I never make the same mistake twice.

I make it five or six times, just to be sure.

Using Non-Routine Problems to Better Understand Students' Reasoning

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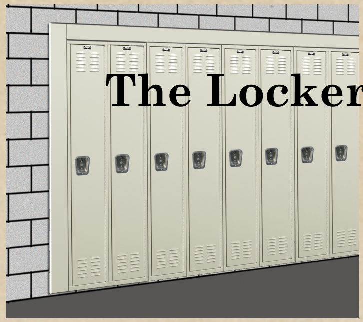
Name:

Class:



The Locker Problem

Imagine you are at a school that has student lockers. There are exactly 100 lockers, all shut and unlocked, and 100 students.

Here's the problem:
1. Suppose the first student goes along the row and opens every locker.
2.The second student then goes along and shuts every other locker beginning with number 2.
3.The third student changes the state of every third locker beginning with number 3. (If the locker is open the student shuts it, and if closed, the student opens it.)
4.The fourth student changes the state of every fourth locker beginning with number 4.
5.Imagine that this continues until the 100 students have followed the pattern with all 100 lockers.

Assignment:
A. At the end, which lockers will be open and which will be closed?
B. What mathematical explanation can you give for your answer?
C. At what point did you realize you had found the answer?
D. Using your results, prepare a powerpoint slide presentation to show the class, or write a 3 paragraph report following the habit of mind rubrics, or make an interactive poster

Extension to the locker problem.

- Discuss your solution with others at your table group.
- What other methods or patterns did you discover?
- Can you use your solution to predict which lockers would remain open if there were 200 lockers and 200 students?
- Our school has 902 lockers. This year we have only 808 students. Does your method still work for finding the number of open lockers? Are there other things to consider? Explain.

Abby's solution

Mathematical Explanation of the Locker Problem

While working on the problem, I realized that all of the prime numbers were only touched twice. I also realized that all of the other numbers except the perfect squares were touched an even amount of times. I found out that all of the lockers that were touched an even amount of times ended up closed and the lockers that were touched an odd amount of times ended up closed [sic]. You can easily find out if a locker would be open or closed by listing the number's factors. For example, the number 12 has the factors 1, 2, 3, 4, 6 and 12. Since it has an even amount of factors, it will be touched an even amount of ties. Then for the number 9, the factors are 1, 3, and 9. Since the number nine has an odd number of factors, it will be touched an odd amount of times, which will make it end up opened! Therefore, the only lockers left open are the perfect squares because their repeated factors only count as one factor and make them have an odd number of factors.


THE
LOCKER PROBLEM


The
Locker
Problem
Southeast
KNIGHTS

Common Core State Standards-Math

- The middle school standards are **robust and provide a coherent and rich *preparation*** *for high school mathematics*.

- The high school standards call on students **to practice applying mathematical ways of thinking to real world issues and challenges;** they prepare students to think and reason mathematically.

- The high school standards set a *rigorous definition of college and career readiness*, by helping students **develop a depth of understanding and ability to apply mathematics to novel situations**, as college students and employees regularly do.

- The standards **stress not only procedural skill but also conceptual understanding**, to make sure students are learning and absorbing the critical information they need to succeed at higher levels

Examples of some Non-Routine Problems

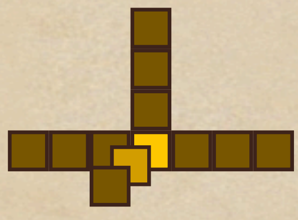
- The Upside-Down T Pattern
  - Use the pattern to answer the following questions. You must show all your work. You will probably need several sheets of paper. Don't erase any work. (It's important to keep track of every strategy, even the incorrect ones.)
  - Step 1 Step 2 Step 3

  - 1. Draw the next two steps in the Upside-Down T Pattern.
  - 2. How many total tiles are in each step 1 through 5?
  - 3. Sketch and describe two steps in the pattern that are larger than the 10th step.
  - 4. Describe a method for finding the total number of tiles in the 50th step.
  - 5. Write a rule to predict the total number of tiles for any step. Give evidence that supports how your rule relates to the pattern.
  - 6. Which step has exactly 97 squares? Prove your answer is correct.

Expected Student Solutions: Upside-Down 'T'

Most students had solutions that followed a line of thinking similar to this:

Step 1 = 1 tile
Step 2 = 4 tiles
Step 3 = 7 tiles
Step 4 = 10 tiles
Step 5 = 13 tiles



Each step adds 3 tiles, one to the left, one to the right, and one on top. Each leg has exactly the same number of tiles as the step number...step number 5 has 5 squares in each leg, or 5 x 3; but, the center square gets counted 2 times too many so 2 needs to be subtracted from the product: (5 x 3) - 2. Therefore, the rule needs to be (step # x 3) - 2, or 3x -2.

Unusual Student Solutions: Upside-Down 'T'

- Jacob's solution

I am supposed to find the answer to which step has exactly 97 squares...First I took the time to figure out a formula. Then I started it by getting all the squares and subtracting one-the center square-because it gets counted more than once. 97 - 1 = 96. Then I divided by 3 and got 32. Then I added back the square in the middle. There were 32 squares on each leg, so the 33rd (32 + 1) step has 97 tiles

Here is my rule to find the step number of squares: (N -1)/3, then add 1 is the number of the step.

- Jenna's solution

The problem is about finding out the pattern and how many tiles are in each step...First we knew that step 1 had 1 tile. As the pattern progressed we saw that it adds one tile to each point (top, left side and right side). Next we figured out how many tiles were in each step. We found that step 2 had 4 tiles, step 3 had 7 tiles, step 4 had 10 tiles, step 5 had 13 tiles and step 6 had 16 tiles. After that...we figured out how to find the amount of tiles in step 50. We concluded there were 148 tiles.

On my data sheet, I found there were 34 tiles in the 12th step. Then we found the rule: You subtract/add the last step you did from the one you want: 50 - 12 = 38. Then you multiply 38 by 3 = 114. Then you add the number of tiles from the last (12th) step you did. 114 + 34 = 148. So there are 148 tiles in the 50th step.

To figure out which step has exactly 97 tiles I figured using this rule: 3y + 1 = 97. 3y + 1 - 1 = 97 - 1. 3y = 96. y = 32. There are 32 squares in each stem of step 33 plus the one in the center.

Habits of Mind Guide

- Follow these directions exactly to create a data sheet.
  - Write your name and the date at the top of a sheet of paper.
  - Write the title of the Habits of Mind with the words *Data Sheet*.
  - Show all the work you did to solve the problem using words, pictures, and numbers. Don't erase. It is important to show your errors and how you overcome them.
  - Write a concluding sentence at the end of your data sheet that clearly states the answer to the problem.

On a separate page, record your write-up.

- Write your name and the date at the top of a sheet of paper.
- Write the title of the Habits of Mind with the words *Write-up*.
- Copy the paragraph titles shown below, and use the sentence starters provided to complete your write-up. Everything you write must refer to the math content, the procedures you followed, and the strategies you used to solve the problem. You can print, type, or use cursive writing. Write in paragraph form. These are suggested ways to begin you paragraphs:

Paragraph 1
Problem Statement

- This problem is about …
- I am supposed to find …

Paragraph 2
Work Write-up

Explain step by step and in detail everything you did to complete your Data Sheet and arrive at your answer. Think of it as a recipe for someone to follow or as directions to your house. You may include strategies that you used that didn't work.

- First I …
- Then I …
- Next I …
- After that I …

Finally I …

Paragraph 3
Answer

Verify, or prove, your answer by referring to the math you did. Don't write that you checked it on a calculator, you did it twice, or a friend or family member told you.

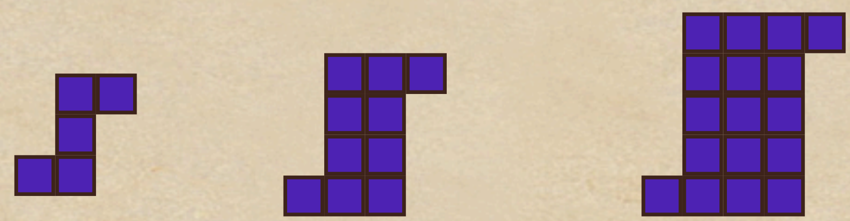
- My answer is …
- I think my answer makes sense

because …

Another Non-routine problem

The S-Pattern

Directions: Use the pattern to answer the following questions. You must show all your work. You will probably need several sheets of paper. Don't erase any work.

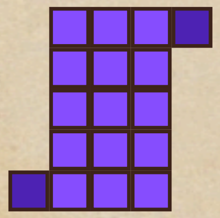


1. Draw the next two steps in the S Pattern.
2. How many total tiles are in each step 1 through 5?
3. Sketch and describe two steps in the pattern that are larger than the 10^{th} step.
4. Describe a method for finding the total number of tiles in the 50^{th} step.
5. Write a rule to predict the total number of tiles for any step. Explain how your rule relates to the pattern.
6. Can you write a different rule by counting the arrangement of tiles from a different view?

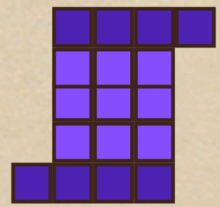
Expected Student Solutions: S-Pattern

There are several near-solutions to this and the majority of students find at least one variation on their initial solution.

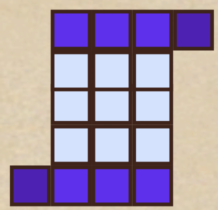
Step 3



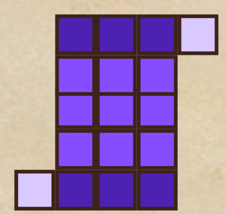
n x (n + 2) + 2



(n x n) + 2(n + 1)



(n x n) + (n + 1) + (n + 1)



n^2 + 2n + 2

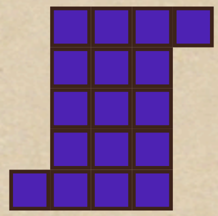
Students' discussions about which solutions are correct lead to some interesting insights into multiple solution-steps and a need to simplify the final answers.

Students frequently express amazement that they can approach the problem in different ways, yet they get the same answers.

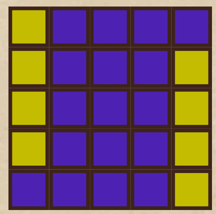
They should be able to see that all these answers, once simplified, are the same: n^2 + 2n + 2.

One unusual solution: S-Pattern

Step 3



Step 3



5 x 5 - (4 + 4)

(3 + 2)(3 + 2) - 2(3 + 1)

Bailey solved this by filling in the 'empty spaces.' She recognized that this made a square and to find the area of the square you take the Step number (n) and add 2 because it is two columns wider and two rows higher than the step number. Then to figure out how many to subtract you take the step number and add one for each side.

"Say you are trying to find how many tiles to Step 1, the S doesn't seem even, so to make it even, add the blocks as needed. Find the area, then subtract the blocks you added.:

Step 1: 3 x 3 = 9

- 4

= 5

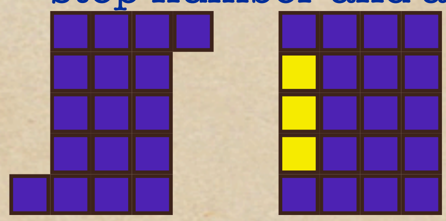
Rule: (n + 2)^2 - 2(n + 1)

n^2 + 4n + 4 - 2n - 2

n^2 + 2n + 2

Another **unusual** **solution**: S-Pattern

Josie solved this by sliding the top row one square to the left, then she filled in the ‘empty spaces.’ She recognized that this made a square and to find the area of the square you take the Step number (n) and add 2 because it is two columns wider and two rows higher than the step number. Then to figure out how many to subtract you take the step number and add one for each side.



(n + 1) (n + 2) – n

(3 + 1) (3 + 2) - 3

& Another unusual solution: S-Pattern

Sometimes the differences in solutions are found in the algebra rather than the geometric representations.

Isis wrote this completely correct solution:

N = Step Number

N + 2 = Y

N x Y = X + 2

"*To use the equation in step 3 would be 3 + 2 = 5. Then 3 x 5 = 15. Then add 2 to get 17, and 17 equals the number of tiles in step 3.*"

Chicken Nuggets

A certain food-chain sells chicken nuggets in boxes of six, nine, or twenty nuggets. What is the largest number of nuggets you can not purchase given any combination of boxes?

Make a prediction of what you think your answer will be.

Work together to develop a strategy and then solve this problem

How can you explain/show your answer is correct?

How does your answer compare to your prediction?

Taxation Vexation

The country of Taxico has a different method of taxing income. If you make $1000 per year, your tax rate is 1%. For each additional $1000 you earn per year, the rate goes up by 1%. For example, if you made $24,000 per year, you paid 24% in taxes. If you could set your salary anywhere between $0 and $100,000, what would you want it to be and why?

For this assignment, you do not need to write a 3-paragraph write-up; but you do need to show all work and answer the question completely.

Taxation Vexation-Katelyn's Solution

In the city of Taxation, 1% taxes are charged for every $1000 you earn in salary. For example, if you earn $30,000, you would pay 30% in taxes. I wanted to know where you would want your salary from $0 to $100,000. To start, I found a formula that could be used to find the money you get after taxes are subtracted from your salary. Next, I made a list of the money you get from a $0 salary to $6000, and from $48,000 to $55,000. I found that from $0-$50,000 your pay increases, and from $50,000 to $100,000 it decreases. I also found that the second delta of the list changes by $20 each time.

I would want my salary to be $50,000 because you would get the most money with the tax subtracted. From a $0 salary to a $50,000 salary, the pay you get keeps increasing. After $50,000, pay decreases. 50,000 is the middle mark, which means if you earned $50,000 in salary, you would keep $25,000, which is the most money you can possibly get. If you earned more than $50,000, you would get to keep less.

Katelyn's Table

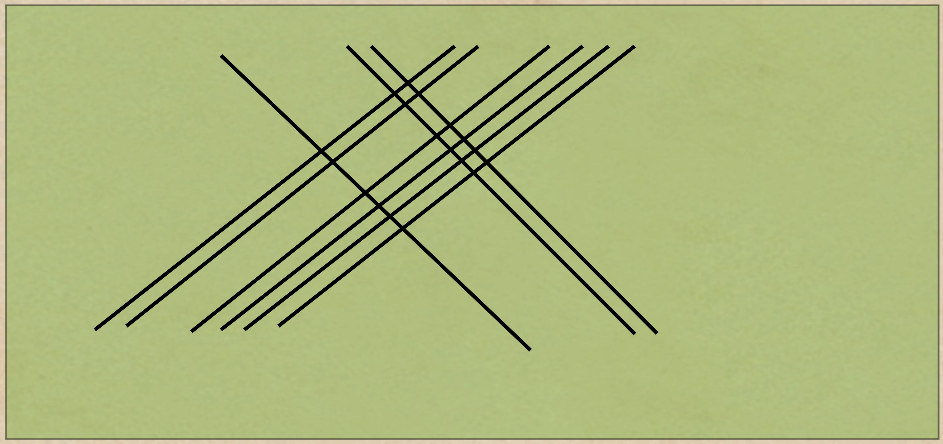
Salary-x	Salary-f(x)-after tax is subtracted	Delta (x)	Delta2 (x)
0	0		
1000	990	990	
2000	1960	970	20
3000	2910	950	20
4000	3840	930	20
5000	4750	910	20

48,000	24,960	50	20
49,000	24,990	30	20
50,000	25,000	10	20
51,000	24,990	-10	20
52,000	24, 960	-30	20
53,000	24,910	-50	20

Kyle's Table and solution

$$$	Net Profit
5,000	4,750
10,000	9,000
15,000	12,750
20,000	16,000
25,000	18,750
30,000	21,000
35,000	22,750
40,000	24,000
45,000	24,750
50,000	25,000
55,000	24,750
60,000	24,000
65,000	22,750

A 'Stick'y Problem



As the story goes, a teacher in China shared this problem with his students. Just by examining this figure, can you solve the mystery?

What is the solution?

WORK SPACE

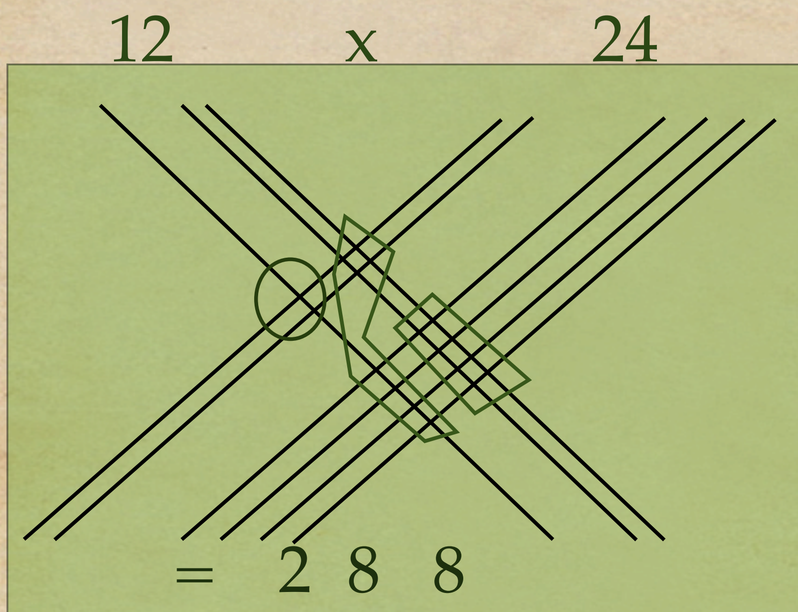
Work Space

Hint #1: The solution is 288. Can you determine the solution process?

Hint #2: The line segments represent the factors 12 and 24.

Hint #3: The numbers are represented by diagonal lines broken into place values of tens and ones.

Hint #4: The lines intersect to form arrays of points.



The first array of points symbolizes where the tens places of the numbers 12 and 24 cross.

The second array shows where the units place of the number 12 intersects with the tens place of 24 and where the tens place of 24 x the units place of 12.

The third array is where the units place of 12 x the units place of 24.

Student Thinking Gone Awry

Several student erred with this rate-problem:

Hot Tub problem.

- **Gary needs to fill his hot tub tonight. If he uses a large hose (A), it will take only 3 hours. The smaller hose (B) takes 5 hours to fill the tub. How long will it take Gary if he uses both hoses at the same time?**

Predictably, the common error was to simply average the two numbers; (3 + 5)/2 hours is 4 hours. Students who answered this way failed to take the time to conceptualize the problem. They realized their error during class discussion time, even though, initially, some couldn't see how to correctly solve it.

The common error is that students try to find the average rate by adding the given rates and finding the mean. You can add distances, you can add quantities, and you can add times, but **you cannot add rates**.

The common error is that students try to find the average rate by adding the given rates and finding the mean. You can add distances, you can add quantities, and you can add times, but **you cannot add rates**. Think about it: If you drive 20 mph on one street, and 40 mph on another street, does that mean you drove 60 mph?

Answer: Think about how much of the tub each hose will fill in only 1 hour.

Hose A can fill 1/3 of the tub. Hose B will fill only 1/5 of the tub. Together they can fill 1/3 + 1/5, or 8/15 of the tub in one hour. Now he needs to fill the remaining 7/15 of the tub. 7/15 / 8/15 = 7/8. It will take 7/8 of an hour to fill the rest of the tub; which is 60 x 7/8 or 52 ½ minutes.

Back-on-Track

Activities Used To Assess Student Growth In Solving Rate Problems

- Suppose one painter can paint the entire house in twelve hours, and the second painter takes eight hours. How long would it take the two painters together to paint the house?

- One pipe can fill a pool 1.25 times faster than a second pipe. When both pipes are opened, they fill the pool in five hours. How long would it take to fill the pool if only the slower pipe is used?

- A 555-mile, 5-hour plane trip was flown at two speeds. For the first part of the trip, the average speed was 105 mph. Then the tailwind picked up, and the remainder of the trip was flown at an average speed of 115 mph. For how long did the plane fly at each speed?

Solutions

Answer: If the first painter can do the entire job in twelve hours and the second painter can do it in eight hours, then (this here is the trick!) the first guy can do $^1/_{12}$ of the job *per hour*, and the second guy can do $^1/_8$ *per hour*. How much then can they do *per hour* if they work together?

To find out how much they can do together *per hour*, I add together what they can do individually *per hour*: $^1/_{12} + {^1/_8} = {^5/_{24}}$. They can do $^5/_{24}$ of the job *per hour*. Now I'll let "t" stand for how long they take to do the job together. Then they can do $^1/_t$ *per hour*, so $^5/_{24} = {^1/_t}$. Flip the equation, and you get that $t = {^{24}/_5} = 4.8$ hours. That is:

hours to complete job:	completed per hour:
first painter: 12	first painter: $^1/_{12}$
second painter: 8	second painter: $^1/_8$
together: t	together: $^1/t$

$^1/_{12} + {^1/_8} = {^1/_t}$ $\qquad$ $^5/_{24} = {^1/_t}$ $\qquad$ $^{24}/_5 = t$

They can complete the job together in just under five hours.

Convert to rates:

hours to complete job:	completed per hour:
fast pipe: f	fast pipe: $^1/_f$
slow pipe: $1.25f$	slow pipe: $^1/_{1.25f}$
together: 5	together: $^1/_5$

adding their labor: $^1/_f + {^1/_{1.25f}} = {^1/_5}$

multiplying through by $5f$: $\quad 5 + 5/1.25 = f \qquad 5 + 4 = f = 9$

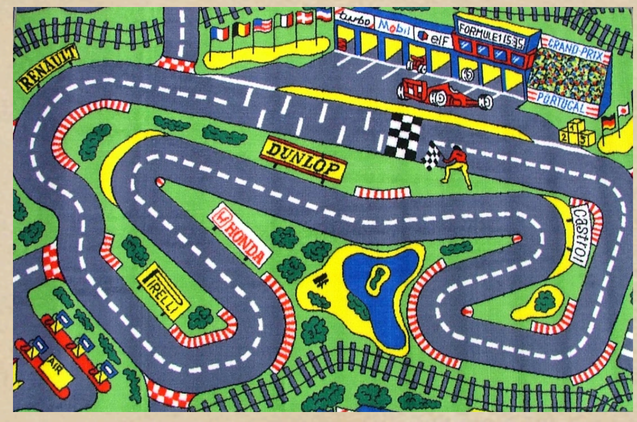
Then $1.25f = 11.25$, so **the slower pipe takes 11.25 hours**.

If you're not sure how I derived the rate for the slow pipe, think about it this way: If someone goes twice as fast as you, then you take twice as long as he does; if he goes three times as fast, then you take three times as long. In this case, if he goes 1.25 times as fast, then you take 1.25 times as long.

Thinking Deeply

I have adopted *"Think About Simple Things Deeply"* as my class mantra. It's a statement frequently made by Dr. Jim Lewis, UNL.

The following problem illustrates how important it is to think about a problem before spending time trying to solve it.

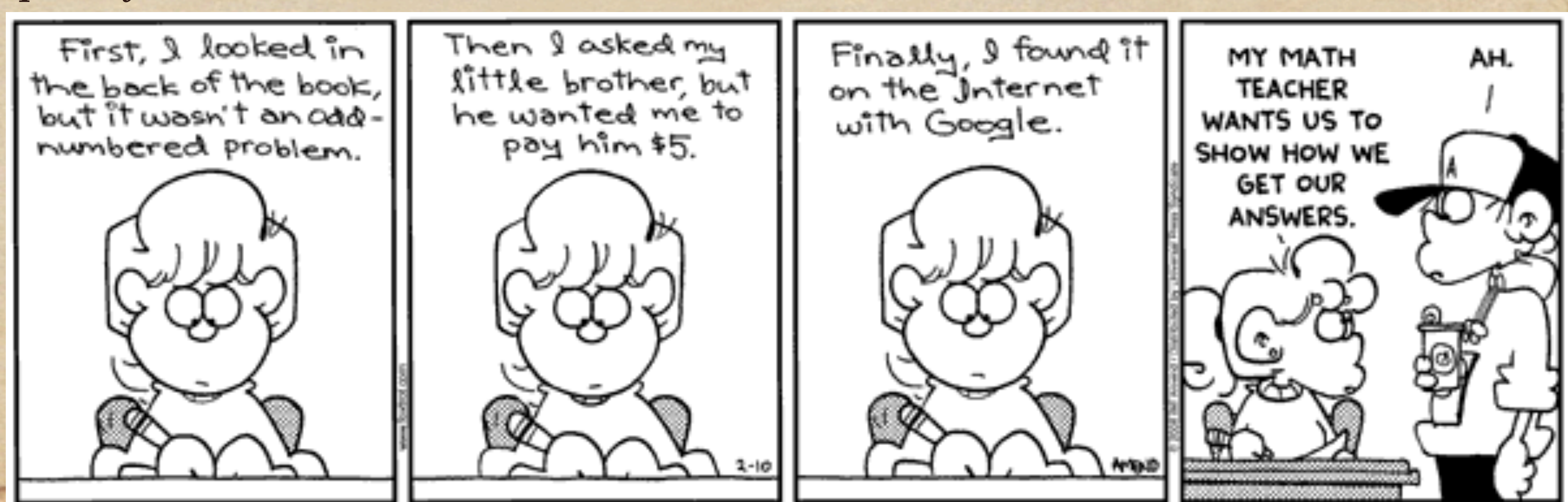


I first saw a version of this problem in a Foxtrot cartoon:

- **To qualify for a race, you need to average 60 mph driving two laps around a 1 mile long track. You have some sort of engine difficulty the first lap so that you only average 30 mph during that lap; how fast do you have to drive the second lap to average 60 for both of them?**

Thinking Deeply

Sometimes the right answers will seem counter-intuitive, so it is really important to think about your components methodically and systematically. Intuitively it would seem you need to drive 90, but this turns out to be wrong. The answer is that NO MATTER HOW FAST you do the second lap, you can't make it. And this SEEMS really odd and that it can't possibly be right, but it is. The reason is that in order to average at least 60 mph over two one-mile laps, since 60 mph is one mile per minute, you will need to do the whole two miles in two minutes or less. But if you drove the first mile at only 30, you used up the whole two minutes just doing IT. So you have **no time left** to qualify.



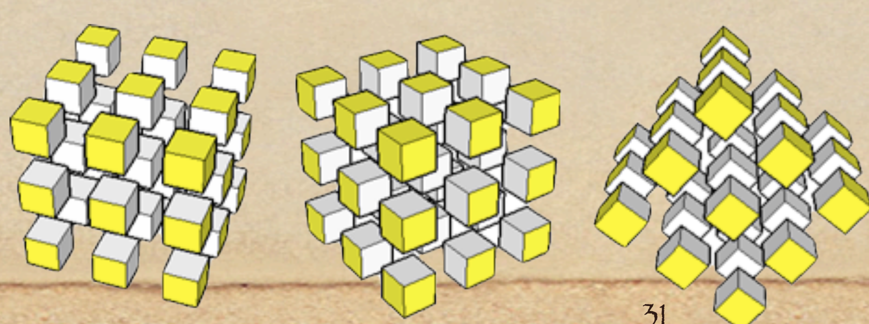
Solve It
The Thinking of Students
A Habit-of-Mind Problem

The Painted Cube Problem

1. A cube with edges of length 2 centimeters is built from 1-cm cubes. If you paint this cube and then break it into 1-cm cubes,

A. How many cubes will be painted on exactly 3 sides?
B. How many cubes will be painted on exactly 2 sides?
C. How many cubes will be painted on only one side?
D. How many cubes will be unpainted?

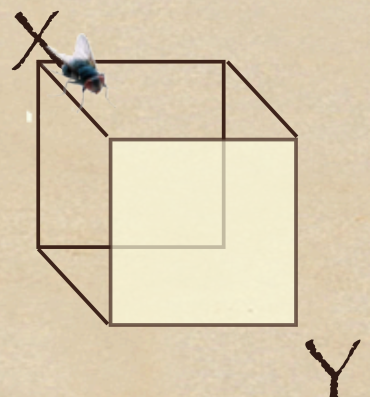
2. What if the cube has a length different from 2? What would the answers be to A-D if the cube had a length of 3-cm?
3. What would be the answers for A-D if the cube had a length of 50-cm?
4. What if the cube had a length of n-cm?



A Walking Fly

A fly is sitting on one corner of a 1-cubic inch sugar cube at X. By walking only, what is the shortest distance from X to Y?

What are the paths that it might take?



Sketch at least 4 distinct paths and label them. Order them from longest to shortest.

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Lincoln Public Schools is the second largest public school district in Nebraska, located in the heart of the plains, renowned for its long-standing legacy of educational excellence and tradition of rigorous academic achievement. The school district is growing and thriving, serving more than 38,900 students in more than 60 schools and programs.

This work is funded [in part] by the National Science Foundation [NebraskaMATH grant DUE-0831835] [NebraskaNOYCE grant DUE-1035268]. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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Robert Noyce NSF Master Teaching Fellowships

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