Making, Generalizing, and Justifying Conjectures about Number and Operations

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Overview

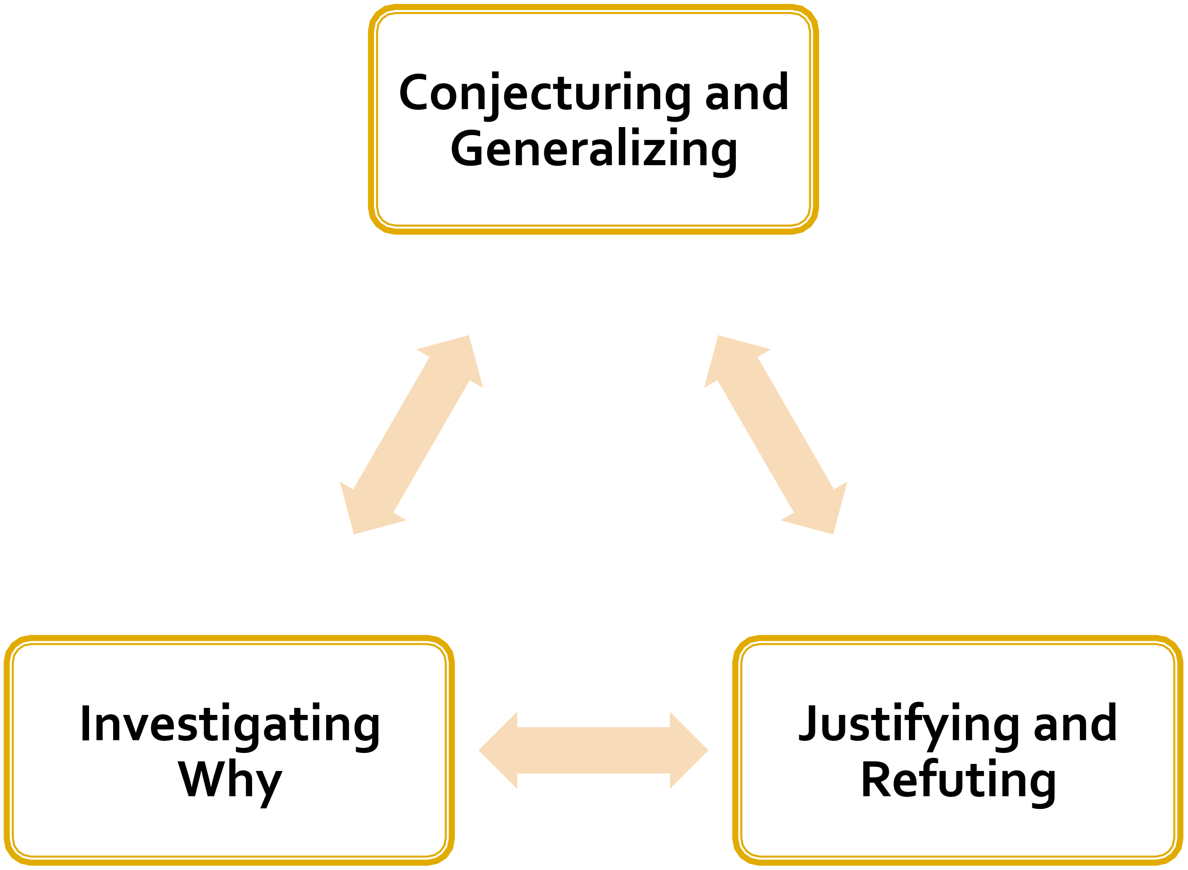
- Background
- Key Aspects of Mathematical Reasoning
- Examples
- Discussion
- Summary

Reasoning in the Common Core Standards (2011)

Students should:

- Reason abstractly and quantitatively (attending to the meaning of numbers as quantities)
- Construct valid arguments and critique the reasoning of others
- Look for and express regularity in repeated reasoning

Key Components of Reasoning



Reasoning Process

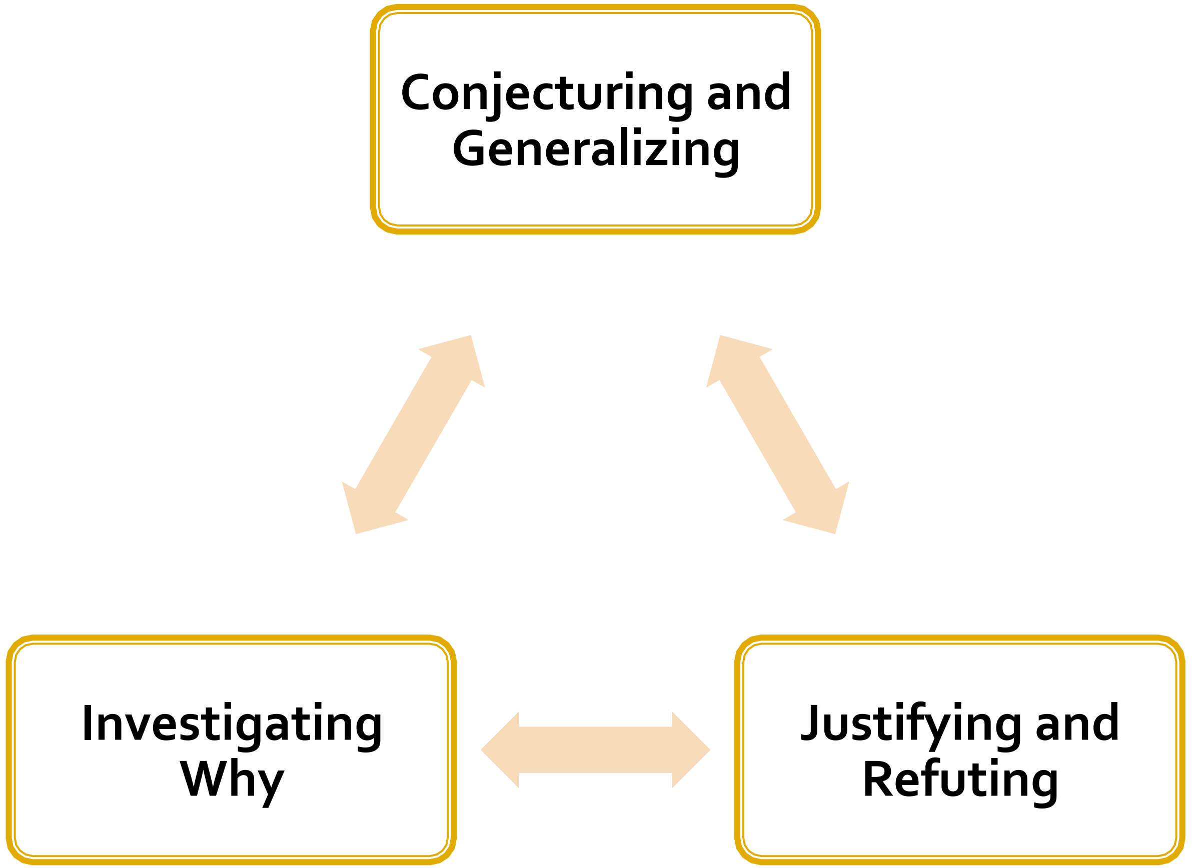
What happens when we add 1 to the numerator and the denominator for a positive fraction? Make a conjecture and attempt to support or refute your conjecture.

For example:

$^{7}/_{8} \rightarrow {}^{8}/_{9}$

$^{24}/_{37} \rightarrow {}^{25}/_{38}$

Key Components of Reasoning



Conjecturing

- Conjectures – statements thought to be true but not known to be true.
  - The taller the person, the older the person.
  - 7/8 is more than ½.
  - If you toss 3 heads in a row, the next toss will probably be tails.
  - $2/3 + 7/8 < 2$

Whether a statement is or is not a conjecture depends on the view of the person.

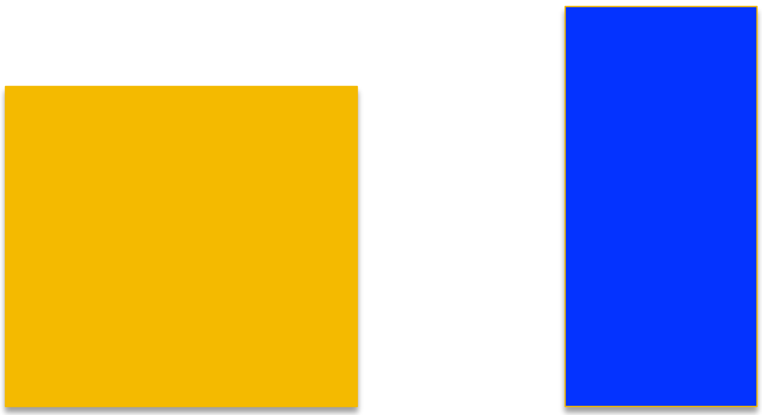
Generalizing

- Generalizing involves identifying commonalities or extending reasoning to new situations.
  - $2/3 + 7/8 < 2$
  - Figures with “pointy tops” are triangles.
  - 9 + 6 is the same as 9 + 1 + 5 which is 10 + 5 = 15, extending to 98 + 45 is the same as 98 + 2 + 43 which is 100 + 43 = 143.

Need to clarify when a generalization does and does not apply.

Conjecturing and Generalizing

- Students naturally conjecture and generalize
- Need to clarify their conjectures and generalizations
- For example, what makes something "bigger?"



Investigating Why

- Trying to figure out why a statement may be true or false.

Investigating why

Maria's Conjecture

I multiplied 3 *x* 5 and noticed that this is one less than 4 *x* 4. I tried this with some other numbers, and it seems to work for all numbers.

- 4 *x* 6 = 24, and that is one less than 5 *x* 5 = 25
- 7 *x* 9 = 63, and that is one less than 8 *x* 8 = 64
- 6 *x* 8 = 48, and that is one less than 7 *x* 7 = 49

I think that whenever you multiply two numbers that are two apart, the result is one less than the number in the middle times itself.

Investigating why

Look at examples

10 x 10 = 100 & 9 x 11 = 99

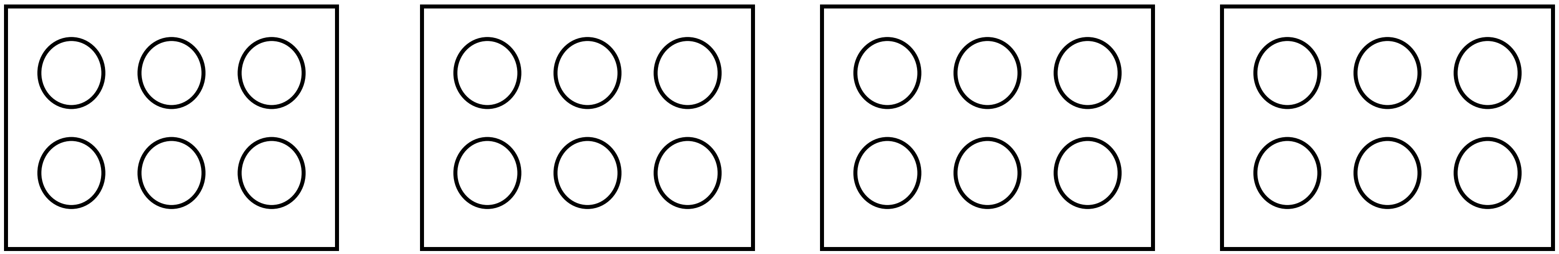
25 x 25 = 625 & 24 x 26 = 624

78 x 78 = 6084 & 77 x 79 = 6083

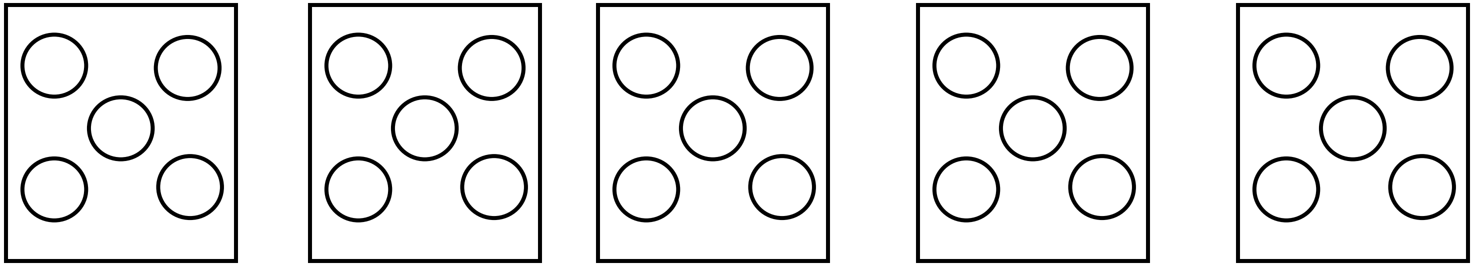
1 x 1 = 1 & 0 x 2 = 0

Investigating why

- Four Groups of Six: 4 x 6

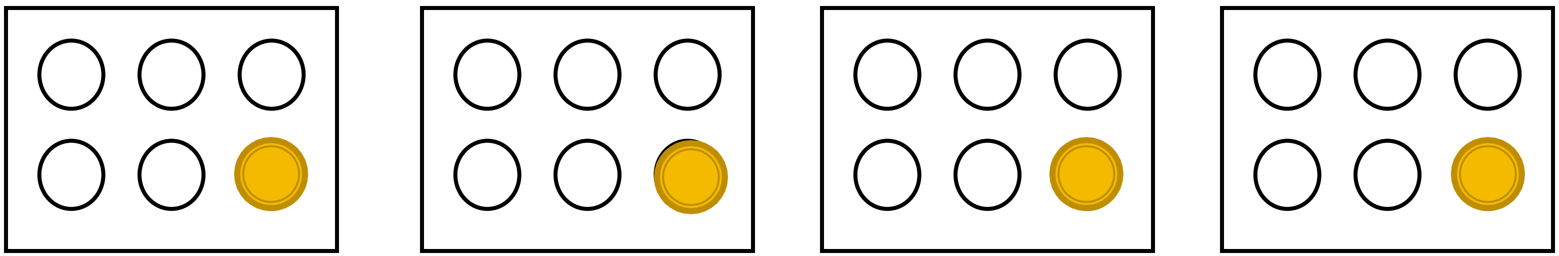


- Five Groups of Five: 5 x 5



Investigating why

- Four Groups of Six: 4 x 6



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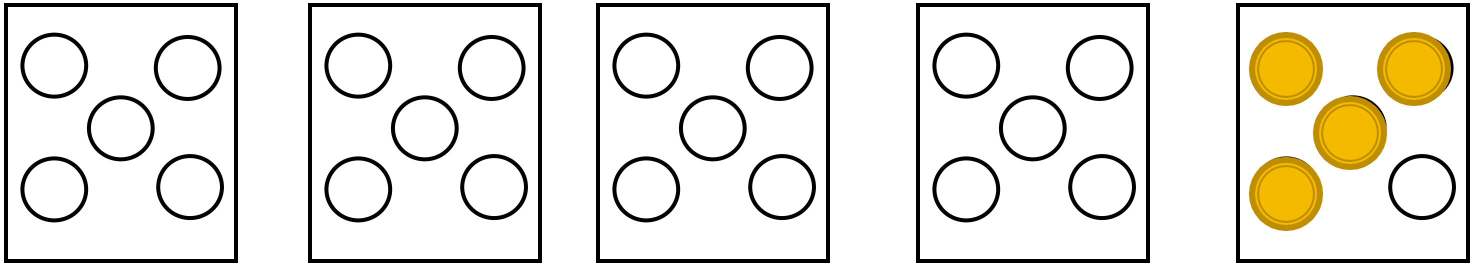
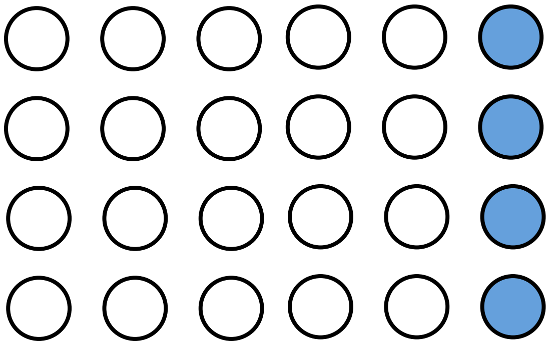
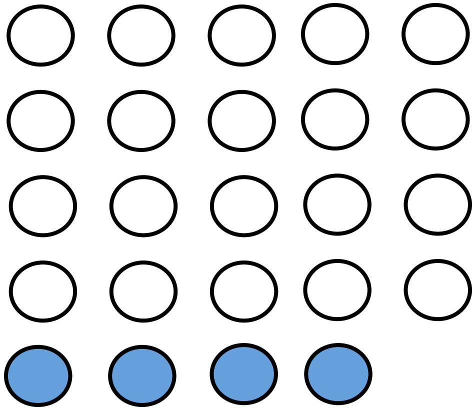


Investigating why

- 4 x 6 array



- 5 x 5 array



Investigating why

- Consider what we could examine further.
- What relationship exists that could explain why?
- What can we investigate?

Important Distinction

- Students need to recognize when a statement is true or false.

- Students need to recognize when a valid argument or invalid argument is being provided.

Justifying or Refuting

- Justifying involves providing a logical argument to support a conjecture.
- The justification should show why a statement is true for every instance in a generalization.
- Easy to state, but difficult to recognize what is valid.

Justifying or Refuting

Common Core Standards

- Recognize and generate simple equivalent fractions (e.g., 1/2 = 2/4, 4/6 = 2/3). Explain why the fractions are equivalent, e.g., by using a visual fraction model. (Grade 3)

- Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. (Grade 4)

Four Justifications for 3/4 = 6/8

- Decide which of the justifications (see handout) are valid and which are not. Why do you think so?

Justifying or Refuting

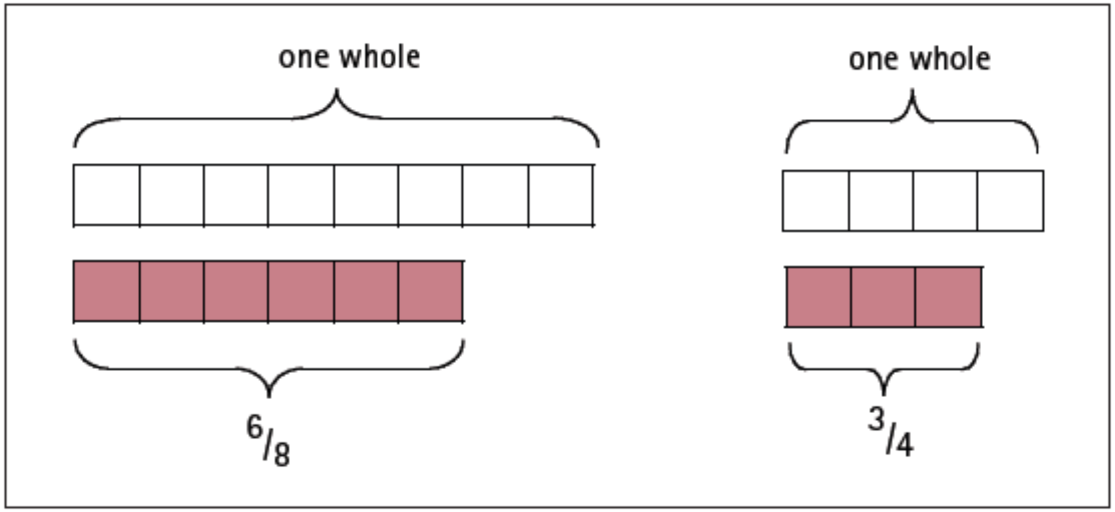
Abby's Justification

To show two fractions are equivalent, look at *3/4* and *6/8* . Three-fourths is equivalent to *6/8* because you can divide the numerator and denominator for *6/8* by 2, making *3/4* the same number as *6/8*. You can do this for any two fractions.

Justifying or Refuting

Briana's Justification

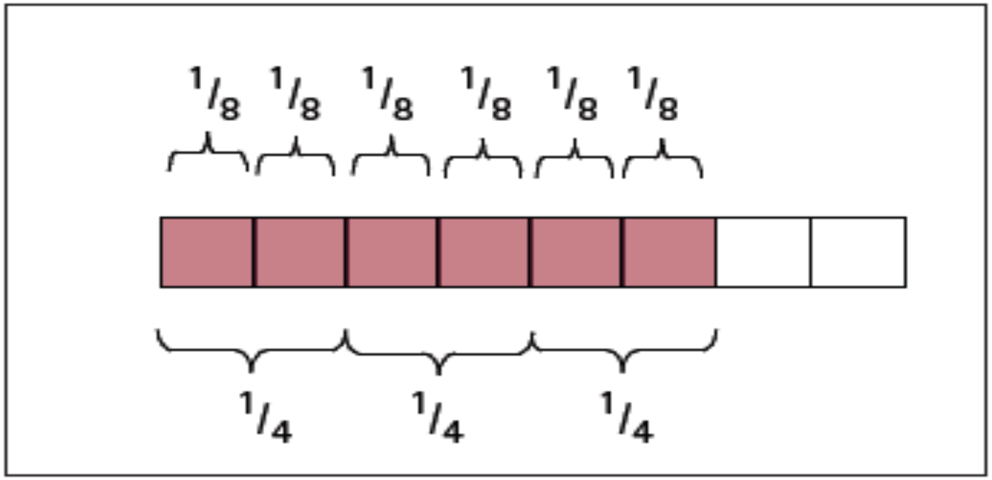
Two fractions are equivalent when you can divide one fraction by something to make the other fraction. Three-fourths is equivalent to *6/8* because you can divide *6/8* in half to make *3/4*. See my picture where *6/8* divided by 2, or cut in half, is the same as *3/4*.



Justifying or Refuting

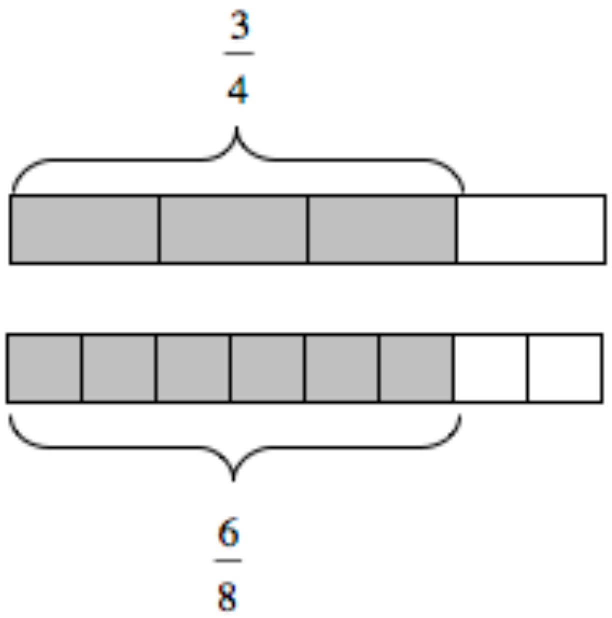
Candy's Justification

Three-fourths is equivalent to *6/8* because if you draw both pictures you can see that two of the eighths is the same as one-fourth. So you can see that the shaded *3/4* is the same as *6/8*. See my picture below. For equivalent fractions, you can always split parts to generate the same fraction. For example, *3/4* is the same As *6/8*, because you can split each of the fourths into two parts, making eighths.



Justifying or Refuting

Debbie's Justification
Three-fourths is equivalent to *6/8* because if you draw both pictures *6/8* looks like it is the same as *3/4*.



Using an invalid argument

- A general argument may be invalid for a number of reasons
  - Uses support that involves mistaken components
  - Does not address all aspects of the generalization
  - Supported by examples rather than general relationships

Refuting a false statement

- Important that students learn how to show statements are false.
- Recognize the role of counterexamples in generalizations

Example:

3/4 is the same as *7/8* because both fractions are one part away from the whole.

Refuting a false statement

- When you multiply two positive numbers, the answer is always larger than the original numbers.

Refuting a false statement

Suppose two one-gallon containers are filled with liquids that are mixtures of grape juice and water. The 1/8 of the first gallon is grape juice, and 3/8 of the second gallon is grape juice. If the two gallons are mixed together, what fraction of the combined mixture will be grape juice?

- A student conjectures that 1/2 of the mixture will be grape juice, since 1/8 + 3/8 = 1/2.

Modifying or Extending Tasks

- "Show this is true" or "Show this is false"
- Instead: Determine whether the statement is true or false and explain why the statement is true or false.
- Use statements that are not obvious to students. (e.g., attempting to use compensation with multiplication)

Discussion Activity (on handout)

- How can these problems be revised or extended (if necessary) to encourage students to make conjectures, generalize, and justify?

1. Fill in the blanks with the correct symbols (>, =, or <):

$$(a)\ \frac{5}{9}[\quad]\frac{7}{9}$$

$$(b)\ \frac{3}{5}[\quad]\frac{12}{20}$$

$$(c)\ \frac{7}{15}[\quad]\frac{7}{16}$$

2. Compute the following:

$$(a)\ 2 \times \frac{1}{5}$$

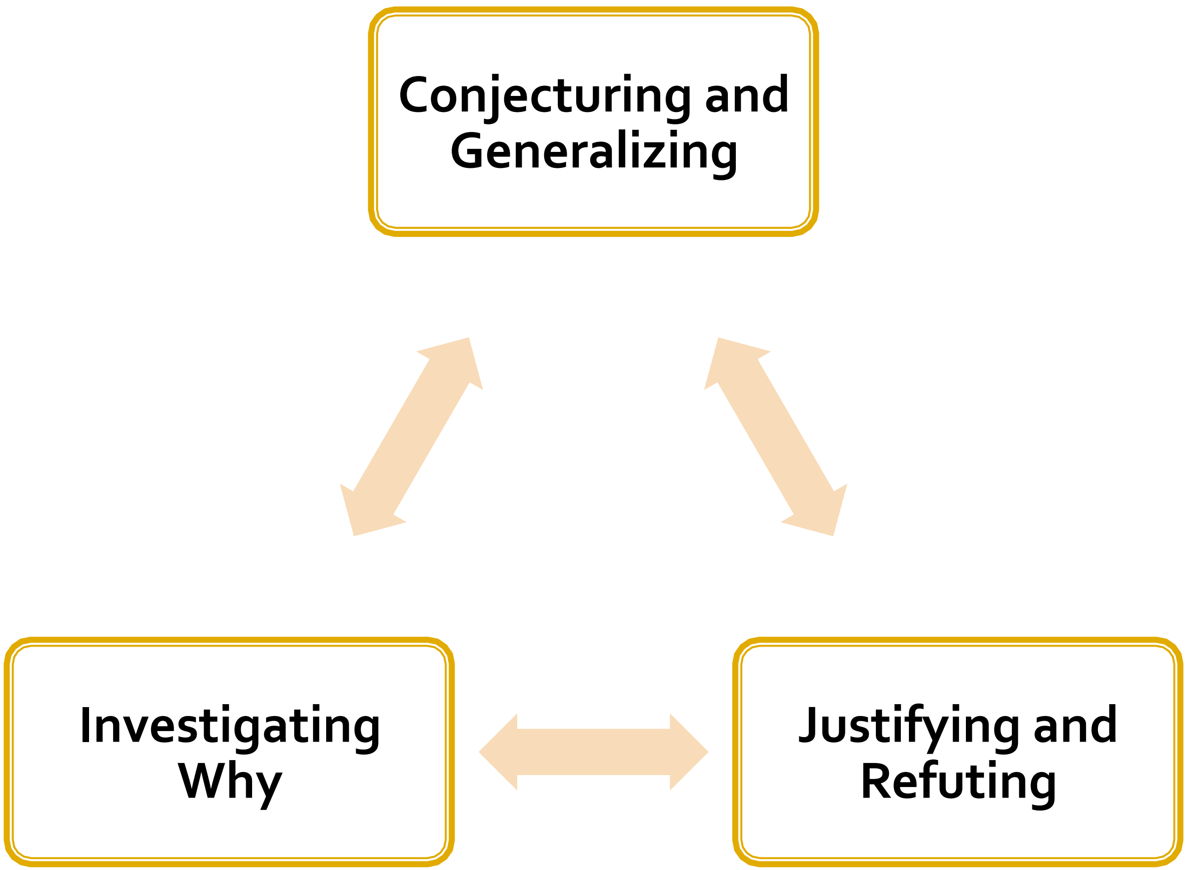
$$(b)\ \frac{1}{3} \times \frac{1}{5}$$

$$(c)\ \frac{2}{3} \times \frac{1}{5}$$

Problem 3

- Brandon used his calculator to multiply 19/18 by 6000 and got 5333.333… Does that answer seem reasonable? Why or why not?

Key Components of Reasoning



Big Ideas about Reasoning

- Students need opportunities to share their conjectures and generalizations
- Teachers need to assist students in exploring reasons why a statement may be true or false.
- Students need to examine what constitutes a valid justification or refutation.

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Thanks!

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