# What do teachers of mathematics know? 

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## Teaching is complex!

Look at the (Australian) curriculum

- TQA
- Content, assessment criteria and standard
- Elaborations
- Past exam papers with comments from examiners (outcomes aspired to)
Look at current textbook (teachers and students)
- Sequencing information
- Some activities
- content

Consider time frame available

- Timetabling
- Have I got time to trial it?

How will I engage students?
Make links with previous teaching episodes

- Is the new resource better than what I've used in the past
Depth of learning required
- Conversation with colleagues
- Past exam papers with comments from examiners (outcomes aspired to)

Students' prior learning

- Level

Student cohort

- Science background
- Conversation with colleagues in college and across colleges
- EAL students
- Community focus

Is it an active learning experience for my students?
Experiential aspects e.g. laboratory work involved
Do I have all the equipment I need?
What don't I want to teach?
Hierarchy of priorities?
What do I know?
What do I need to know / do?
How do I work out what I don't know?
Who can help me?
Is this going to take me into areas that I don't want to go?
How do I explain the relevance of this to my students?
How do I know what is relevant?
Do I need IT support, online access?
How does this framework help/build upon already established pedagogies?

## Teaching is complex!

Has it been reliable?
Has it worked each time I've used it?

- How has it not working contributed to a teachable moment?
- Have I used it in the way it was meant to be used?
- Do I understand why it didn't work?
- How can I find out?

How much time does it take to utilise it?
What does it rely on (e.g. internet dependent)
What IT support is required?

- KISS - keep it simple

How much support do the kids need?
What are the kids doing with it?
Are they learning?
How might I use formative assessment with this resource?
How did I need to change my teaching strategies?
Are my children engaged?
How might I use this resource to move into other related learning?

- Students taking ownership of their learning / opening new pathways to learning.

Is the activity safe?

- What are the risks?


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How might it be improved?

- What might I add to it?

How did students go on the assessment task?

- Did they learn what we wanted them to learn?

Why didn't it work with this group?

- It worked last year with the previous group.
Would I use it again?
How long did it take?
- What did I do to adapt for errors?
- Did I need to reteach?

Why did the resource not work? Do I know why it didn't work?
How should I have better managed the environment?
What responsibility should I have required of students?

Is it worth persisting with?
Could I use this resource for something else/some other learning?
What concerns/questions did the students have that I couldn't answer?
What else do I need to do / know in order for this resource to be better utilised?
Should I re-order my unit plan?
How might I help students who has missed the session to catch up?
What OHS issues came up that were unexpected?
What behaviour management issues came up?

- Did I handle this well?

What might I have done differently?

- A student says that $\frac{1}{4}+\frac{1}{4}=\frac{2}{8}$. She uses counters to show this as follows.

- What representation would you use to help this student?
- A student says that $\frac{1}{4}+\frac{1}{4}=\frac{2}{8}$. She uses counters to show this as follows.

- And we'd have to go back to what our understanding is, and if we've already talked about our understanding of fractions is, "How many equal parts make that whole, so how many do you need to make that whole? And what part of our fraction tells us how many equal parts make the whole?" So they still have to link it to their understanding as well. So you might have to challenge them further if you did that, and take them back to previous discussions.
- But it also depends what they consider the whole to be. Because, you know, if a child said to me, "Well here's my whole."... ...you know, but then you'd be saying, "Well what fraction is that? If that's your whole, what's your fraction?" You know, "What's that fraction?" so, "Really is it a quarter plus a quarter?" You know, and that sort of thing. So it's making sure that they really understand, that you're on the same wavelength in what you each consider the whole to be?
- I don't think the counters are a good tool to use it, but ... well, you know, you've got to really address that because she's selected them
- The students come along and have shown this with counters, we all know that that's not the best model. Now if we use a fraction wall or a paper or paper folding or any of those other kinds of models, which I agree are a much, much better for teaching this concept, is there a danger that she will go away saying, "Ah yes, well, it works for counters, but it doesn't work for bits of paper".
- So something is wrong with either the model or how we instruct them to trust in the models. And there's a conceptual thing, "does this one do it?" Because that's what they're thinking, it's about the whole.
- And so that's why I can give and show you some of the fraction that will change this existing understanding because she feels, there's the quarter, there's the quarter, now here's two eighths. "Now, are they the same?" You know, so you can actually physically see that. And so that will get them start thinking they're going to have the cognitive conflict that that's not what my thinking fits...
- You got to get them to look at something like that or the fraction model...so they can see, "oh, my thinking is wrong, I need to rethink" not tell them they need to.
- How do we provide a representation that can help the student to see that concept that we're trying to... what are the alternative ways, if this isn't working what might be a better representation, a clearer representation?
- And to understand that a whole can change.
- Until that light flashes, you know. Until they go "Aha!" and they understand, so you're trying different things...
- Where you have to have a fairly good understanding. Fractions... and the stages of acquiring fraction knowledge, in order for us to apply that to teaching them in the first place and then to determining where their understanding is falling to pieces.
- It's been a week and a half and we haven't put pen to paper yet, ... but you know we've used play dough and we've used paper and we've used the small skipping rope and we've used a large skipping rope and we've used number lines on the board and we've used chocolate and holding the chair over our heads...[And we've used Iollies and we've made fractions about the class .... I think a lot goes back to the teacher's understanding and the teacher's understanding of pedagogy and the teacher's understanding that fractions mean more than one thing,
- And it's that importance of not rushing, and making sure that something is secure before you rush on, and I think as 5-6 teacher sometimes you start to panic and think, "oh gosh, I should have them here", but...
- Well, they're even scared too from the experience, you know, they're not going to get it. There is no biological way they could get it because they're frightened up...so um, "Okay, you need a break, that's okay, we'll do this again tomorrow", you know, whatever.
- And that's what my scenario was this morning, was they added, and one of the girls-, and it was eleven eighths, and she was writing, and she wrote eleven eighths, and she went, "Equals, one and three eighths." And one of the boys said, "Well it equals one and three eighths, it doesn't equal eleven eighths." And she said, "Well no, it can be the same. They're both the same." And he said, "No, because as soon as you get to eight eighths it's a whole, so it becomes the one in the three eighths." And she's going, "No, because if I..." and she had the equipment out, she had her eighths there, and she said, "If I go like that, it's still eleven eighths. But it doesn't look like a whole." And he's going, "Yeah, but you wouldn't do that, you'd put it together neatly."
- when I'm looking for a problem that's what I'll often be looking for, ... a rich task that has more than one answer, or more than one way of getting it. And, you know, my key catchphrase is, "Prove it."

Go to slide 29 (Shulman)

- A class was asked to calculate:


## $\frac{2}{3} \div \frac{3}{4}$

- Jessica wrote the following on the whiteboard:

$$
\begin{aligned}
& \frac{2}{3} \div \frac{3}{4} \\
& =\frac{8}{12} \div \frac{9}{12} \\
& =8 \div 9 \\
& =\frac{8}{9}
\end{aligned}
$$

- my first thought was, I'd probably ask them why they put the twelve, ... I'd like to hear from the students why they have changed it to eight over twelve divided by nine over twelve.
- Because that's what you do when you add.
- I think that's a wonderful answer
- Absolutely right. That is a perfectly valid way to do it.
- Well, firstly, I would ask them why they did what they did. You know, I think that's really important that kids understand how to do something, why it works and when you use it. And in this case I am intrigued that they've got the right solution by not the standard algorithm so I would ask them why they did it like that and then I think you could explore with them, and mathematicians do this all the time, okay, let's have a look at some other ways that you might do this and look at why they work and then engage them in a conversation about, "Let's weigh up so some of the advantages and disadvantages of the different strategies you're looking at now."
- So using equivalent fractions, and then, because are they ... I think the student was saying, I think, from memory, it was a while ago, they went twelve divided by twelve is one
- ... eight divided by nine is, ... and twelve divided by twelve is one, I think that's what ... this is a while ago. However, it works.
- Is it similar to where a student takes an algebra problem and does a big long complicated way to do it, and there's an elegant way that's quite, quite short? And you have to decide, you know, are you going to let them persevere and take the big long way, or just go to the [elegant] short one.
- do you react to this as being less elegant than another procedure for dividing fractions, or, is it actually showing, is it more elegant in showing a higher level of understanding?
- It's hard to tell, ... Either it's someone who's like really switched on, or it's someone that's blundered.
- I would put this up on the board for them and say, "Does this work? What do you think?"
- I think I would use it to challenge their thinking and say...
- So you wouldn't explore other fractions and see if it works?
- Definitely, I would see that as an open thing. I don't know that I would show them this method, but I think I would have them explore it.
- I'd get the students to come up with numbers and explore them themselves
- I'd have to play around with, um, myself. ... they certainly do a lot of different things, like I don't, teach them to flip the fractions, I do teach them to divide fractions, you know, to find out how many three-quarters are in twothirds.
- So you'd actually do that by pictures, you know, so that...
- It also emphasises the whole doesn't it
- The teacher response might be important to the kid's progression.
- I think perhaps an inexperienced teacher might just go, "Right answer, wrong working." And just dismiss it. Whereas, it's made us think, hasn't it?
- And you also can, you can say, "I need time to think about this." Because the students have to understand that some of these things can take time, people want to go away and work with them, you might like to go away and work with them, and so that, you don't always have instant answers. And you might say, "Well that's a really interesting method, I can't quite, can you explain it?" If they can't explain it, you know, or you just want time to come back the next day and have a look at it
- It's the understanding that, we can take a day or two, to work on something
- Sometimes students will come up with a nice way of doing it, but their method may not take them further up, you know, it might only be done for that particular question, but actually they won't be able to use that for sort of the harder problems or other problems, so that's what you've got to check, to see whether it will work, their method
- I think though as teachers we need to develop a library of different ways of helping kids understand things, and if this kid knew why they were doing that and they explained it in a way that makes sense to me, I want to add that to my library because in many ways it makes more sense than what we teach them because it is linked and connected to the addition ... because if you actually do the eight divided by nine and the twelve divided by twelve, you've got your, you know, so you're nearly there and it might even be less confusing than what we currently do. So we sort of have to be aware of those possibilities.
- The following is part of a set of cards for students to sort:



- If the purpose of the cards is to help students to understand the definition of 'function', what cards might be useful additions?
(Source: Malcolm Swan)
- What about some non-functions?
- a circle centred on the origin
- A parabola on it's side, but I've forgot what you call it...
- kids have a bit of trouble with it, why is the independent variable on the horizontal axis, that is just a convention, there's not necessarily a good reason for that. That would actually get it, but I think probably I should have some tables with numeral values along side some of those as well, so that I could see, oh, hang on, here is an $x$ value mapped into two different $Y$ values
- the absolute value function
- another good example then of a non-function is one of the step functions with gaps there, that you know there's some $X$ values that don't map to a $Y$ value
$\infty$
$\square$ PRer
- Well I think it's good to contrast the notion of asymptotic behaviour and functional behaviour because I think students do get the two things confused. So we haven't got any other function with an asymptote so therefore an asymptotic function is a good idea, because then you can, get away from that whole concept that asymptotic functions are somehow not functions because they're going along the same sort of level and therefore they're somehow not progressing and therefore students tie that up with, "Oh, they're on the same number, and...", so they get a bit confused. So I would probably actually use a log function as opposed to an exponential function for that reason because if they're looking and thinking about functions being vertical line tests and that sort of thing, they get to the log function and go, "Oh well, that's going to be linear at the bottom, so therefore it's not a function." So I really like throwing a log function in for that reason.
- the thing that strikes me is, in one respect there's really no difference between the line $x=0$ and $y=$ 0 , and yet one's a function and one is not. And it's really interesting just sort of appreciating, ... what's the same, what's different, so the whole idea of invariance in mathematics.
- And even choosing $y=x$ can be really interesting, you know, because it is different again in it's own way ... there's nothing happening, so therefore it's somehow that can't be the same as the others... it's sort of a nothingness which is so, ... the beautiful, beautiful function, but, for students it's just sort of a waste of a function.


## What knowledge did these teachers

## have?

- Importance of connecting current and prior learning
- Content including beyond the year level they're teaching
- How to explain mathematical ideas
- The importance of the whole in fractions
- How to question
- Importance of building on student thinking
- Representations and their limitations
- Student thinking and typical learning trajectories
- Power of cognitive conflict
- How to learn from their teaching
- When to move on, what's mathematically important
- Individual student personalities and likely reactions
- Selecting and evaluating tasks
- Directing mathematically profitable discussions
- How mathematicians work
- Evaluating solutions
- What mathematics is - an appreciation of the discipline
- How to get students thinking
- How to explore/learn mathematics for themselves
- Their own impact - cognitively and affectively
- Choosing examples and nonexamples


## Knowledge for teaching Mathematics

- Shulman (1987)
- Content knowledge
- Pedagogical knowledge
- Pedagogical content knowledge
- Curriculum knowledge
- Knowledge of students
- Knowledge of context
- Knowledge of educational ends and values


## Knowledge for teaching Mathematics

- Ball and Colleagues


## Domains of Mathematical Knowledge for Teaching



## Knowledge for teaching Mathematics

- Chick and colleagues:
- 3 categories of knowledge
- Clearly PCK
- Content knowledge in a pedagogical context
- Pedagogical knowledge in a content context


## Knowledge for teaching Mathematics

$\left.\begin{array}{ll}\hline \text { PCK Category } & \text { Evident when the teacher ... } \\ \hline \text { Clearly PCK } \\ \text { Teaching Strategies } & \begin{array}{l}\text { Discusses or uses strategies or approaches for teaching a } \\ \text { mathematical concept } \\ \text { Discusses or addresses student ways of thinking about a } \\ \text { concept or typical levels of understanding } \\ \text { Student Thinking } \\ \text { Student Thinking - } \\ \text { Misconceptions } \\ \text { Explanations } \\ \text { concept }\end{array} \\ \begin{array}{l}\text { Appropriate and Detailed } \\ \text { Representations of Concepts }\end{array} & \begin{array}{l}\text { Explains a topic, concept or procedure } \\ \text { Identifies aspects of the task that affect its complexity } \\ \text { Describes or demonstrates ways to model or illustrate a } \\ \text { concept (can include materials or diagrams) } \\ \text { Curriculum Knowledge }\end{array} \\ \begin{array}{l}\text { Discusses/uses resources available to support teaching } \\ \text { Purpose of Content Knowledge }\end{array} \\ \hline\end{array} \begin{array}{l}\text { Discusses how topics fit into the curriculum } \\ \text { Discusses reasons for content being included in the } \\ \text { curriculum or how it might be used }\end{array}\right]$

## Knowledge for teaching Mathematics

| Content Knowledge in a Pedagogical Context |  |
| :--- | :--- |
| Profound Understanding of | Exhibits deep and thorough conceptual understanding of |
| Fundamental Mathematics | identified aspects of mathematics |
| Deconstructing Content to | Identifies critical mathematical components within a <br> concept that are fundamental for understanding and <br> Key Components |
|  | applying that concept |
| Mathematical Structure and | Makes connections between concepts and topics, including <br> interdependence of concepts |
| Connections | Displays skills for solving mathematical problems <br> (conceptual understanding need not be evident) |
| Procedural Knowledge | Demonstrates a method for solving a maths problem |

## Knowledge for teaching Mathematics

Pedagogical Knowledge in a Content Context

Goals for Learning
Describes a goal for students' learning (may or may not be related to specific mathematics content)
Getting and Maintaining Student Focus
Classroom Techniques

Discusses strategies for engaging students
Discusses generic classroom practices

## What the models seem not to cover

- Appreciating the discipline
- Knowing how mathematicians work
- Knowledge of students as individual people
- Knowledge of own impact - cognitively and affectively
- How to learn


## How do teachers learn (acquire knowledge)?

- How can you keep adding to your knowledge throughout your career?
- That's my opinion, after reading what I have read, and I can understand ... but I've only understood that from doing the readings and the postgraduate work that I've done. And I wish as a teacher I would have had a bit more explanation about why we've been shifting [assessment practices]. And I can see why we have, but, a few years ago I didn't understand that as a teacher. And as a teacher I would have liked to have, so yeah, maybe every teacher should do postgraduate
- I know what the connections are and I can actually get down and see the world from their point of view and think, "Oh yeah, you're here." You know, I can analyse a kid's understanding mathematically. It's not so much my content knowledge, it's about my knowledge of the stages people go through and how they come to understand.
- that person [an ineffective teacher] can't get inside the kid's head and go, "I can see what you do get, and I can see what your misconceptions are and I can do something to help you correct that misconception and then you're going to feel like I've helped you." That's what the kid feels like. The teacher's not helping.


## How do teachers learn (acquire knowledge)?

- How can you keep adding to your knowledge throughout your career?
- How can you learn from your teaching?


$\frac{2}{3}$

$\frac{3}{4}$



