## Mathematical Reasoning Through the Grades

An introduction to the video series

The Common Core State Standards for Mathematical Practice (MP) are part of the expectations for students at every grade level. They require students to learn mathematical content with understanding, including to:

- Reason quantitatively and abstractly (MP2); and
- Construct viable arguments and critique the reasoning of others (MP3).

We set out in this series of lessons to try to capture how students grow and mature in these areas over time. You will be able to observe the dramatic changes that take place-from a kindergartner's first tentative attempts to connect bits of information, to an intermediate student's wrestling with concepts of fraction operations, and culminating in a high school student's application of trigonometry in understanding how to build and fly drones.

1. Hyman Bass (mathematician): Reasoning "requires two foundations: a body of prior knowledge, either inherited or assumed," and the verbal ability and logic to justify claims. ${ }^{1}$ NCTM agrees: "A mathematical justification is a logical argument based on already-understood ideas," and "a successful justification ... explains why by providing insight into underlying relationships that hold in every instance."2

Since valid justifications should extend beyond particular problems, it is often necessary in the earliest years for the teacher to help children understand and/or extend what they see and frame those underlying relationships. The teacher can do this with questions, examples, and sometimes restatements of what children say. In Mathematical Reasoning through the Grades, children are in the process of learning and each insight moves them closer to understanding.
2. Words like "since" and "because" indicate a related chain of thinking or reasoning. "If...then..." statements recognize that one fact or event follows from another.

This video series Mathematical Reasoning through the Grades begins in kindergarten with the numbers 11 to 19 and ends in grade 11 with trigonometry. Below, we have identified some moments from the lessons that we think qualify as mathematical reasoning. How close they come to mathematical reasoning as envisioned in the Common Core State Standards is a topic that should yield rich

[^0]discussion among teachers and perhaps point the way to digging deeper into the topic.

## Elementary-Grades Videos

- The Common Core State Standards say this about constructing viable arguments in the early grades: "Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions." ${ }^{4}$
- "Justifications at this level [K-2] ... may simply be a way for students to share their strategies when solving a problem rather than explaining why something always works."5
- "Students in grades 3-5 can develop justifications that are more general in nature than the ones typically developed by students in prekindergarten-grade 2. ... Students' abilities to make sense of one another's arguments and to make decisions about the validity of other justifications increase as they progress through the grades. Furthermore, student conjectures and generalizations will increase in sophistication."6

Kindergarten Lesson: Beyond Fingers. In this lesson, it is difficult to pinpoint with certainty what the reasoning is for understanding the numbers 11 to 19 . Children say there are 10 because a 10 -frame can have 10. It is not easy to tell whether they think of a 10 -frame as a measure of 10 or they think, "I know there are two rows of 5 in the 10 -frame, so l know it has 10 ," or they simply subitized the pattern. They may know that the frame was just filled with $2+8$ and so there are 10 . A child may remember that when asked to "prove" that $8+2$ is 10 , Alexia places 8 blue and 2 red objects in the 10 -frame and fills it up. Is knowing that a full 10 -frame holds 10 , and thinking "10 plus 2 more ones is 12 ," considered an early step to reasoning? Children are using commonly held knowledge that the frame holds 10 and building on it. to build on? While we can't know exactly what is going on inside the children's heads, these thoughts are at least small preparatory steps to clearer statements that can be accepted as reasoning or building an argument.
First-Grade Lesson: Leprechaun Traps. We are more confident of the instances that show reasoning in this first-grade lesson. When Logan identifies a mystery number located in the third row and third column of a 100 -grid as 33 , and is asked how many to the next "friendly" number (a multiple of 10 and identifies the friendly number as 40 "because it goes 3,4 ." The child is buildingon his knowledge of the structure of the number system. The teacher scaffolds the language and clarifies his thinking of 3 tens and 4 tens for other children.

[^1]Gabriella relates the solution of the previous problem $(24+6)$ to $25+6$. "I heard Demetrius say 30 (when adding $24+6$ ) and then 24 plus one more is 25 , and then you add one more from the 24 and then it's one more from the 30 ." She clearly sees a relationship between the two equations and uses her first-grade language to explain that 24 is one more than 25 , so the sum of $25+6$ has to be one more than the sum of $24+6$.

Justin adds $25+6$ and gets 31 mentally. He reasons, "Because I knew ... I took away 5 from the 6 and then I made a 30. Then I had 1 left, then I made 31." 5 and 10

When solving the situational problem about leprechaun traps, Jean Marie explains that others had 3 traps in each of 10 containers and thought that they could just put "two piles" of 3 together (6) and that would take 5 containers. We also see Briella the budding economist reason about how many containers Mrs. Wright should buy. "I think you should buy three contain ers because it's the cheapest way. They're containers and each one must cost the same amount of money and like something plus something plus something must equal something, like if it's a bigger number it would be an even bigger number and l'm thinking everyone wants to save money." So generalizations start even though they may lack precision.

Fourth-Grade Lesson: Square-Foot Garden. By intermediate grades, it is much easier to identify students' reasoning. Children now have a better vocabulary, have more mathematical knowledge, and can more easily connect statements. They are beginning to use words such as "because" and "since" and "so"" that mark the building of conclusions on a known or just-realized idea. In this lesson, fourth-grade children reason using their knowledge of yards and fractions. A child explains, "A yard is 3 feet. Two-thirds of a yard would have to be 2 feet, and since she has one ribbon in each corner she has to multiply 2 feet by 4 , which is 8 yards [sic]." While wrestling initially with units of both yards and feet, the student finally reaches the conclusion, "You have to purchase 9 ... no ... you could purchase 3 yards." These statements are built on common knowledge of what a yard is and what the problem requires.
Mrs. Spies teases out an explanation of why, with repeated addition of fractions, the denominator stays the same. This same argument is connected to multiplying a fraction by a whole number: You multiply the numerator but not the denominator. Students appear to be developing enough understanding so they will be able to build more precise language over time with the help of the teacher. "You can change the top because you can add as many as you need, but you keep the bottom the same because that's as many as are in one whole." The teacher helps clarify that the denominator is telling us how many pieces the whole has been divided into, modeling a clearer description of what they seem to understand.

Students are beginning to build a generalization that will be true for every case in which a whole number and a fraction are multiplied. Such understandings are not built in a day. They become clearer and stronger over time. John Carroll said long

[^2]ago that learning is a function of time spent over time needed. Now that the Common Core has provided more time to spend developing deeper understanding, students can become more adept.

A Passion for Fractions. In this class with fourth- and fifth-grade students, reasoning is becoming clearer. In fifth grade, students multiply two fractions with different denominators, a much more complex idea than a whole number and a fraction. Their reasoning revolves around representations of a situational problem. Which representations are correct? Which are incorrect? The diagrams use color to help students see and talk about the representations and the problem. One student is asked to explain what the plain orange block represents and searches for the right words: "The only orange block, like, yeah, block, represents-they represent a part of the third, the final third, because the riddle is only on ${ }^{2 / 3}$ of them and there's a last third to it. So you still need to include it."

Edward provides a reason when asked to explain why one of the representations is wrong. "The reason I think it's wrong is because they counted the $\frac{3}{3}$ from each part of the $3 / 4$. The $6 / 9$ is not of a mile; it's of the track. There's actually three more parts here, and if you do $3 \times 4$, it's 12 . It would actually be ${ }^{6} / 12$ of a mile, which is also $1 / 2$, and they only have $\%$, which is wrong!"

Better language, clearer connections, and the ability to deal with two fractions that refer to two different entities (one mile and the track), take students to a different level.

## Middle and High School Videos

"The conjectures that students in grades 6-8 create are more sophisticated and general in nature. ... [They] have more tools at their disposal," including algebraic ideas that allow them to rely on symbolic and algebraic representations. ${ }^{8}$ Use of formal mathematical symbols continues to increase in high school, when students produce proofs, and the range of proof types (indirect, proof by induction) continues to expand. What is very clear is that a strong foundation of core content knowledge is essential for students to reason mathematically. Enabling constant connections among mathematical ideas is a crucial support for the use of these and the other Common Core Standards for Mathematical Practice. That will help build high-level reasoning.

Eighth-Grade Lesson: Conjecturing about a Function. In this lesson, Mrs. McPhillips' purpose is to take students beyond generalizing the functions with an equation, to looking at sets of functions and drawing conclusions that hold true for all functions of that type. The video captures only day one of the lesson during which students are noticing things that will help them make conjectures. She emphasizes the critical role that knowledge of mathematics content plays in reasoning. She is excited, therefore, when Michael says his group looked for and made use of

[^3]structure. Mrs. McPhillips asks what their group was discussing. "We were discussing on how to find the beginning and constant rate of change on each problem." She rephrases that: "So Michael's saying, 'We had to look for important math structures while we were analyzing this function and noted things like rate of change, and $y$ intercept, and beginning.' They were examining all those structures to kind of make connections. ... So I think people who make good conjectures, use structure really well."

Zachary and Steven begin to formalize a conjecture that patterns with a constant rate of change but different beginnings will always result in graphs that are collinear and parallel. The next day the class tested it with a wide range of linear equations to try to find a counterexample (but they could not). Zachary and Steven also justified their conjecture with the following statements: 1. This will always work because the constant rate of change for each linear function means the lines will have the exact same slope. 2. This will also always work because the y-intercepts are the start of the function in the first quadrant, and this will just move them up and down. Another group added on to this with a third piece: "We discovered that if you have the same rate of change and different y-intercepts that by comparing just the y-intercepts, we know where the line (collinear points) will fall compared to the other one. For example, if the $y$-intercepts are 5 and -2 , the first one will end up above the second one.

Students are beginning to extend their thinking beyond the specific problem they are solving with the help of specific questions the teacher poses to help them crystallize the mathematics and make explicit what they have noticed.

11th-Grade Lesson: Sine and Cosine-Trigonometry in Flight. In this class focused on sines and cosines, perhaps the most commonly used question to help the students "own" their learning related to building quad-copters is "Why?" Why must the propellers turn in opposite directions? Why is amplitude the constant in all the graphs? Why do you have to calculate the longest distance? Why wouldn't you just use the Pythagorean theorem? When a student models a graph that shows a left turn, and explains that it has the right propeller going fast (high frequency), then slowing down to match the frequency of the left propeller, the teacher asks him why. All the answers to the teacher's why questions are based in content and involve reasoning; for example, this student explains, "We want it to level out again so it can go forward and, to do that, one side can't be going faster." "How?" was also used regularly in the lesson. For example: How do you know you got a number that's too big? How is that graph useful?

## The Teacher's Role at Every Level

In each instance, teachers prod and probe to help the students see important aspects of the situations so they can use them in their explanations and become better at articulating what is important mathematically. Teachers help them recognize the underlying concepts that are broader than particular problems or examples. Mrs. Wright, the first-grade teacher, does a lot of revoicing of what children say, using slightly different language, providing models from which students can learn. Mrs. Spies and Mrs. Pittard provide worked examples for their fourth- and fifth-grade students to think about and discuss, providing some clues within questions along the way that students may or may not have thought about themselves.

In her lesson with eighth-graders, Mrs. McPhillips' focus on using the Common Core Standards for Mathematical Practice helps students think about the problems and see their usefulness. She also encourages the use of color and precisely defined variables with labels to help students recognize key patterns and make their reasoning and justifications transparent and clear to others.
Eleventh-grade teachers Mrs. Brookins and Mr. James press hard for students to connect mathematics and physics principles to an ambitious long-term, real-world task; they use the motivation of the task itself.
At every level, these teachers direct questions to important aspects of the content students are learning, to some tools that power their learning, and to the usefulness of math in real life. The journey from understanding 13 as 10 and some more ones, to creating and flying useful quad-copters, is a long and intense one for teachers as well as students. Students will take different paths and progress at differing paces, and that freedom to think differently will contribute to their deep learning. Thus, it is important for teachers to continue to grow in their understanding of mathematics, of how children learn, and of what the Common Core State Standards for Mathematical Practice require of them and their students.


[^0]:    ${ }^{1}$ Hyman Bass, 1999 workshop on Teacher Content Knowledge.
    ${ }^{2}$ Lannin et al., Developing Essential Understanding of Mathematical Reasoning, 35.
    ${ }^{3}$ Gaea Leinhardt, (presentation at the AFT Educational Research and Dissemination winter meeting, Washington, DC, 1995).

[^1]:    ${ }^{4}$ Common Core State Standards for Mathematical Practice, MP3 (Washington, DC: Council of Chief State School Officers and National Governors Association, 2010).
    ${ }^{5}$ Lannin et al., Developing Essential Understanding of Mathematical Reasoning, 61.
    ${ }^{6}$ Lannin et al., Developing Essential Understanding of Mathematical Reasoning, 65.

[^2]:    7 Leinhardt (presentation at AFT ER\&D winter meeting, 1995).

[^3]:    8 Lannin et al., Developing Essential Understanding of Mathematical Reasoning, 70 .

