2014 NCTM Regional Conference

Andy Clark
andyclark@qwest.net

How Singapore's Visual Models (and Visualization) Enable Algebraic Thinking

8th

	average	advanced	high	intermediate	low
South Korea	613	47	77		99
Singapore	611	48	78		99
Taiwan	609		73	88	96
Hong Kong	586	34	71	89	97
Japan	570	27	61	87	97
Russia	539	14	47	78	95
Israel	516	12	40	68	87
Finland	514	4	30	73	96
United States	509	7	30	68	92
England	507	8	32	65	88
International	500	3	17	46	75

Example of 8th Grade TIMSS Problem

Pat has red tiles and black tiles. Pat uses the tiles to make square shapes.

The 3 × 3 shape has 1 black tile and 8 red tiles.

The 4 × 4 shape has 4 black tiles and 12 red tiles.

The table below shows the number of tiles for the first three shapes Pat made. Pat continued making shapes using this pattern. Complete the table for the 6 × 6 and 7 × 7 shapes.

Shape	Number of Black Tiles	Number of Red Tiles	Total Number of Tiles
3 × 3	1	8	9
4 × 4	4	12	16
5 × 5	9	16	25
6 × 6	16		
7 × 7	25		

% Received full credit
United States 57%

Korea, Japan, Singapore 88-90%

Example of 8th Grade TIMSS Problem

ID: M032761 | Mathematics Grade 8

Pat wanted to add a line to the table showing how to find the number of tiles needed to make a square of any size. Use the patterns in the table on the opposite page to help you complete the line for shape $n \times n$ in the table below.

Shape	Number of Black Tiles	Number of Red Tiles	Total Number of Tiles
$n \times n$	$(n-2)^2$		

End of Red and Black Tiles section.

% Received full credit

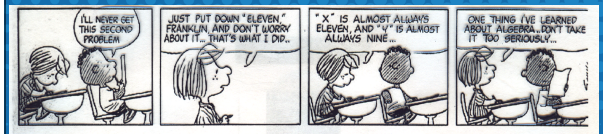
United States 8%

Korea, Taipei, Singapore 45-54%

Why does math become increasingly difficult?

- Students lack understanding of number relationships
- New content is more abstract, requires generalization
- There is a great deal of new content and often dealt with superficially
- Fewer manipulatives and visual images are used
- Teachers may not always be able to explain procedures (e.g. division of fractions or decimals)
- Students don't see relevance of math
- Parents less able to help

Attitude matters



Return to Algebra Comics

There are many difficult transitions for middle school students

- Whole number to rational numbers
- New definitions for fractions: ratios and division
- Natural numbers to integers
- Positive to signed numbers
- Number to variable
- Patterns to functions
- Arithmetic to generalizations

Balance understanding and procedural fluency

But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic comes from... Understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

COMMON CORE STATE STANDARDS FOR Mathematics

An important feature of the Singapore curriculum is its emphasis on visualizatio as a major strategy for developing understanding and problem solving.

One often hears teachers there saying: "Can You see it?"

Diezmann, C. M. (2000). The difficulties students experience in generating diagrams for novel problems. In T. Nakahara & M. Koyama (Eds.), Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 241-248). Hiroshima, Japan: PME.

Diezmann, C. M., & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. A. Cuoco (Ed.), 2001 National Council of Teachers of Mathematics Yearbook: The role of representation in school mathematics (pp.77-89). Reston, VA: National Council of Teachers of Mathematics.

Novick, L. R. (2001). Spatial diagrams: Key instruments in the toolbox for thought. In D. L. Merlin (Ed.), The psychology of learning and motivation, 40, 279-325.

van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. Journal of Educational Research, 83(6), 301-312.

Lowrie, T., & Kay, R. (2001). Relationship between visual and nonvisual solution methods and difficulty in elementary mathematics. Journal of Educational Research, 94(4), 248-255.

When you read the number -5, what images do you have?

- Location on a number line 5 units to the left or 5units below 0.
- The integer between -4 and -6
- The action of removing 5 from a set
- The action of moving left or down 5 units
- The additive inverse of 5, the number when added to +5 = 0

Less than 40% of of 12-13 year olds were able to subtract integers

Garladdo (1995)

More than 25% of 12-13 year olds can't add a positive and negative number

Kloosterman (2012)

50% of 12-13 year olds can't divide integers correctly.

Kloosterman (2012)

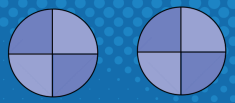
When you see the division of a fraction, what mental image do you have?

How can students develop an understanding of why the division of a fraction by a fraction is solved by multiplying by the reciprocal?

Greater than one or less than one?

1 ÷ 2/3
3/4 ÷ 1/8
3/4 ÷ 7/8
3/4 ÷ 2/3
2/3 ÷ 3/4
4/13 ÷ 2/7

Visualize whole number divided by a unit fraction



How many fourths in 1 whole? In 2? 2 ÷ 1/4 = 2 x 4

How many fourths in 6 wholes? In 20?

If you know one whole, how can you find the number in any number of wholes ? n ÷ 1/4 = n x 4

#wholes x 4 = number of fourths

Visualize a whole number divided by a proper fraction

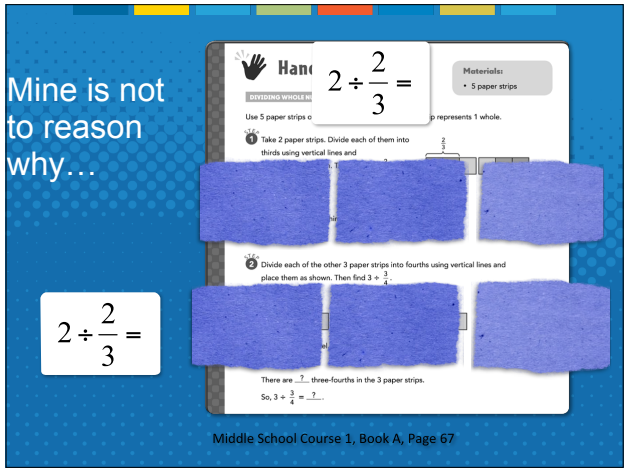
How many 2/3 in 1 whole?

How many 2/3 in 2 wholes?

Now how many 2/3 in 1 whole? Half of 3 or 3/2

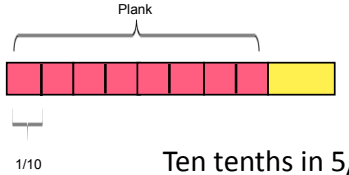
So if we know how many in 1, how can we find how many 2/3 in 5 wholes?

5 x 3/2 = 5÷ 2/3
7 x 3/2 = 7 ÷ 2/3



Visualize a larger proper fraction divided by a smaller unit fraction

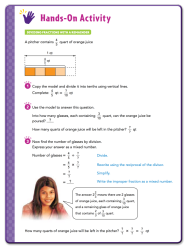
A plank is $\frac{4}{5}$ meter in length. A worker cuts it into some pieces, each of which is $\frac{1}{10}$ meter long. Into how many pieces did he cut the plank?



Ten tenths in 5/5, so
4/5 x 10 in 4/5 of a yard

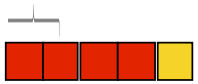
Visualize larger proper fraction divided by a smaller proper fraction

A pitcher contains $\frac{4}{5}$ quart of juice. If the juice is poured into glasses that hold $\frac{3}{10}$ quart, how many glasses can be filled? How much juice is left in the pitcher?



Visualize larger proper fraction divided by a smaller proper fraction

3/10



How many 3/10 in 4/5 of a quart

How many 3/10 in 1?

10/3

How many 3/10 in 4/5?

4/5 x 10/3

Visualize larger fraction divided by a smaller fraction

$1/2 \div 1/4$

1/2

4 fourths in 1, so 1/2 x 4 in 1/2 of a whole

Visualize smaller fraction divided by a larger fraction

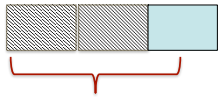
$1/4 \div 1/2$

1/2

Two halves in 1, so 1/4 times 2 = 1/2 in a quarter

Visualize larger fraction into smaller

$2/3 \div 3/4$

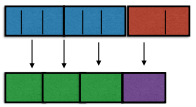


3/4

How many 3/4 in 1, (4/3)
so 2/3 x 4/3 = 2/3÷ 3/4 = 8/9

Partitive Division of fractions

If Barclay swims 2/3 of a mile in 3/4 of an hour, how far can he swim in one hour at that same rate?



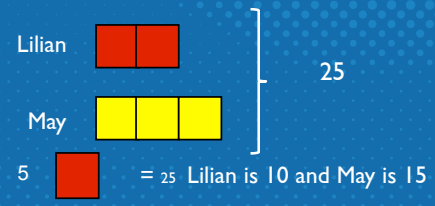
- Why visualization?

•Lilian's present age is 2/3 times May's age.

•a) Find the ratio of Mays age to Lilian's age.

•b) How many times the total age of the two girls is Lilian's age?

•c) Their combined age is 25 years. Find the age of



Math in Focus Singapore Math by Marshall Cavendish

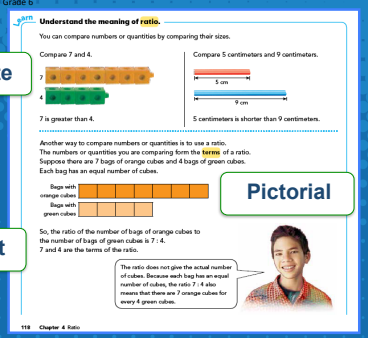
Visualization: Is this it?

Developing visualization

Concrete

Pictorial

Abstract

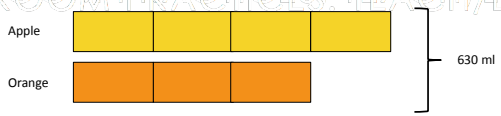


Model For Ratio

Learn **Draw models to solve problems involving ratios.**

Megan prepares a fruit punch using apple juice and orange juice in the ratio 4 : 3.

a) If the total volume of the fruit punch is 630 milliliters, find the volume of apple juice Megan uses.



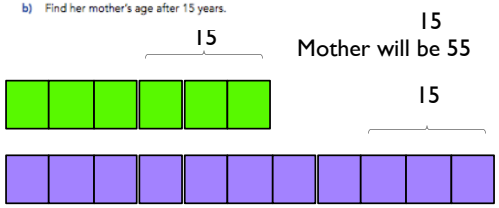
7 units equals 630 ml
1 unit equals 90
4 x 90ml = 360ml = apple juice

From fractions to ratio

14 Today the ratio of Elinor's age to her mother's age is 3 : 8. After 15 years, the ratio will become 6 : 11.

a) Find Elinor's age today.

b) Find her mother's age after 15 years.



Mother will be 55

Ratio problems

- The ratio of the number of beads Karen had to # of beads Patricia had was 2:5. After Patricia bought another 75 beads, the ratio became 4:15 How many beads did each girl have at first?

75

75 ÷ 5 = 15

= 15

Karen = 60
Patricia had 150

Visualization so students can do independent practice

Basic

1. A rope is cut into three pieces P, Q, and R. The lengths of the pieces are in the ratio 3 : 5 : 7. If the rope is 33 feet 9 inches long, find the lengths of P, Q, and R.

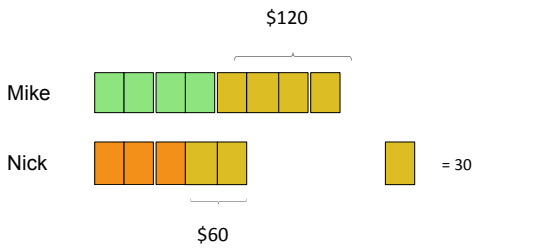
termediate

8. In a school gym, the ratio of the number of boys to the number of girls was 4 : 3. After 160 boys left the gym, the ratio became 4 : 5. How many girls were there in the gym?

dvanced

15. The ratio of Mike's savings to Nick's savings was 4 : 3. After Mike saved another $120 and Nick saved another $60, the ratio became 8 : 5. What was their combined savings before each of them saved the additional money?

Visualize ratio



15. The ratio of Mike's savings to Nick's savings was 4 : 3. After Mike saved another $120 and Nick saved another $60, the ratio became 8 : 5. What wa their combined savings before each of them saved the additional money?

Percents: Given a quantity and its percent, find the whole

1. Ana has 8% of her CD collection in a box. If there are 96 CDs in the box, how many CDs are in Ana's whole collection?

96 CDs (8%)

Whole bar is 100%
8 % = 96
1% = 96÷8 = 12
100% = 100 x 12 = 1200

2. 250% of what number is 60?

Visualize rates

"Solve unit rates problems including those involving unit pricing and constant speed."

COMMON CORE STATE STANDARDS FOR Mathematics

Visualize rates

A racing car can travel at a speed of 175 km per hour. How far can the racing car travel in 3 hours?

175

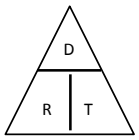
1 h ▭ unit rate

3 h ▭▭▭ 3 x 175 = 525 km

When you use this method, can you visualize?

or method 2:

Distance = Rate x Time
Distance = 175 km/hour x 3 hours = 525 km



Thinking, Fast and Slow

Mathematics is an excellent medium for "slow thinking."

Car travels at 60 mph for two hours. On the return trip, there is a snowstorm and the car travels at 40 mph the whole way. What is the average mph for the whole trip?

From ratios to rates

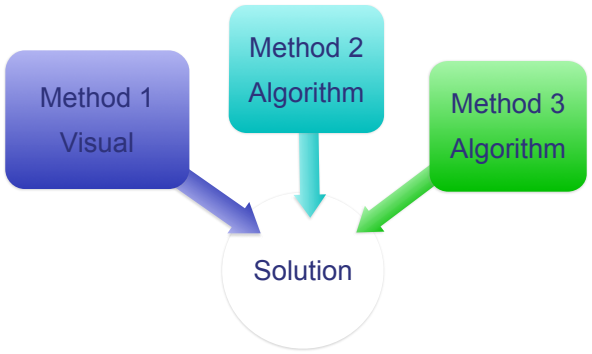
11 Mr. Alan drove for $2\frac{1}{5}$ hours at a speed of 70 kilometers per hour. He then drove another 224 kilometers. He took 5 hours for the whole journey. What was Mr. Alan's average speed for the whole journey?

12 A family took 2 hours to drive from City A to City B at a speed of 55 miles per hour. On the return trip, due to a snowstorm, the family took 3 hours to travel back to City A.

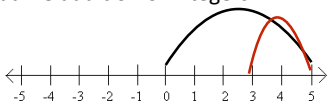
a) How many miles did the family travel in all?

b) What was the average speed for the entire trip?

Visualization: connecting image to the algorithm



Visualize addition of integers

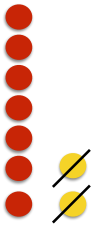


5 + -2

Algorithm:
Opposite signs, subtract absolute value and use sign of larger value.

Visualize subtraction integers

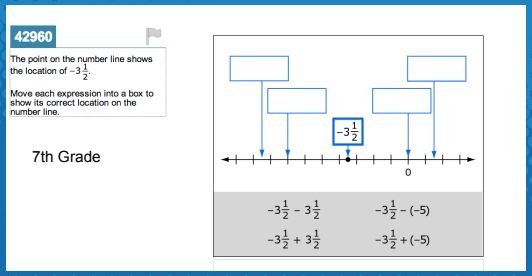
5 - (-2) means how far is it from -2 to +5



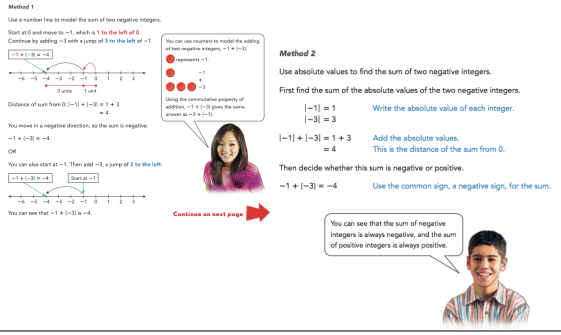
5 -2 = 3
- 5 - 2 = -3
- 5 -2 = -7
5 - (-2) = 7

Algorithm:
Change the sign.

Preparing for CCSS Assessments



Concrete to Pictorial to Abstract- Integer Addition



Method 2

Use absolute values to find the sum of two negative integers.

First find the sum of the absolute values of the two negative integers.

|−1| = 1 Write the absolute value of each integer.
|−3| = 3

|−1| + |−3| = 1 + 3 Add the absolute values.
= 4 This is the distance of the sum from 0.

Then decide whether this sum is negative or positive.

−1 + (−3) = −4 Use the common sign, a negative sign, for the sum.

Prior to middle school, students represent word problems

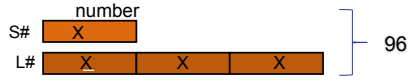
- The sum of two numbers is 96. The smaller number is 1/3 the size of the larger number. What is the smaller number

96÷ 4 = 24
3 x 24 = 72

The smaller number is 24

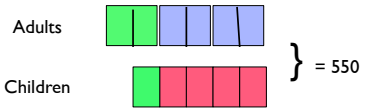
Visualize word problems with unknowns

- The sum of two numbers is 96. The smaller number is 1/3 the size of the larger number. What is the smaller number



4 X = 96
x = 96/4

There are 550 people at the tournament. 1/3 of the adults and 1/5 of the children wear jerseys .The number of adult and children not wearing jerseys are equal. How many children do not wear jersey?



Does this picture make sense, the green are the people wearing jerseys. Notice that the 4/5s of children and 2/3 of adults must be the same size.

4 parts of the children equal 2 parts of the adults.
mmm so we can make the fifths the unit

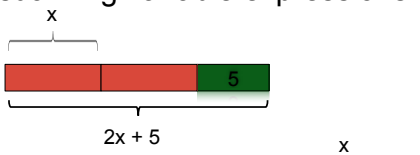
11 units = 550, 1 unit = 50 so 4 units of children don't wear jerseys or 200 children. Check your answer.

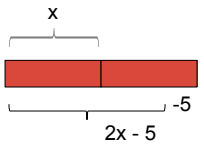
Visualizing variable expressions and equations

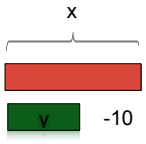
Caleb is x years old. His sister is ten years older. She is y years old. Write an equation that relates their two ages.

y = x + 10

Visualizing variable expressions



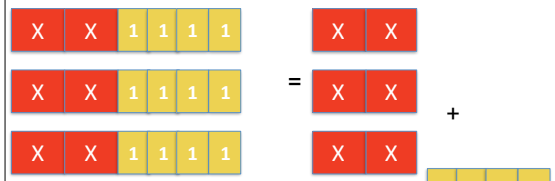




y = x - 10

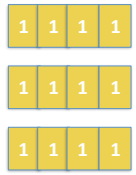
Visualize: Distributive property

3 (2x + 4)



=

+



6x + 12

- Visualize word problems

Guided Practice

Solve. Copy and complete.

1 Mark wrote a riddle: A negative number is $\frac{2}{5}$ of another negative number. If th sum of the two negative numbers is −35, find the two negative numbers.

- Teach to mastery and proficiency

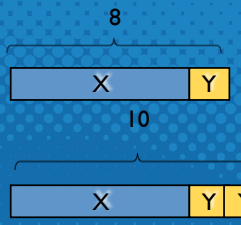
Guided Practice

Solve. Copy and complete.

1 Mark wrote a riddle: A negative number is $\frac{2}{5}$ of another negative number. If th sum of the two negative numbers is −35, find the two negative numbers.

x + 2/5x = -35
7/5 x = -35
divide both sides by 7/5
x = -25 2/5 (- 25) = -10

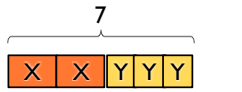
x + y = 8
x + 2y = 10



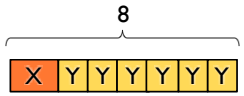
Solve algebraically, first with elimination, then with substitution

(x + 2 y) - (x + y) = 10 -8 = 2

Trajectory Algebra 8th Grade

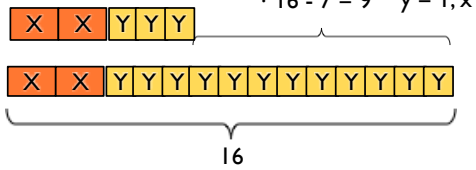


2x + 3y =7
x + 6y = 8

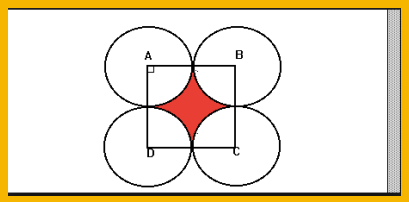


If we double the bottom equation, we will have 2 x a 3 y to subtract

? 16 - 7 = 9 y = 1, x = 2

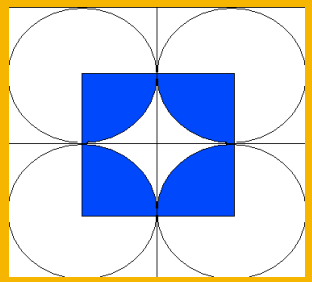


Visualization: spatial relationships



Area of shaded area

Visualization: Spatial relationships



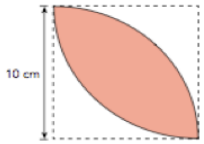
Area of shaded area

Visualization: spatial relationships

10 The petal of a paper flower is created by cutting along the outlines of two overlapping quadrants within a square. Use 3.14 as an approximation for π.

a) Find the distance around the shaded part.

b) Find the area of the shaded part.



- **The Power of visualization**
- To understand the problem: see relationships
- To simplify the problem
- To see connections to a related problem
- To cater to individual learning styles
- As a substitute for computation
- As a tool to check the solution
- To transform the problem into a mathematical form

Visualization: Enabling students to persevere & have entry points