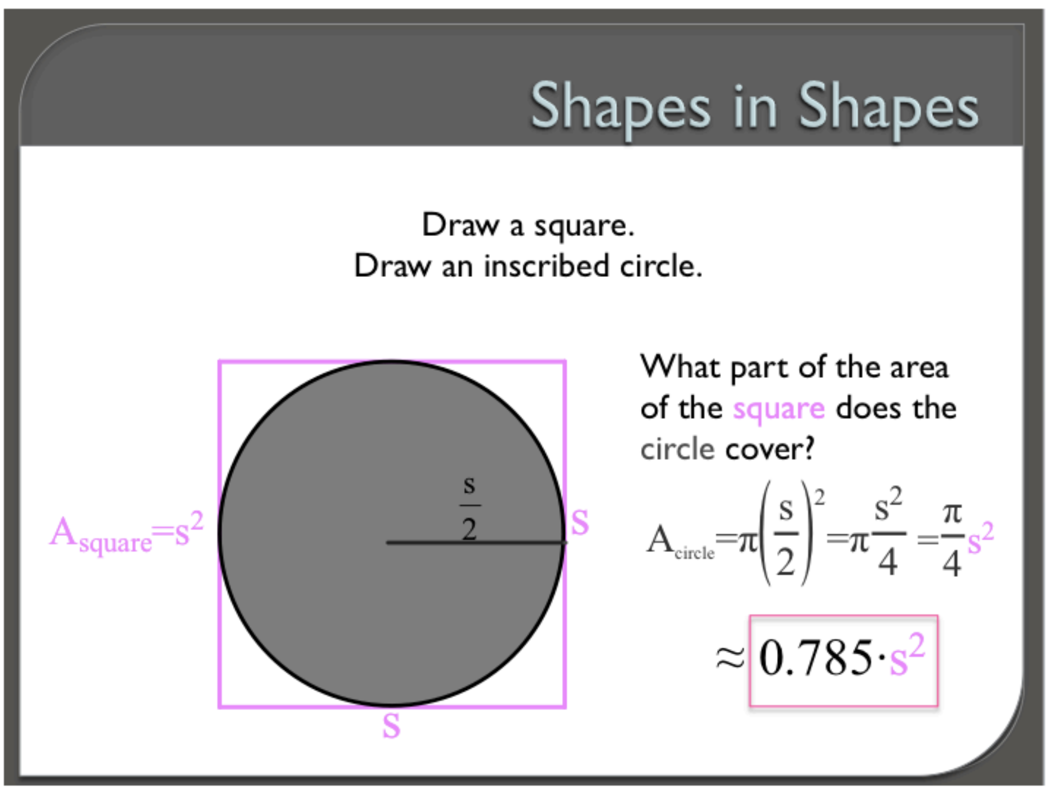


Answer:

The ratio of Areas of an Equilateral Triangle to a Square is $\frac{\sqrt{3}}{4}:1$.

A Unit Equilateral Triangle takes up 43.3% of the area of a Square.

Question:
What part of the area of a Square does a Circle cover?



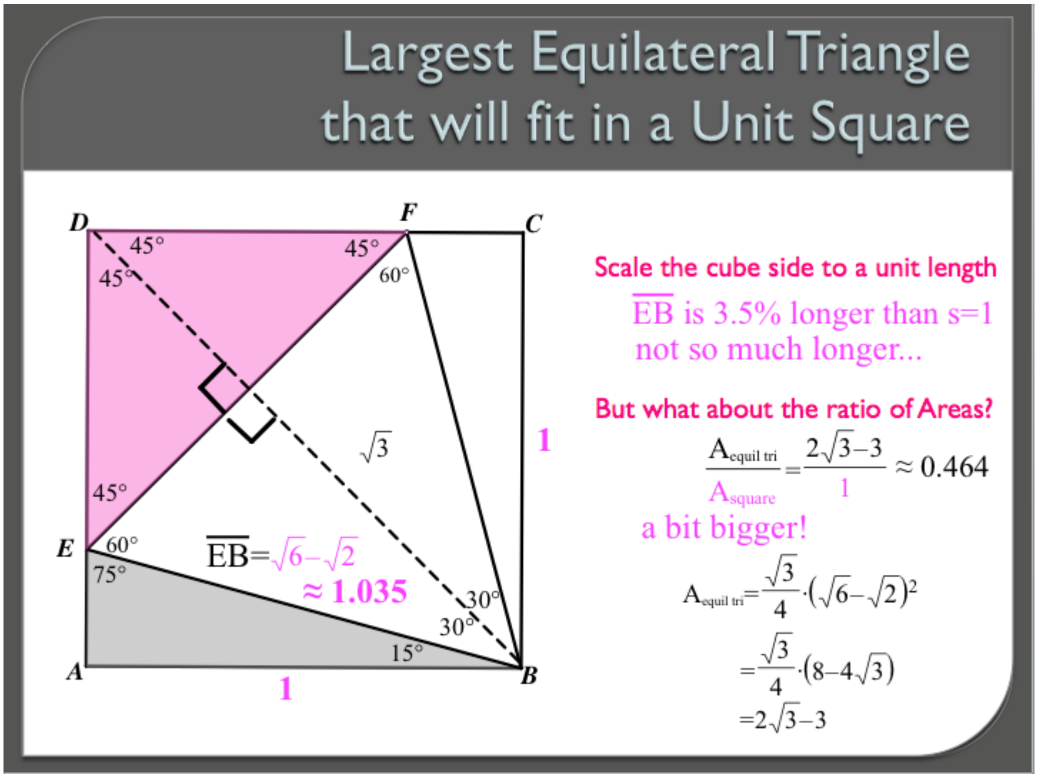
Answer:

The ratio of Areas of a Circle to a Square is $\frac{\pi}{4}:1$.

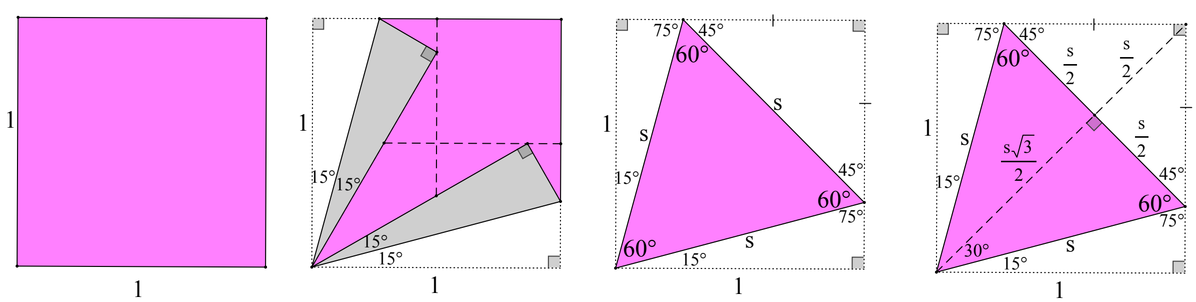
A Circle takes up 78.5% of the area of a Square.

Question:

What is the largest Equilateral Triangle that will fit in a Unit Square with $s = 1$?



To fold the largest equilateral triangle in a square (patty paper works great):



$$d = \frac{s\sqrt{3}}{2} + \frac{s}{2} = \sqrt{2}$$

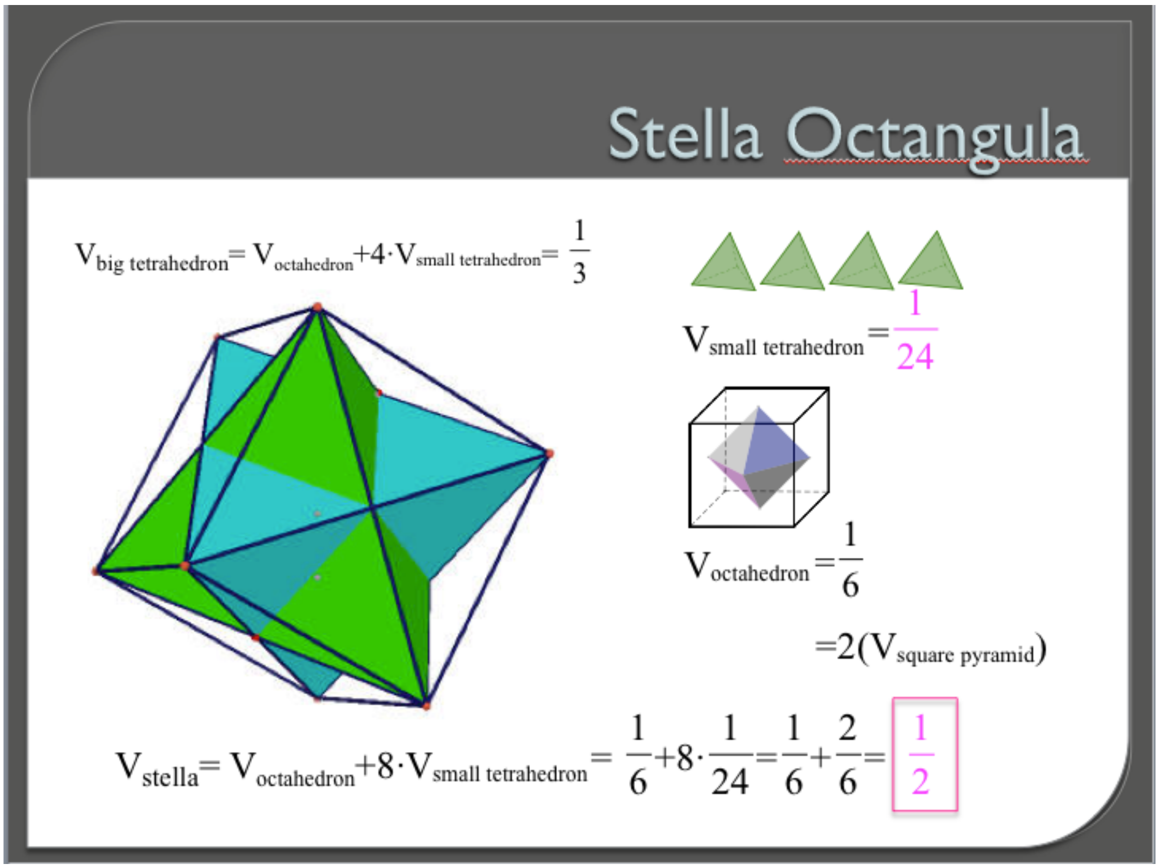
$$\left(1+\sqrt{3}\right)s = 2\sqrt{2}$$

$$s = \frac{2\sqrt{2}}{1+\sqrt{3}} = \frac{2\sqrt{2}}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{2\sqrt{2}-2\sqrt{6}}{-2} = \sqrt{6}-\sqrt{2}$$

Answer:

The largest Equilateral Triangle that will fit in a Unit Square has side length $s = \sqrt{6} - \sqrt{2} = 1.035$ and takes up 46.4% of the area of the Square

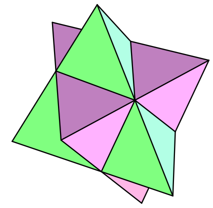
Question:
What is the volume of the Stella Octangula in a Cube of side length, $s = 1$?



Answer:
The Stella Octangula has a side length of $s = \sqrt{2}$.

The volume of the Stella Octangula is exactly $\frac{1}{2}$ the volume of the Cube.

$$V_{stella\,octangula,\,s} = \frac{\sqrt{2}}{8}s^3$$



Volume of a Stella Octangula with side length, $s = 1$

$$V_{stella\,octangula,\,s=1} = V_{octahedron,\,s=1/2} + 8V_{tetrahedron,\,s=1/2} \quad = \frac{\sqrt{2}}{24} + 8\left(\frac{\sqrt{2}}{12}\left(\frac{1}{2}\right)^3\right) = \frac{\sqrt{2}}{24} + \frac{2\sqrt{2}}{24} = \frac{\sqrt{2}}{8}$$

Volume of a Stella Octangula with side length, $s = \sqrt{2}$

$$V_{stella\,octangula,\,s=\sqrt{2}} = \frac{\sqrt{2}}{8}s^3 = \frac{\sqrt{2}}{8}\sqrt{2}^3 = \frac{1}{2}$$

which is pretty remarkable!

Summary

- The volume of a Stella Octangula is $1/2$ that of the smallest cube into which it can be placed.

- The solid common to both intersecting tetrahedron is a regular octahedron of edge length s.

- The volume of a Stella Octangula composed of two intersecting tetrahedron of edge length $2s$, equals 12 times the volume of the smaller tetrahedron of edge length s

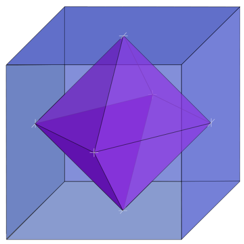
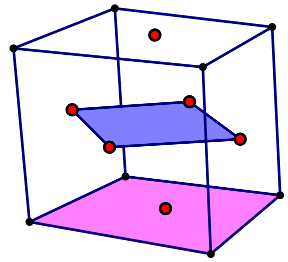
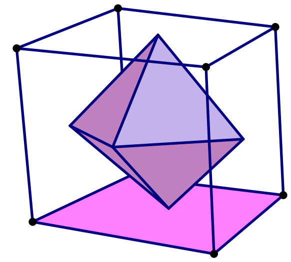
  The Stella Octngula is 3 times the volume of the regular octahedron of edge length s.

- The volume of a regular octahedron of edge length s is 4 times the volume of the smaller tetrahedron of edge length s, or $1/2$ the volume of the tetrahedron of edge length $2s$.

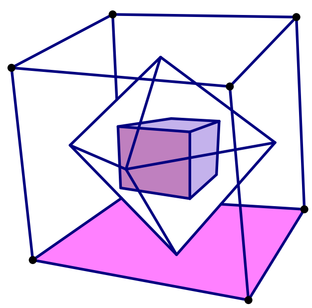
  It is also $1/6$ the volume of the cube in which it is the dual, where it sits with its vertices at the midpoints of each face of the cube.

Duals

Explore the numerical relationship that exist within sequences of duals

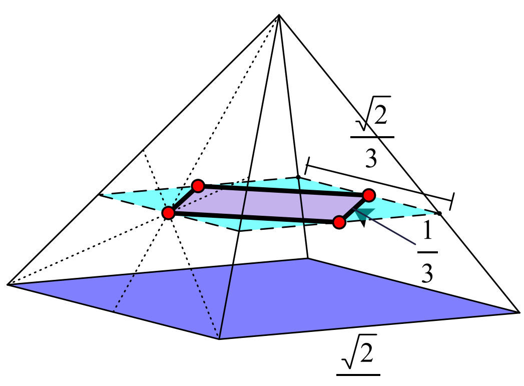




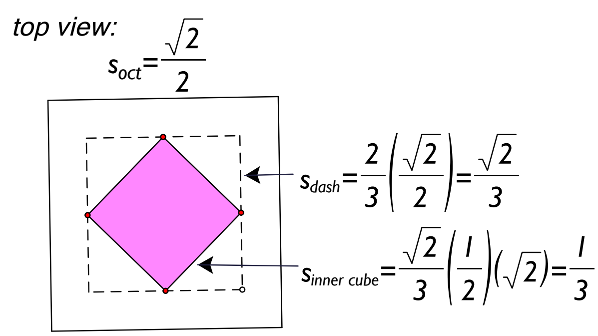
A dual is created by connecting the center of every neighboring face. The dual of a cube is an octahedron, and the dual of an octahedron is a cube. When they are nested together, it is interesting to conjecture what is the ratio of side lengths of successive cubes? Successive octahedra? How do the volumes of successive cubes relate? Successive octahedra?



Starting with a unit outer Cube with $s_{cube} = 1$, we find the inscribed octahedron has side length, $s_{oct} = \frac{\sqrt{2}}{2}$.
Consider the octahedron as two square based pyramids. First consider the dashed square connecting the center of the lateral faces on just the top half of the octahedron. It connects the midpoints of a square that goes along each face, through the centers of the faces, parallel to the base of the top square pyramid.

This dashed square below is one-third of the way up a triangular face, making its side length proportionally two-thirds of the octahedron side length.

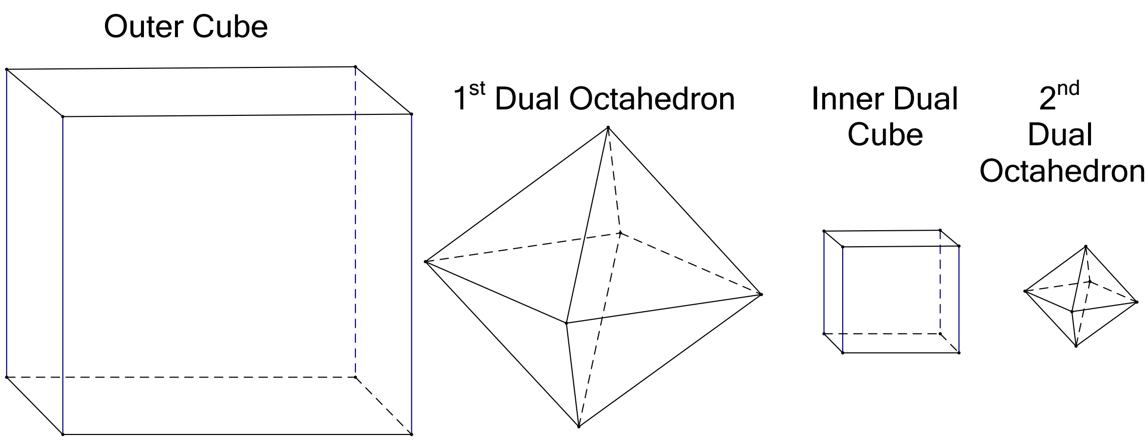




The inner cube side length is the hypotenuse of a 45-45-90 triangle. Multiply half the dashed side length $\frac{\sqrt{2}}{3}\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{6}$ by $\sqrt{2}$ to get $s_{inner\,cube} = \frac{1}{3}$.

Shapes in Shapes in Shapes
Duals

Question:
What is the ratio of side lengths of successive Cubes? Successive Octahedron?
What is the ratio of Volumes of successive Cubes? Successive Octahedron?



cube ➔ $\cdot\frac{\sqrt{2}}{2}$ ➔ *octahedron* ➔ $\cdot\frac{\sqrt{2}}{3}$ ➔ *cube* ➔ $\cdot\frac{\sqrt{2}}{2}$ ➔ *octahedron*

Ratio of Similarity is **3:1**

$s_{cube1} = 1$ $s_{oct1} = \frac{\sqrt{2}}{2}$ $s_{cube2} = \frac{1}{3}$ $s_{oct2} = \frac{\sqrt{2}}{6}$

Ratio of Volumes is **27:1**

$V_{cube1} = 1$ $V_{oct1} = \frac{1}{6}$ $V_{cube2} = \frac{1}{27}$ $V_{oct2} = \frac{1}{162}$

Answer:
The ratio of similarity from Cube to Cube is simply $3:1$.
The ratio of similarity from Octahedron to Octahedron is also $3:1$.

Beautiful!