Voting for Better Reasoning in a Math Classroom

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NCTM Regional Conference
Las Vegas, NV Oct. 2013

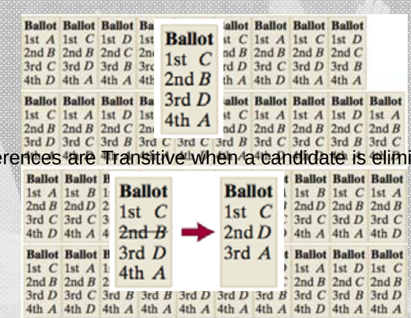
Voting Theory Topics

- Preference Ballots and Preference Schedules
- The Plurality Method
- The Borda Count Method
- The Plurality-with-Elimination Method (Instant Runoff Voting)
- The Method of Pair wise Comparisons
- Fairness Criteria
- Rankings

Example: The Math Club Election

The Math Appreciation Society (MAS) is a student organization dedicated to an unsung but worthy cause, that of fostering the enjoyment and appreciation of mathematics among college students. The Tasmania State University chapter of MAS is holding its annual election for president. There are four candidates running for president: Alisha, Boris, Carmen, and Dave (A, B, C, and D for short). Each of the 37 members of the club votes by means of a ballot indicating his or her first, second, third, and fourth choice. The 37 ballots submitted are shown on the next slide. Once the ballots are in, it's decision time. Who should be the winner of the election? Why?

Preference Ballots



Preferences are Transitive when a candidate is eliminated

Preference Schedule

TABLE 1-1 Preference Schedule for the Math Club Election

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Plurality Method

- Candidate with the most first-place votes (called the plurality candidate) wins
- Don't need each voter to rank the candidates - need only the voter's first choice
- Vast majority of elections for political office in the United States are decided using the plurality method
- Many drawbacks - other than its utter simplicity, the plurality method has little else going in its favor

The Math Club: Plurality Version

TABLE 1-1 Preference Schedule for the Math Club Election

Number of voters	14	10	8	4	1
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*
3rd choice	*C*	*D*	*B*	*C*	*B*
4th choice	*D*	*A*	*A*	*A*	*A*

and the results are clear - *A* wins

Borda Count Method

- Each place on a ballot is assigned points
- With *N* candidates, 1 point for *last* place, 2 points for *second from last*, and so on
- *First-place* vote is worth *N* points
- Tally points for each candidate separately
- Candidate with highest total is winner
- Candidate is called the *Borda winner*

The Math Club: Borda Version

TABLE 1-4 Borda Points for the Math Club Election

Number of voters	14	10	8	4	1
1st [illegible] points	*A*: 5[illegible] pts	[illegible]0 pts	*D*: [illegible]2 pts	[illegible] pts	[illegible] pts
2nd choice: 3 points	*B*: 42 pts	*B*: 30 pts	*C*: 24 pts	*D*: 12 pts	*D*: 3 pts
3rd choice: 2 points	*C*: 28 pts	*D*: 20 pts	*B*: 16 pts	*C*: 8 pts	*B*: 2 pts
4th choice: 1 point	*D*: 14 pts	*A*: 10 pts	*A*: 8 pts	*A*: 4 pts	*A*: 1 pt

- *A* gets: 56 + 10 + 8 + 4 + 1 = 79 points
- *B* gets: 42 + 30 + 16 + 16 + 2 = 106 points
- *C* gets: 28 + 40 + 24 + 8 + 4 = 104 points
- *D* gets: 14 + 20 + 32 + 12 + 3 = 81 points

The Borda winner of this election is Boris! Not Alisha?

Plurality-with-Elimination Method

Round 1: Count the first-place votes for each candidate, just as you would in the plurality method. If a candidate has a majority of first-place votes, then that candidate is the winner. Otherwise, eliminate the candidate (or candidates if there is a tie) with the fewest first-place votes.

Plurality-with-Elimination Method

Round 2: Cross out the name(s) of the candidates eliminated from the preference schedule and recount the first-place votes. (Remember that when a candidate is eliminated from the preference schedule, in each column the candidates below it move up a spot.) If a candidate has a majority of first-place votes, then declare that candidate the winner. Otherwise, eliminate the candidate with the fewest first-place votes.

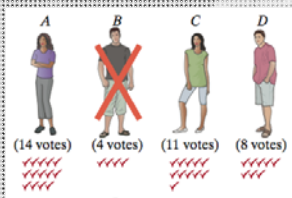
Plurality-with-Elimination Method

Round 3, 4 . . . . Repeat the process, each time eliminating one or more candidates until there is a candidate with a majority of first-place votes. That candidate is the winner of the election.

The Math Club: P-W-E Version

Round 1.

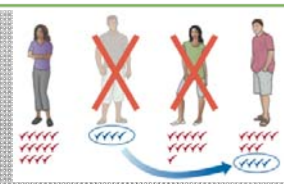
Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	14	4	11	8



The Math Club: P-W-E Version

Round 2. *B*'s 4 votes go to *D*, the next best candidate according to these 4 voters.

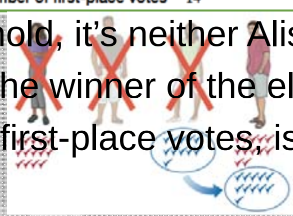
Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	14		11	12



The Math Club: P-W-E Version

Round 3. *C*'s 11 votes go to *D*, the next best candidate according to these 11 voters.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	14			23



We now have a winner, and lo and behold, it's neither Alisha nor Boris. The winner of the election, with 23 first-place votes, is Dave!

The Method of Pairwise Comparison

Every candidate is matched head-to-head against every other candidate. Each of these head-to-head matches is called a pairwise comparison. In a pairwise comparison between *X* and *Y* every vote is assigned to either *X* or *Y*, the vote going to whichever of the two candidates is listed higher on the ballot. The winner is the one with the most votes; if the two candidates split the votes equally, the pairwise comparison ends in a tie.

The Method of Pairwise Comparison

The winner of the pairwise comparison gets 1 point and the loser gets none; in case of a tie each candidate gets 1/2 point. The winner of the election is the candidate with the most points after all the pairwise comparisons are tabulated.

(Overall point ties are common under this method, and, as with other methods, the tie is broken using a predetermined tie-breaking procedure or the tie can stand if multiple winners are allowed.)

The Math Club: Pairwise Comparison

TABLE 1-11 Pairwise Comparison Between *A* and *B*

Number of voters	14	10	8	4	1
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*
3rd choice	*C*	*D*	*B*	*C*	*B*
4th choice	*D*	*A*	*A*	*A*	*A*

A versus *B*: 14 votes to 23 votes (*B* wins)
B gets 1 point.

The Math Club: Pairwise Comparison

TABLE 1-12 Pairwise Comparison Between *C* and *D*

Number of voters	14	10	8	4	1
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*
3rd choice	*C*	*D*	*B*	*C*	*B*
4th choice	*D*	*A*	*A*	*A*	*A*

C vs *D*: 25 to 12 votes (*C* wins) C gets 1 point.

The Math Club: Pairwise Comparison

Comparing in all possible ways two candidates at a time:

A vs *B*: 14 to 23 votes (*B* wins) *B* gets 1 point
A vs *C*: 14 to 23 votes (*C* wins) *C* gets 1 point
A vs *D*: 14 to 23 votes (*D* wins) *D* gets 1 point
B vs *C*: 18 to 19 votes (*C* wins) *C* gets 1 point
B vs *D*: 28 v to 9 votes (*B* wins) *B* gets 1 point
C vs *D*: 25 to 12 votes (*C* wins) *C* gets 1 point

The winner of the election is Carmen!

What Could Go Wrong?

Each method has inherent potential to violate various rules of fairness.

Arrow's Impossibility Theorem states that an error free method does not exist. We therefore must pick a method that poses the least risk and is also reasonable to use.

Let's look at those fairness criteria.

Fairness Criteria

- **The Majority Criterion**
- **The Condorcet Criterion**
- **The Monotonicity Criterion**
- **The Independence-of-Irrelevant-Alternatives Criterion (IIA)**

The Majority Criterion

In a democratic election between two candidates, the candidate with a majority (more than half) of the votes should be the winner.

After all, it seems clearly unfair when a candidate with a majority of the first-place votes does not win.

The Condorcet Criterion

A Condorcet candidate should always win the election.

When the candidates are compared two at a time, the Condorcet candidate beats each of the other candidates. How could it be fair to declare a different candidate as the winner?

The Monotonicity Criterion

Suppose candidate *X* is a winner of the election, but for one reason or another there is a new election. If the only changes in the ballots are changes in favor of candidate *X* (and only *X*), then *X* should win the new election.

The Independence-of-Irrelevant-Alternatives Criterion (IIA)

Suppose candidate *X* is a winner of the election, but for one reason or another there is a new election. If the only changes are that one of the other candidates withdraws or is disqualified, then *X* should win the new election. The flip side of this criterion is that a winner of the election should not be penalized by the introduction of irrelevant new candidates who have no chance of winning.

The Violations

- Plurality: Violates the Condorcet Criterion
- Borda Count: Violates the Majority Criterion and the Condorcet Criterion
- Plurality with Elimination: Violates the Monotonicity Criterion
- Pairwise Comparison: Violates the Independence of Irrelevant Alternatives Criterion
- The violations are possible, not guaranteed.

Rankings

- Each method can be extended to provide ranking of candidates.
- The basic idea is to perform the count according to the method desired. Once a winner is found, eliminate the winner from the preference schedule and re-count. The new “winner” is the second place. Repeat until all candidates are ranked.
- With Elimination methods, the ranking is the reverse from the elimination. In other words, the first candidate eliminated is the last place, and so on.

References

Tannenbaum, Peter, 2010, Excursions in Modern Mathematics, 7th Edition, Pearson Education, Inc. Boston, MA