2014 NCTM Regional Conference: Houston, TX

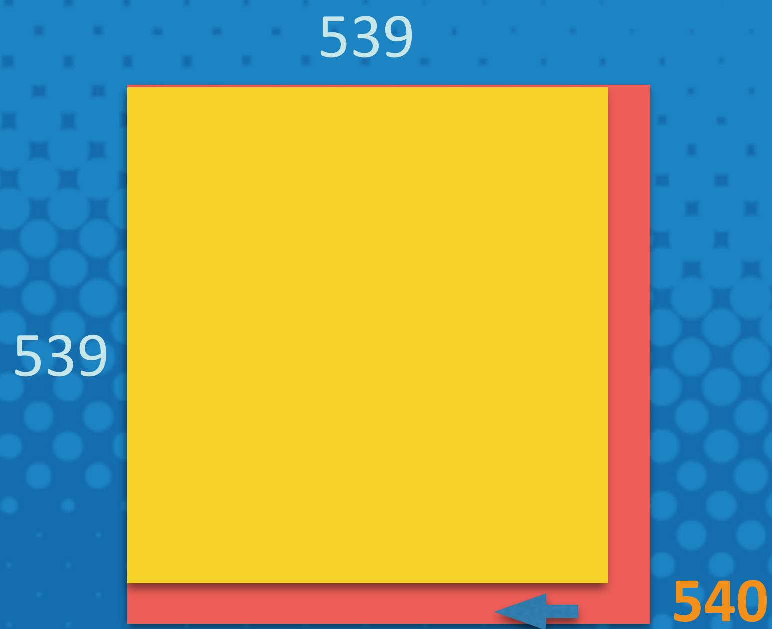
Andy Clark
andyclark@qwest.net

How Singapore's Visual Models (and Visualization) Enable Algebraic Thinking

Try this:
(No calculator and who wants to do all those calculations?)

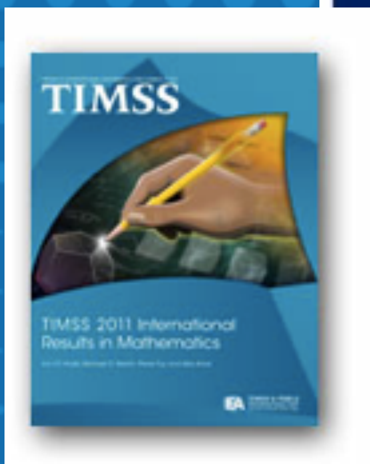
c) What is the value of $540^2 - 539^2$?

Can you visualize the problem?
How did you approach this problem?
What entry points?
Can you visualize the problem?



$$(n + 1)^2 - n^2 = 2n + 1$$

8th



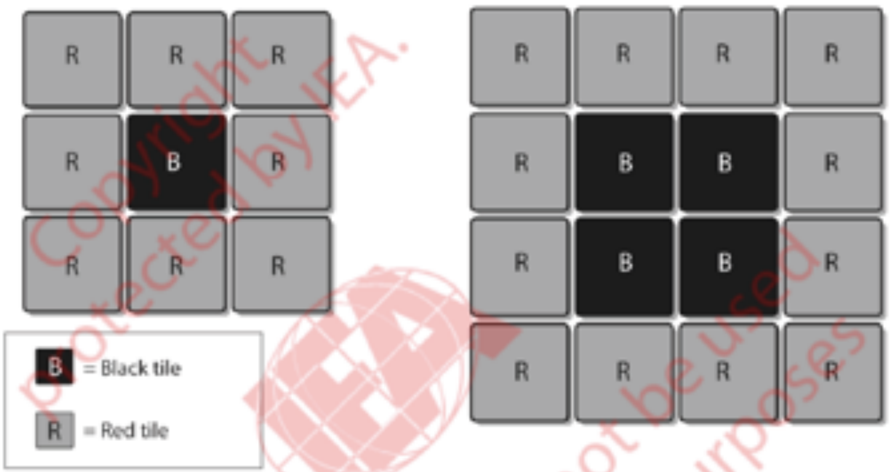
	average	advanced	high	intermediate	low
South Korea	613	47	77	[illegible]	99
Singapore	611	48	78	[illegible]	99
Taiwan	609	[illegible]	73	88	96
Hong Kong	586	34	71	89	97
Japan	570	27	61	87	97
Russia	539	14	47	78	95
Israel	516	12	40	68	87
Finland	514	4	30	73	96
United States	509	7	30	68	92
England	507	8	32	65	88
International	500	3	17	46	75

Example of 8th Grade TIMSS Problem

Pat has red tiles and black tiles. Pat uses the tiles to make square shapes.

The 3 × 3 shape has 1 black tile and 8 red tiles.

The 4 × 4 shape has 4 black tiles and 12 red tiles.



The table below shows the number of tiles for the first three shapes Pat made. Pat continued making shapes using this pattern. Complete the table for the 6 × 6 and 7 × 7 shapes.

Shape	Number of Black Tiles	Number of Red Tiles	Total Number of Tiles
3 × 3	1	8	9
4 × 4	4	12	16
5 × 5	9	16	25
6 × 6	16		
7 × 7	25		

% Received full credit

United States 57%

Korea, Japan, Singapore 88-90%

Example of 8th Grade TIMSS Problem

ID: M032761 | Mathematics Grade 8

Pat wanted to add a line to the table showing how to find the number of tiles needed to make a square of any size. Use the patterns in the table on the opposite page to help you complete the line for shape $n \times n$ in the table below.

Shape	Number of Black Tiles	Number of Red Tiles	Total Number of Tiles
$n \times n$	$(n - 2)^2$		

End of Red and Black Tiles section.

M032761

% Received fu
credit

United States
8%

Korea, Taipei,
Singapore
45-54%

Why does math become increasingly difficult?

- Students lack understanding of number relationships
- New content is more abstract, requires generalization
- There is a great deal of new content and often dealt with superficially
- Fewer manipulatives and visual images are used
- We are not always be able to explain procedures (e.g. division of fractions or decimals)
- Students don’t see relevance of math
- Parents less able to help
- Math anxiety increases in middle school

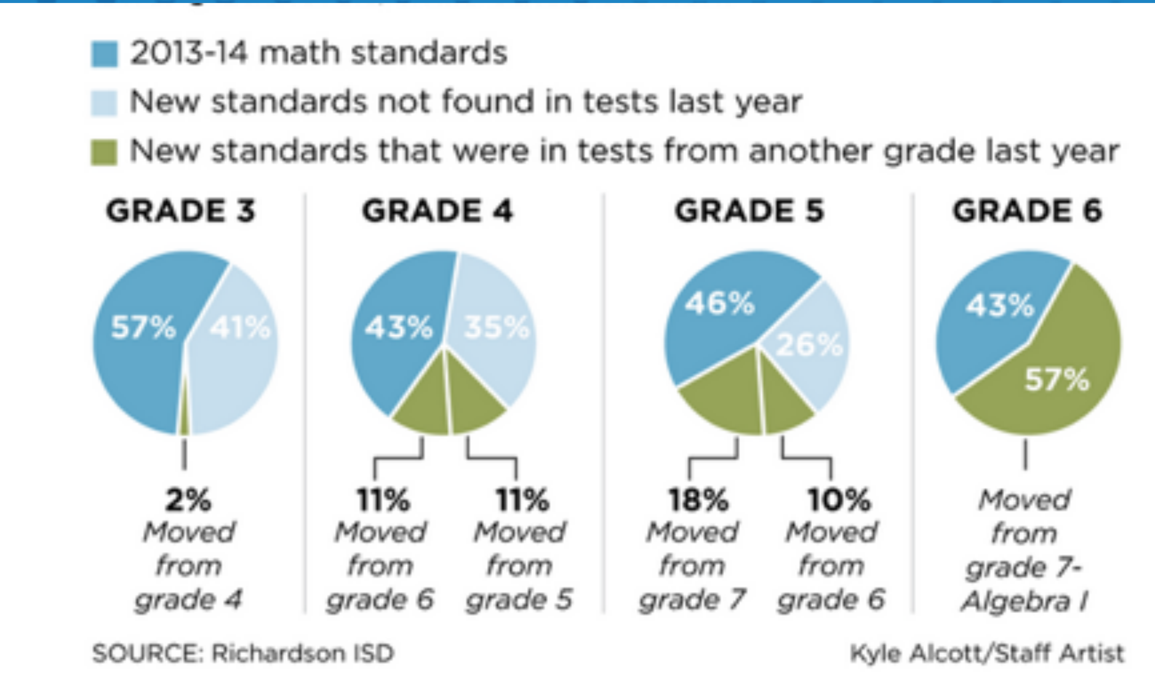
Attitude matters

Return to Algebra Comics

There are many difficult transitions for middle school students

- Whole number to rational numbers
- New definitions for fractions: use fraction symbol for ratios and division
- Natural numbers to integers
- Positive to signed numbers
- Number to variable
- Patterns to functions
- Arithmetic to generalizations

And now students are expected to know even more:



New TEKS

Problem solving in meaningful contexts, language and communication, connections within and outside mathematics, and formal and informal reasoning underlie all content areas in mathematics.

Throughout mathematics in Grades 6-8, students use these processes together with graphing technology and other mathematical tools such as manipulative materials to develop conceptual understanding and solve problems as they do mathematics.

New TEKS

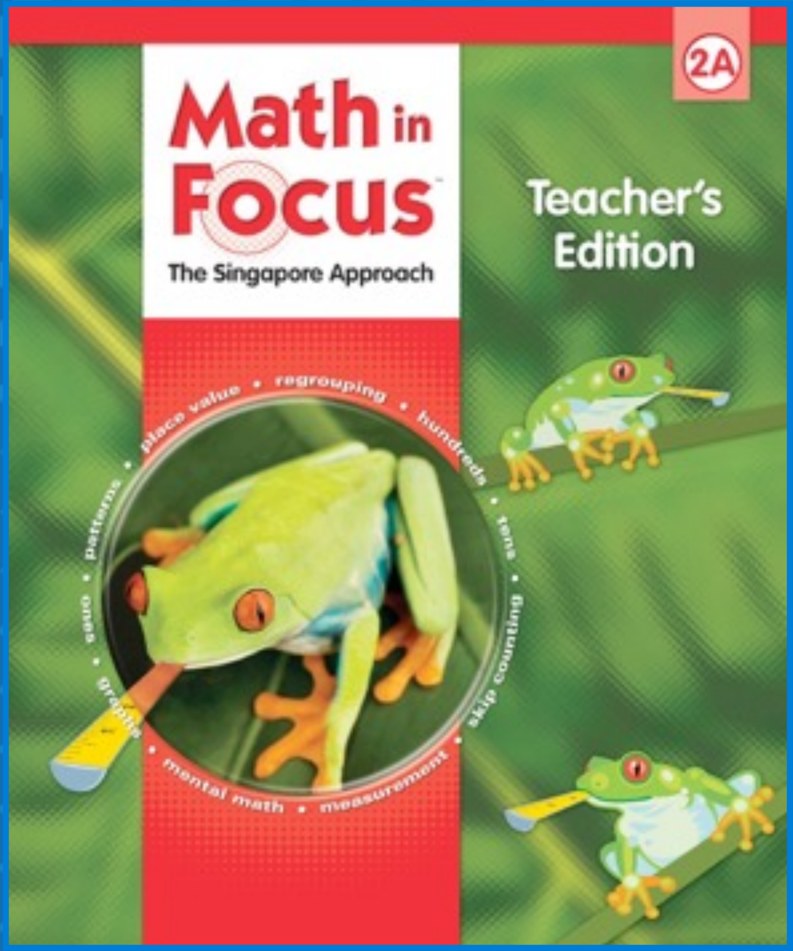
"Mathematical process standards. The student uses mathematical processes to acquire and demonstrate mathematical understanding."

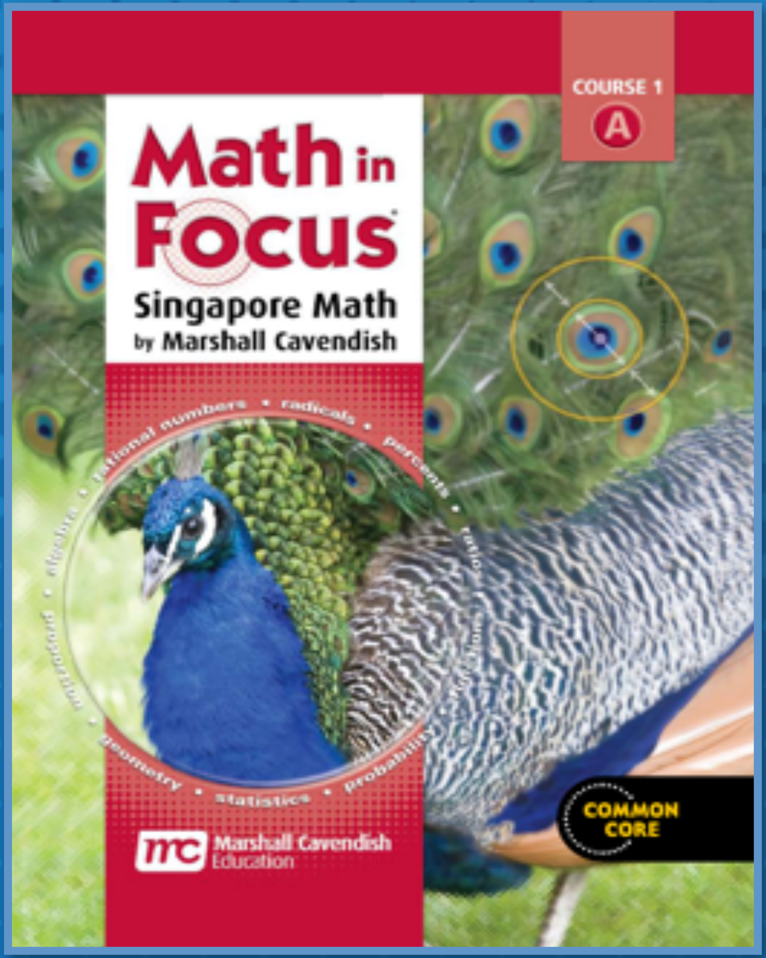
"The process standards describe ways in which students are expectec engage in the content. The placement of the process standards at the beginning of the knowledge and skills listed for each grade and course intentional."

Balance understanding and procedural fluency

"Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily."

COMMON CORE STATE STANDARDS FOR Mathematics


2A
Math in Focus
The Singapore Approach
Teacher's Edition
place value • regrouping • hundreds • tens • skip counting
mental math • measurement
patterns • ones


COURSE 1
A
Math in Focus
Singapore Math
by Marshall Cavendish
rational numbers • radicals • percents
geometry • statistics
Marshall Cavendish Education
COMMON CORE

An important feature of the Singapore curriculum is its emphasis on visualizatior as a major strategy for developing understanding and problem solving.

One often hears teachers there saying: "Can You see it?"

Diezmann, C. M. (2000). The difficulties students experience in generating diagrams for novel problems. In T. Nakahara & M. Koyama (Eds.), Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 241-248). Hiroshima, Japan: PME.

Diezmann, C. M., & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. A. Cuoco (Ed.), 2001 National Council of Teachers of Mathematics Yearbook: The role of representation in school mathematics (pp.77-89). Reston, VA: National Council of Teachers of Mathematics.

Novick, L. R. (2001). Spatial diagrams: Key instruments in the toolbox for thought. In D. L. Merlin (Ed.), The psychology of learning and motivation, 40, 279-325.

van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. Journal of Educational Research, 83(6), 301-312.

Lowrie, T., & Kay, R. (2001). Relationship between visual and nonvisual solution methods and difficulty in elementary mathematics. Journal of Educational Research, 94(4), 248-255.

When you read the number -5, what images do you have?

- Location on a number line 5 units to the left or 5units below 0.
- The integer between -4 and -6
- The action of removing 5 from a set
- The action of moving left or down 5 units
- The additive inverse of 5, the number when added to +5 = 0

Less than 40% of of 12-13 year olds were able to subtract integers

Garladdo (1995)

More than 25% of 12-13 year olds can’t add a positive and negative number

Kloosterman (2012)

50% of 12-13 year olds can’t divide integers correctly.

Kloosterman (2012)

When you see the division of a fraction, what mental image do you have?

How can students develop an understanding of why the division of a fraction by a fraction is solved by multiplying by the reciprocal?

Greater than one or less than one?

$1 \div 2/3$

$3/4 \div 1/8$

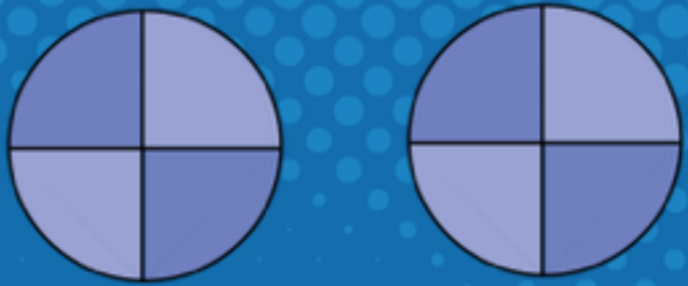
$3/4 \div 7/8$

$3/4 \div 2/3$

$2/3 \div 3/4$

$4/13 \div 2/7$

Visualize whole number divided by a unit fraction



How many fourths in 1 whole? In 2? $2 \div 1/4 = 2 \times 4$

How many fourths in 6 wholes? In 20?

If you know one whole, how can you find the number in any number of wholes ? $n \div 1/4 = n \times 4$

#wholes x 4 = number of fourths

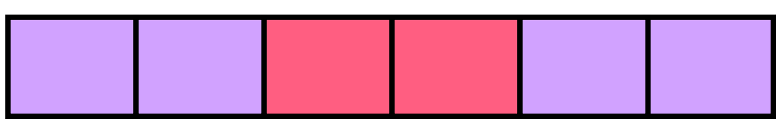
Visualize a whole number divided by a proper fraction

How many 2/3 in 1 whole?

How many 2/3 in 2 wholes?

Now how many 2/3 in 1 whole? Half of 3 or 3/2

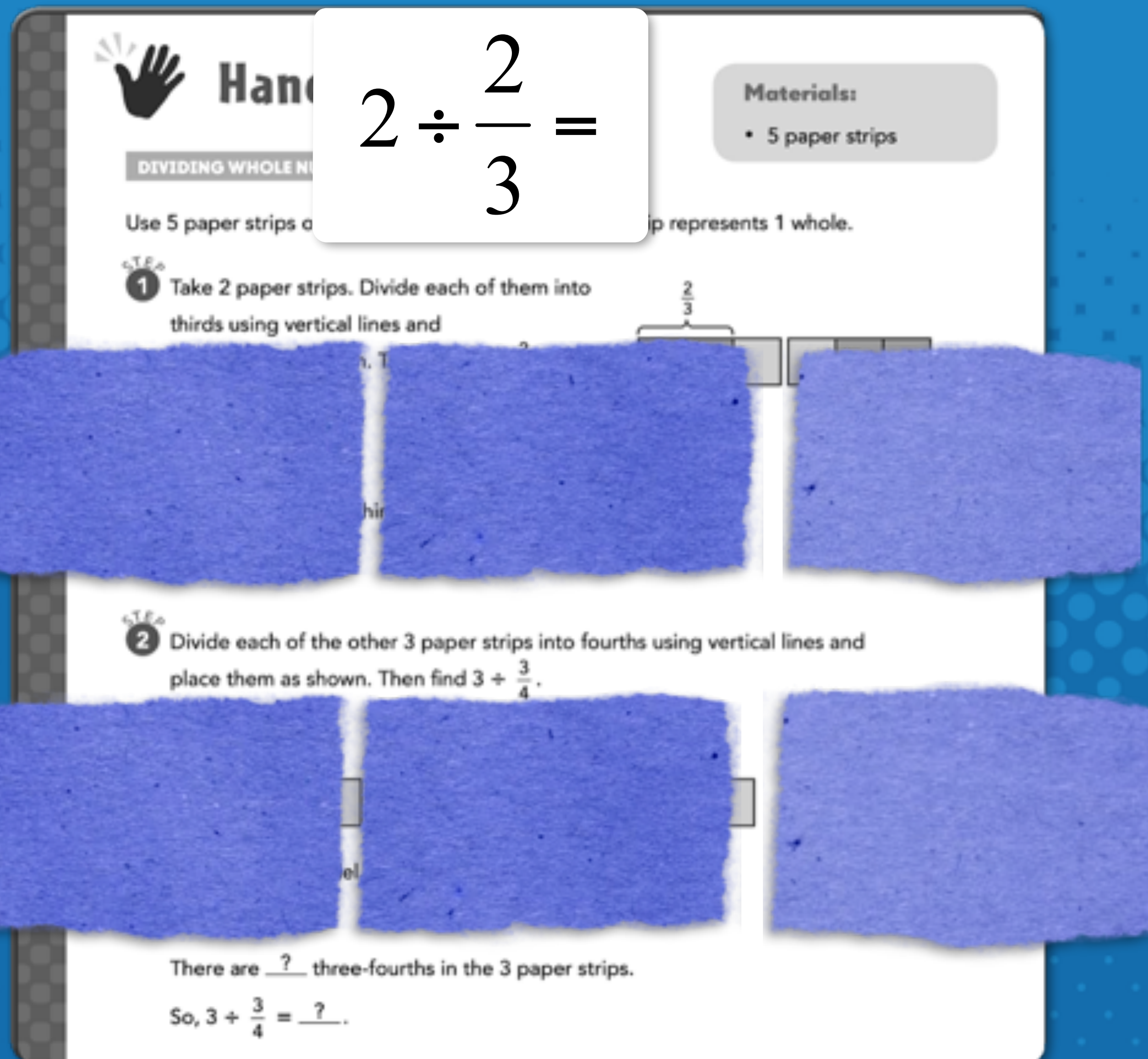
So if we know how many in 1, how can we find how many 2/3 in 5 wholes?



$5 \times 3/2 = 5 \div 2/3$

$7 \times 3/2 = 7 \div 2/3$

Mine is not to reason why…



Han

$$2 \div \frac{2}{3} =$$

Materials:

- 5 paper strips

DIVIDING WHOLE NU

Use 5 paper strips o … p represents 1 whole.

STEP 1 Take 2 paper strips. Divide each of them into thirds using vertical lines and

$\frac{2}{3}$

STEP 2 Divide each of the other 3 paper strips into fourths using vertical lines and place them as shown. Then find $3 \div \frac{3}{4}$.

There are ? three-fourths in the 3 paper strips.

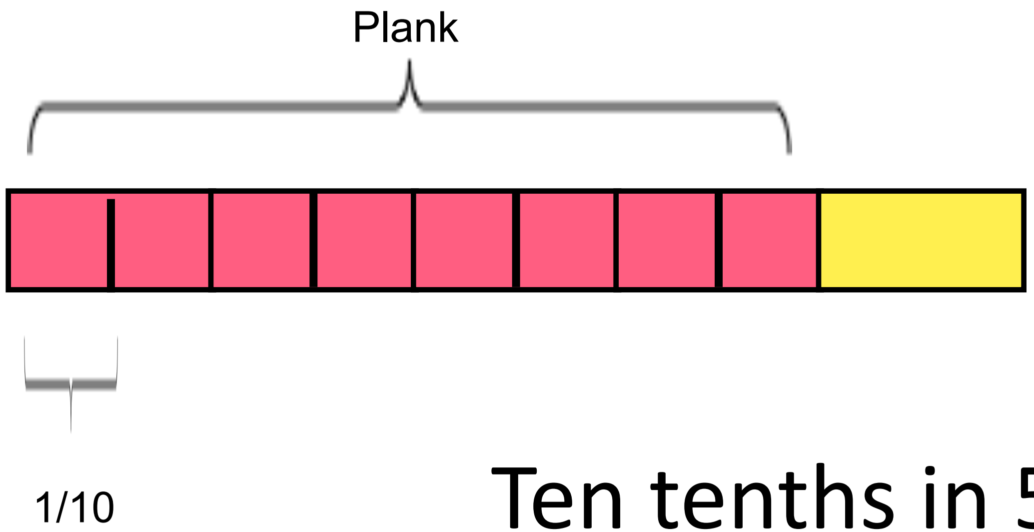
So, $3 \div \frac{3}{4} =$?.

Middle School Course 1, Book A, Page 67

$$2 \div \frac{2}{3} =$$

Visualize a larger proper fraction divided by a smaller unit fraction

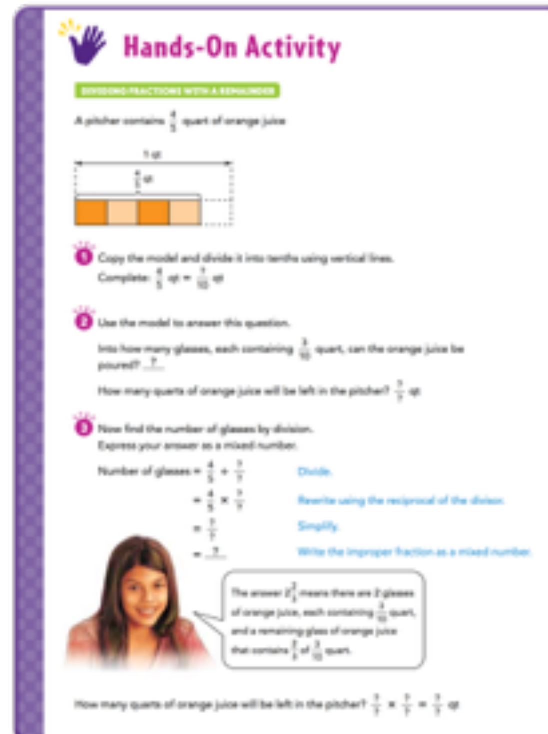
A plank is $\frac{4}{5}$ meter in length. A worker cuts it into some pieces, each of which is $\frac{1}{10}$ meter long. Into how many pieces did he cut the plank?



Ten tenths in 5/5, so
4/5 x 10 in 4/5 of a yard

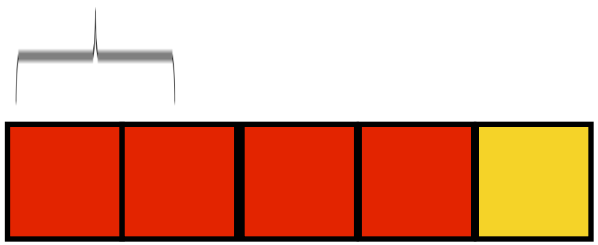
Visualize larger proper fraction divided by a smaller proper fraction

A pitcher contains $\frac{4}{5}$ quart of juice. If the juice is poured into glasses that hold $\frac{3}{10}$ quart, how many glasses can be filled? How much juice is left in the pitcher?



Visualize larger proper fraction divided by a smaller proper fraction

3/10



How many 3/10 in 4/5 of a quart

How many 3/10 in 1?

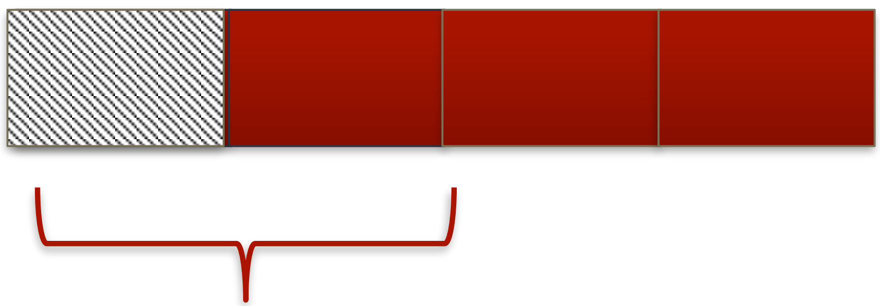
10/3

How many 3/10 in 4/5?

4/5 x 10/3

Visualize larger fraction divided by a smaller fraction

1/2 ÷ 1/4

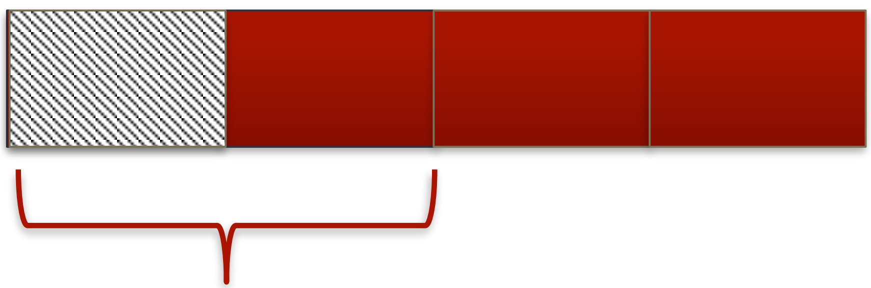


1/2

4 fourths in 1, so 1/2 x 4 in 1/2 of a whole

Visualize smaller fraction divided by a larger fraction

$1/4 \div 1/2$

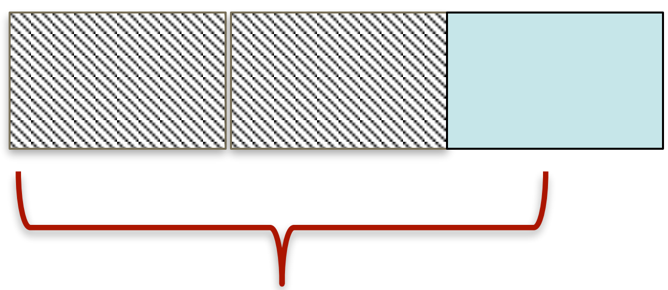


1/2

Two halves in 1, so 1/4 times 2 = 1/2 in a quarter

Visualize larger fraction into smaller

$2/3 \div 3/4$



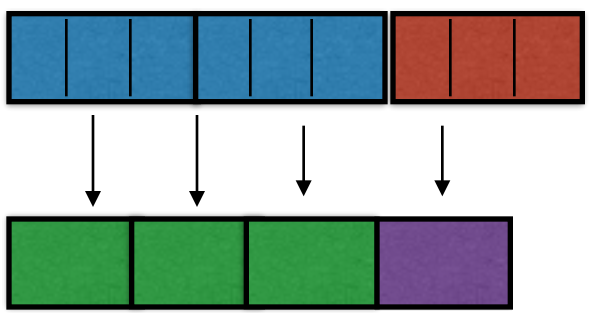
3/4

How many 3/4 in 1, (4/3)

so $2/3 \times 4/3 = 2/3 \div 3/4 = 8/9$

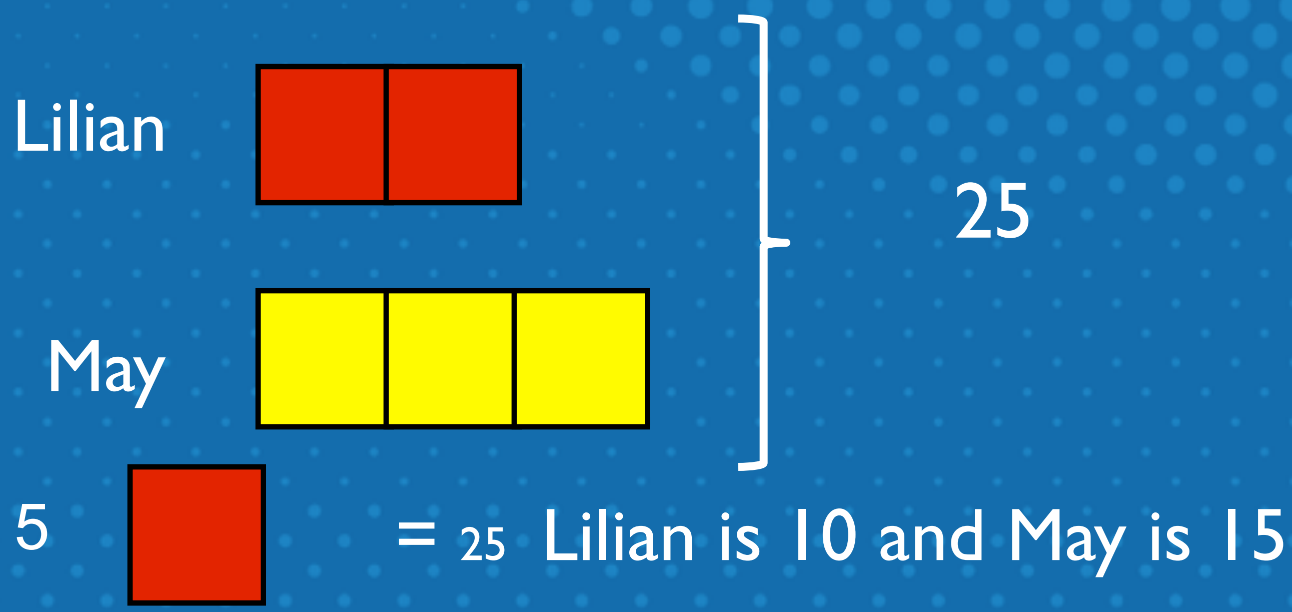
Partitive Division of fractions

If Barclay swims 2/3 of a mile in 3/4 of an hour, how far can he swim in one hour at that same rate?



- Why visualization?

- Lilian’s present age is 2/3 times May’s age.
- a) Find the ratio of Mays age to Lilian’s age.
- b) How many times the total age of the two girls is Lilian’s age?
- c) Their combined age is 25 years. Find the age of



Visualization: Is this it?

Developing visualization

Grade 6

Concrete

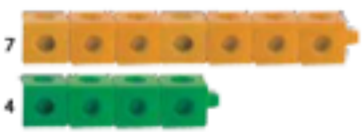
Pictorial

Abstract

Learn Understand the meaning of **ratio**.

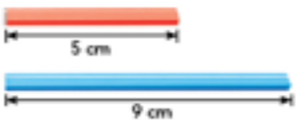
You can compare numbers or quantities by comparing their sizes.

Compare 7 and 4.



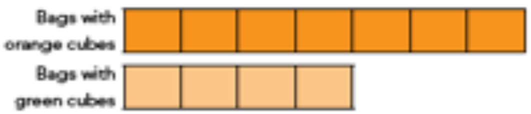
7 is greater than 4.

Compare 5 centimeters and 9 centimeters.



5 centimeters is shorter than 9 centimeters.

Another way to compare numbers or quantities is to use a ratio.
The numbers or quantities you are comparing form the **terms** of a ratio.
Suppose there are 7 bags of orange cubes and 4 bags of green cubes.
Each bag has an equal number of cubes.



So, the ratio of the number of bags of orange cubes to the number of bags of green cubes is 7 : 4.
7 and 4 are the terms of the ratio.

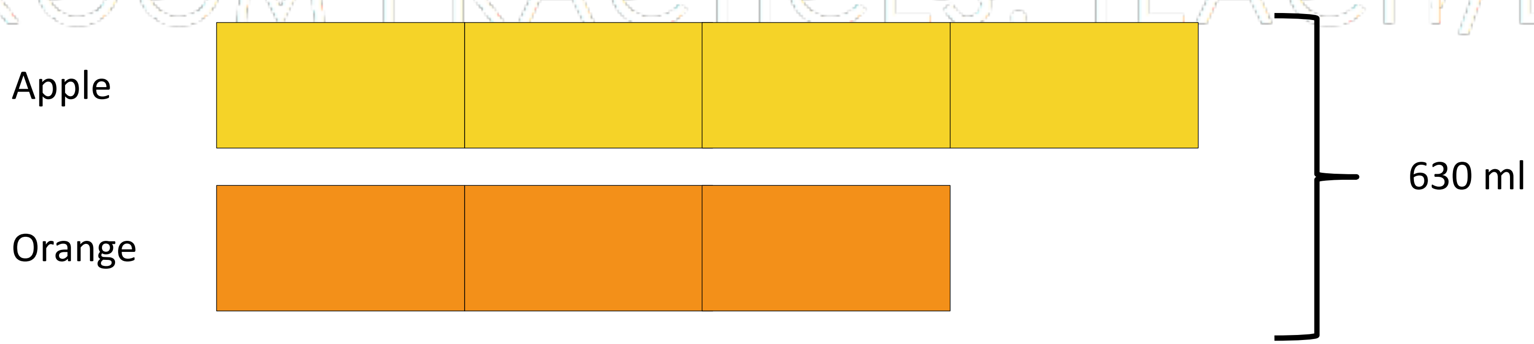
The ratio does not give the actual number of cubes. Because each bag has an equal number of cubes, the ratio 7 : 4 also means that there are 7 orange cubes for every 4 green cubes.

Model For Ratio

Learn **Draw models to solve problems involving ratios.**

Megan prepares a fruit punch using apple juice and orange juice in the ratio 4 : 3.

a) If the total volume of the fruit punch is 630 milliliters, find the volume of apple juice Megan uses.



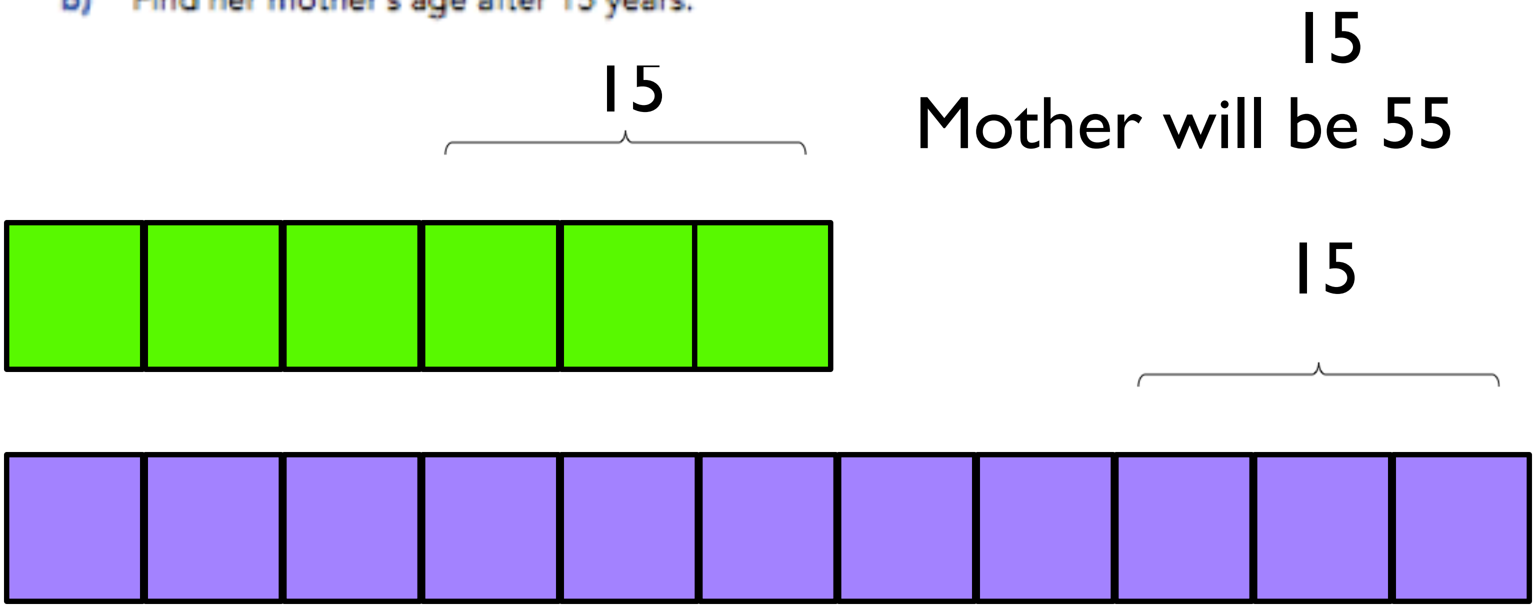
7 units equals 630 ml
1 unit equals 90
4 x 90ml = 360ml = apple juice

From fractions to ratio

14 Today the ratio of Elinor's age to her mother's age is 3 : 8. After 15 years, the ratio will become 6 : 11.

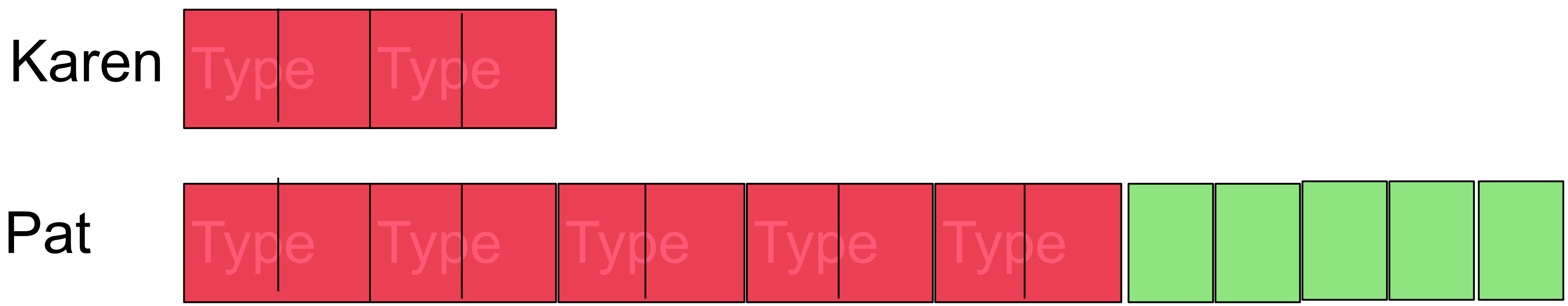
a) Find Elinor's age today.

b) Find her mother's age after 15 years.



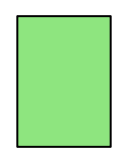
Ratio problems

- The ratio of the number of beads Karen had to # of beads Patricia had was 2:5. After Patricia bought another 75 beads, the ratio became 4:15 How many beads did each girl have at first?



75

75 ÷ 5 = 15

 = 15

Karen = 60
Patricia had 150

Visualization so students can do independent practice

Basic

1. A rope is cut into three pieces P, Q, and R. The lengths of the pieces are in the ratio 3 : 5 : 7. If the rope is 33 feet 9 inches long, find the lengths of P, Q, and R.

termediate

8. In a school gym, the ratio of the number of boys to the number of girls was 4 : 3. After 160 boys left the gym, the ratio became 4 : 5. How many girls were there in the gym?

dvanced

15. The ratio of Mike's savings to Nick's savings was 4 : 3. After Mike saved another $120 and Nick saved another $60, the ratio became 8 : 5. What was their combined savings before each of them saved the additional money?

Visualize ratio

15 The ratio of Mike's savings to Nick's savings was 4 : 3. After Mike saved another $120 and Nick saved another $60, the ratio became 8 : 5. What wa[s] their combined savings before each of them saved the additional money?

Percents: Given a quantity and its percent, find the whole

1. Ana has 8% of her CD collection in a box. If there are 96 CDs in the box, how many CDs are in Ana's whole collection?

96 CDs (8%)

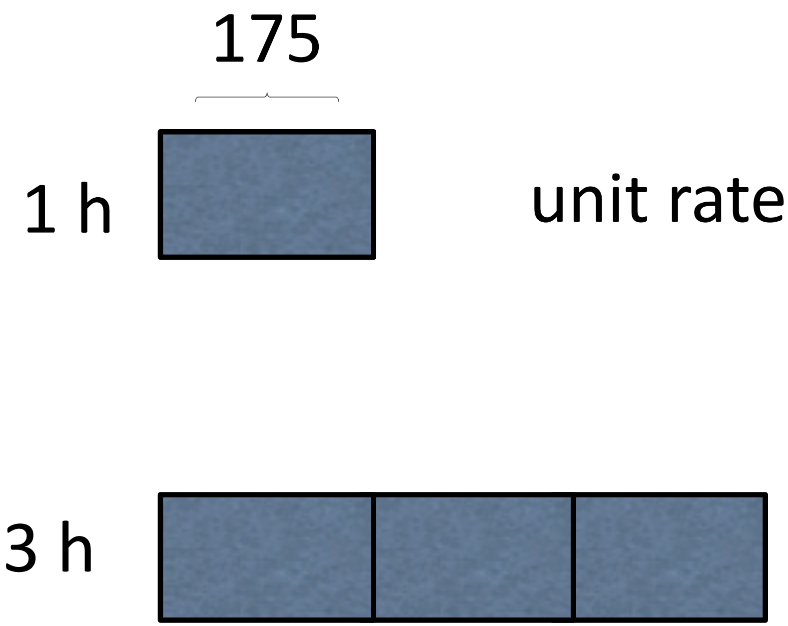
Whole bar is 100%
8 % = 96
1% = 96÷8 = 12
100% = 100 x 12 = 1200

2. 250% of what number is 60?

Visualize rates

A racing car can travel at a speed of 175 km per hour.

How far can the racing car travel in 3 hours?



unit rate

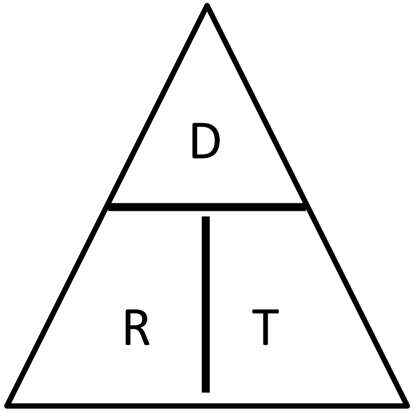
3 x 175 = 525 km

When you use this method, can you visualize?

or method 2:

Distance = Rate x Time

Distance = 175 km/hour x 3 hours = 525 km



Thinking, Fast and Slow

Mathematics is an excellent medium for "slow thinking."

Car travels at 60 mph for two hours. On the return trip, there is a snowstorm and the car travels at 40 mph the whole way. What is the average mph for the whole trip?

From ratios to rates

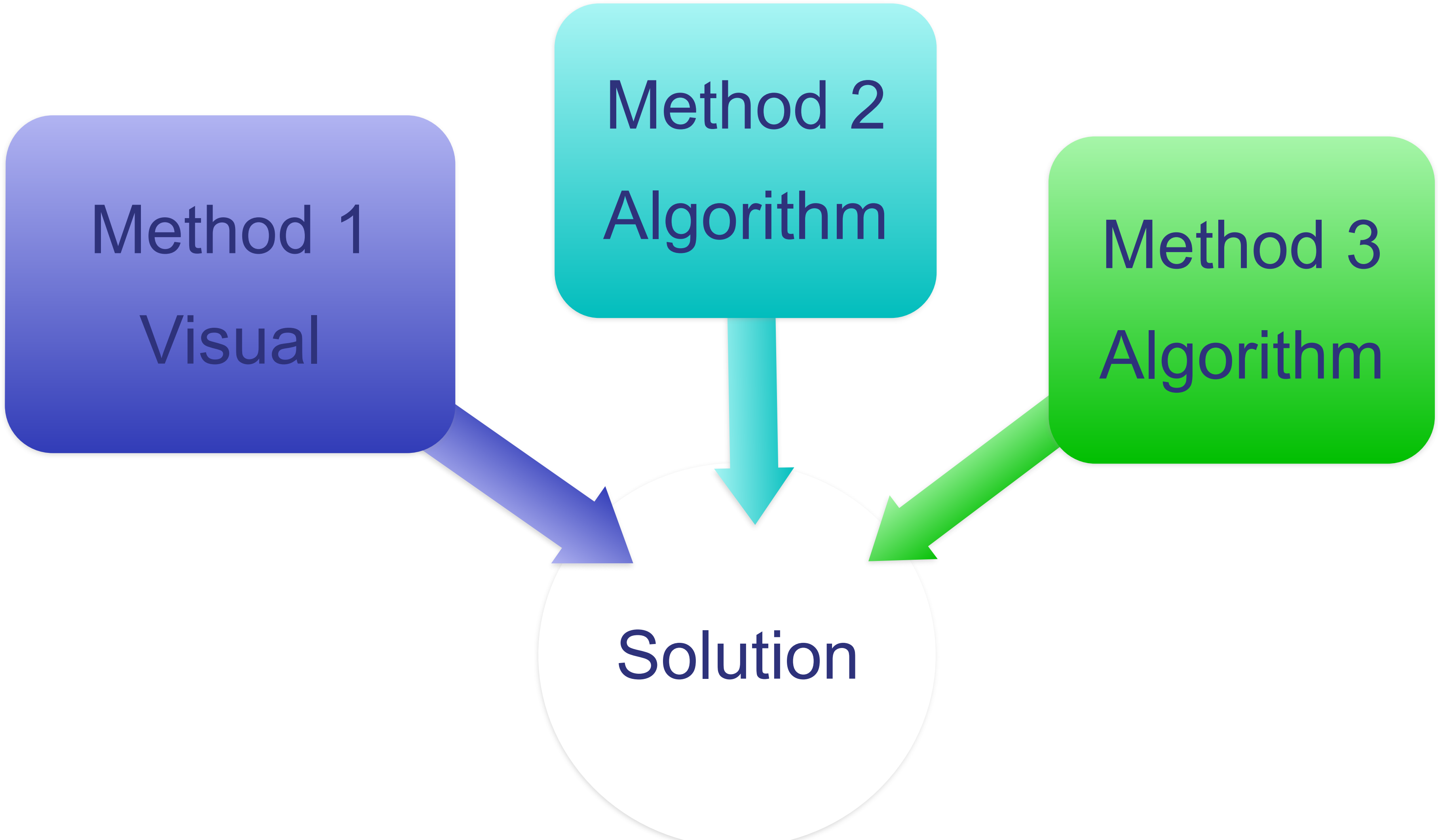
11 Mr. Alan drove for $2\frac{1}{5}$ hours at a speed of 70 kilometers per hour. He then drove another 224 kilometers. He took 5 hours for the whole journey. What was Mr. Alan's average speed for the whole journey?

12 A family took 2 hours to drive from City A to City B at a speed of 55 miles per hour. On the return trip, due to a snowstorm, the family took 3 hours to travel back to City A.

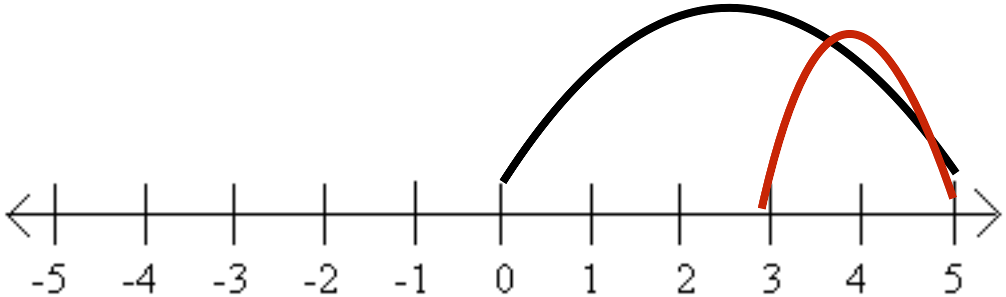
a) How many miles did the family travel in all?

b) What was the average speed for the entire trip?

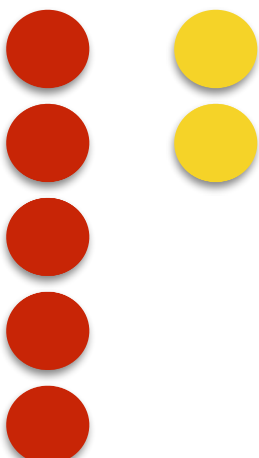
Visualization: connecting image to the algorithm



Visualize addition of integers



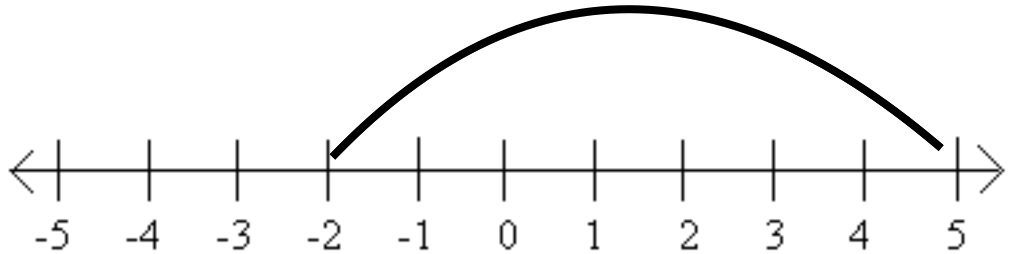
5 + -2



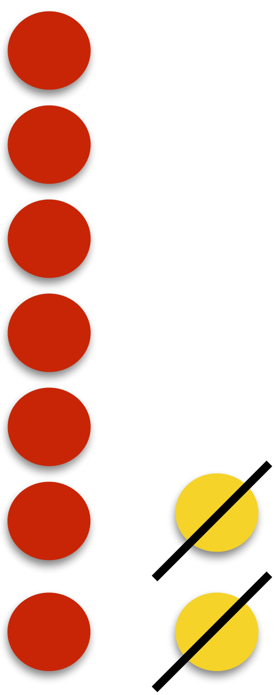
Algorithm:

Opposite signs, subtract absolute value and use sign of larger value.

Visualize subtraction integers



5 - (-2)   means how far is it from -2 to +5



5 -2 = 3

- 5 -2 = -7

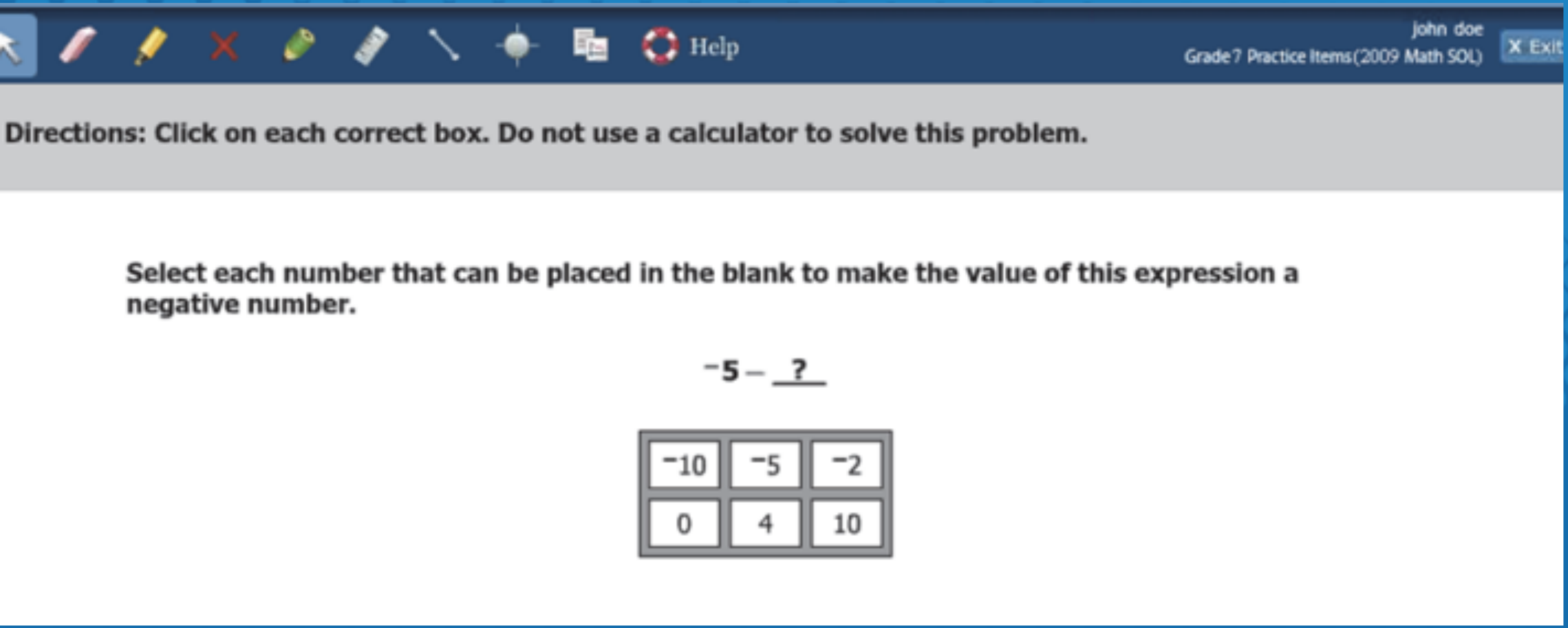
- 5 - (-2)= —3

5 - (-2) = 7

Algorithm:
Change the sign.

Preparing for SOL Assessments

7th Grade

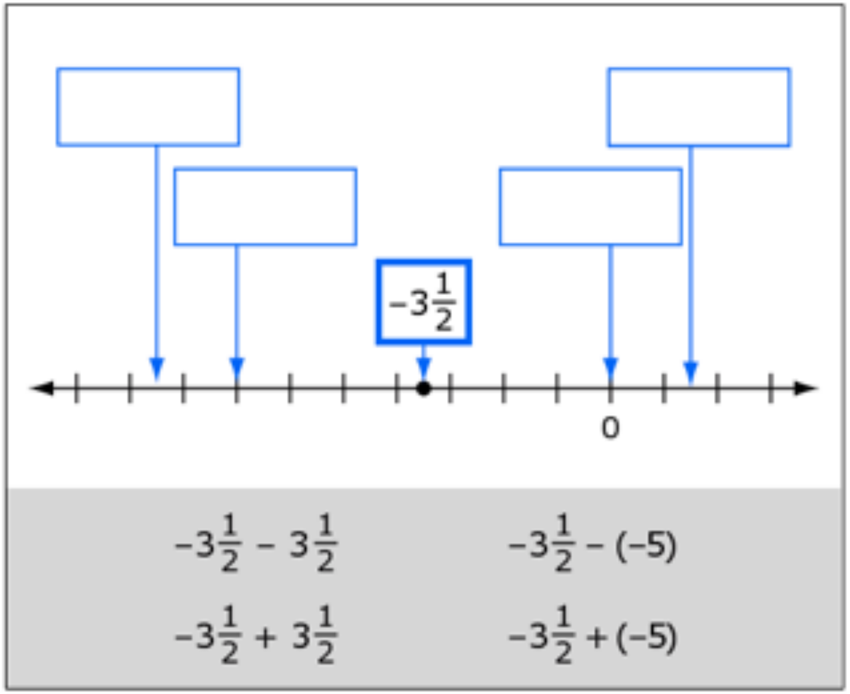


Preparing for CCSS Assessments

42960

The point on the number line shows the location of $-3\frac{1}{2}$.

Move each expression into a box to show its correct location on the number line.



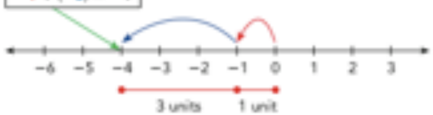
7th Grade

Concrete to Pictorial to Abstract- Integer Addition

Method 1

Use a number line to model the sum of two negative integers.

Start at 0 and move to −1, which is 1 to the left of 0.
Continue by adding −3 with a jump of 3 to the left of −1.



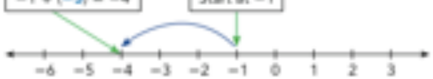
Distance of sum from 0: |−1| + |−3| = 1 + 3
= 4

You move in a negative direction, so the sum is negative.

−1 + (−3) = −4

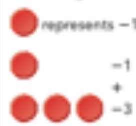
OR

You can also start at −1. Then add −3, a jump of 3 to the left.



You can see that −1 + (−3) is −4.

You can use counters to model the adding of two negative integers, −1 + (−3).



Using the commutative property of addition, −1 + (−3) gives the same answer as −3 + (−1).

Continue on next page

Method 2

Use absolute values to find the sum of two negative integers.

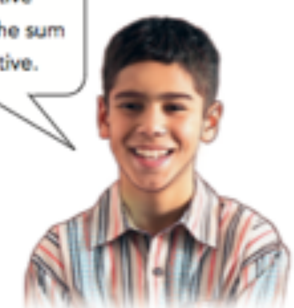
First find the sum of the absolute values of the two negative integers.

$\lvert -1 \rvert = 1$ $\lvert -3 \rvert = 3$	Write the absolute value of each integer.
$\lvert -1 \rvert + \lvert -3 \rvert = 1 + 3$ $= 4$	Add the absolute values. This is the distance of the sum from 0.

Then decide whether this sum is negative or positive.

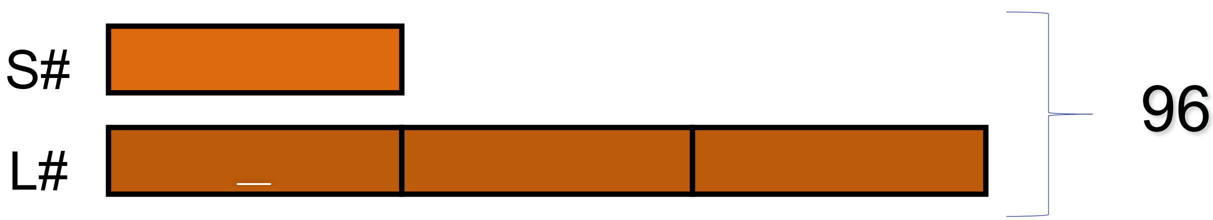
$-1 + (-3) = -4$	Use the common sign, a negative sign, for the sum.

You can see that the sum of negative integers is always negative, and the sum of positive integers is always positive.



Prior to middle school, students represent word problems with bars

- The sum of two numbers is 96. The smaller number is 1/3 the size of the larger number. What is the smaller number

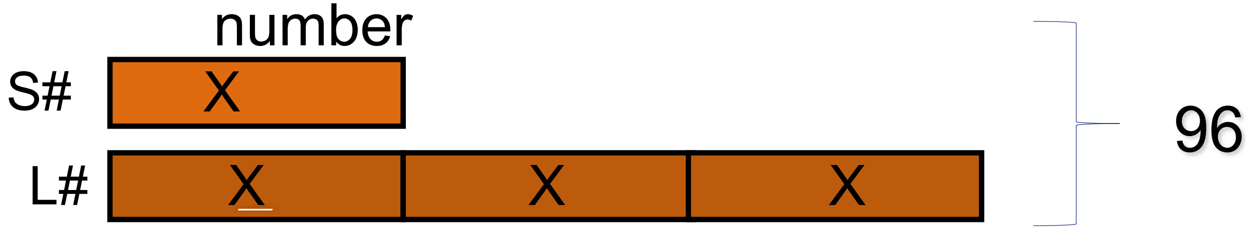


96÷ 4 = 24
3 x 24 = 72

The smaller number is 24

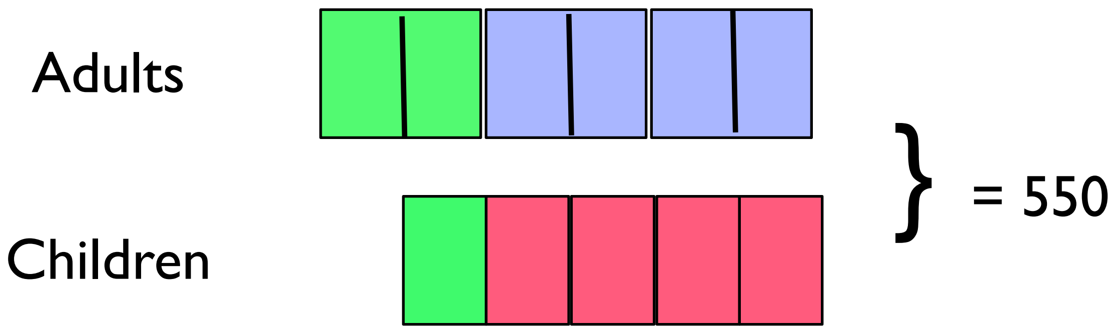
Visualize word problems with unknowns

- The sum of two numbers is 96. The smaller number is 1/3 the size of the larger number. What is the smaller number



$$4X = 96$$
$$x = 96/4$$

There are 550 people at the tournament. 1/3 of the adults and 1/5 of the children wear jerseys .The number of adult and children not wearing jerseys are equal. How many children do not wear jersey?



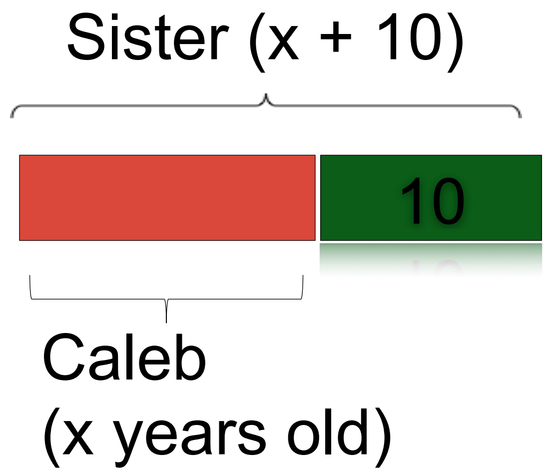
Does this picture make sense, the green are the people wearing jerseys. Notice that the 4/5s of children and 2/3 of adults must be the same size.

4 parts of the children equal 2 parts of the adults.
mmm so we can make the fifths the unit

11 units = 550, 1 unit = 50 so 4 units of children don't wear jerseys or 200 children. Check your answer.

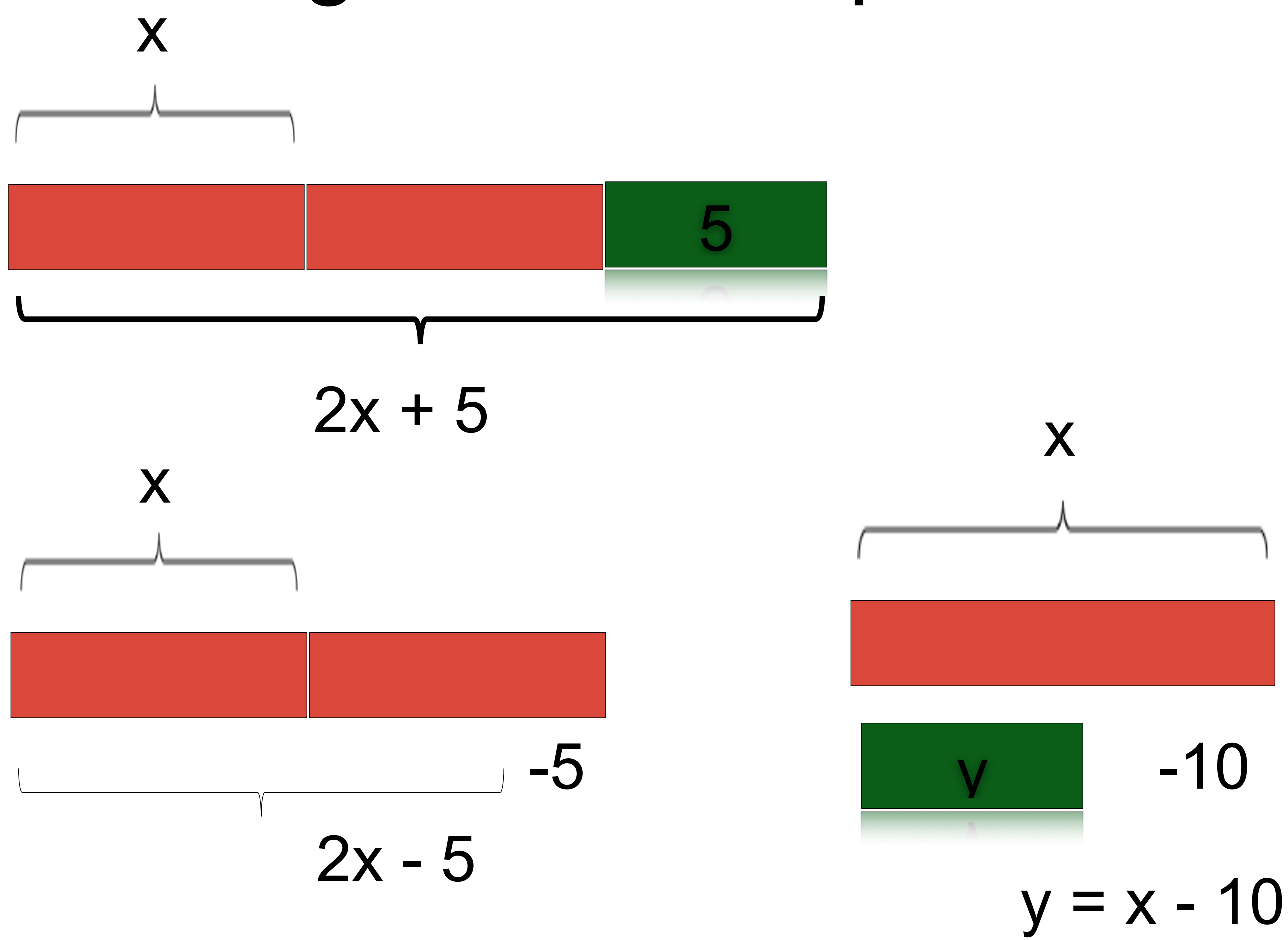
Visualizing variable expressions and equations

Caleb is x years old. His sister is ten years older. She is y years old. Write an equation that relates their two ages.



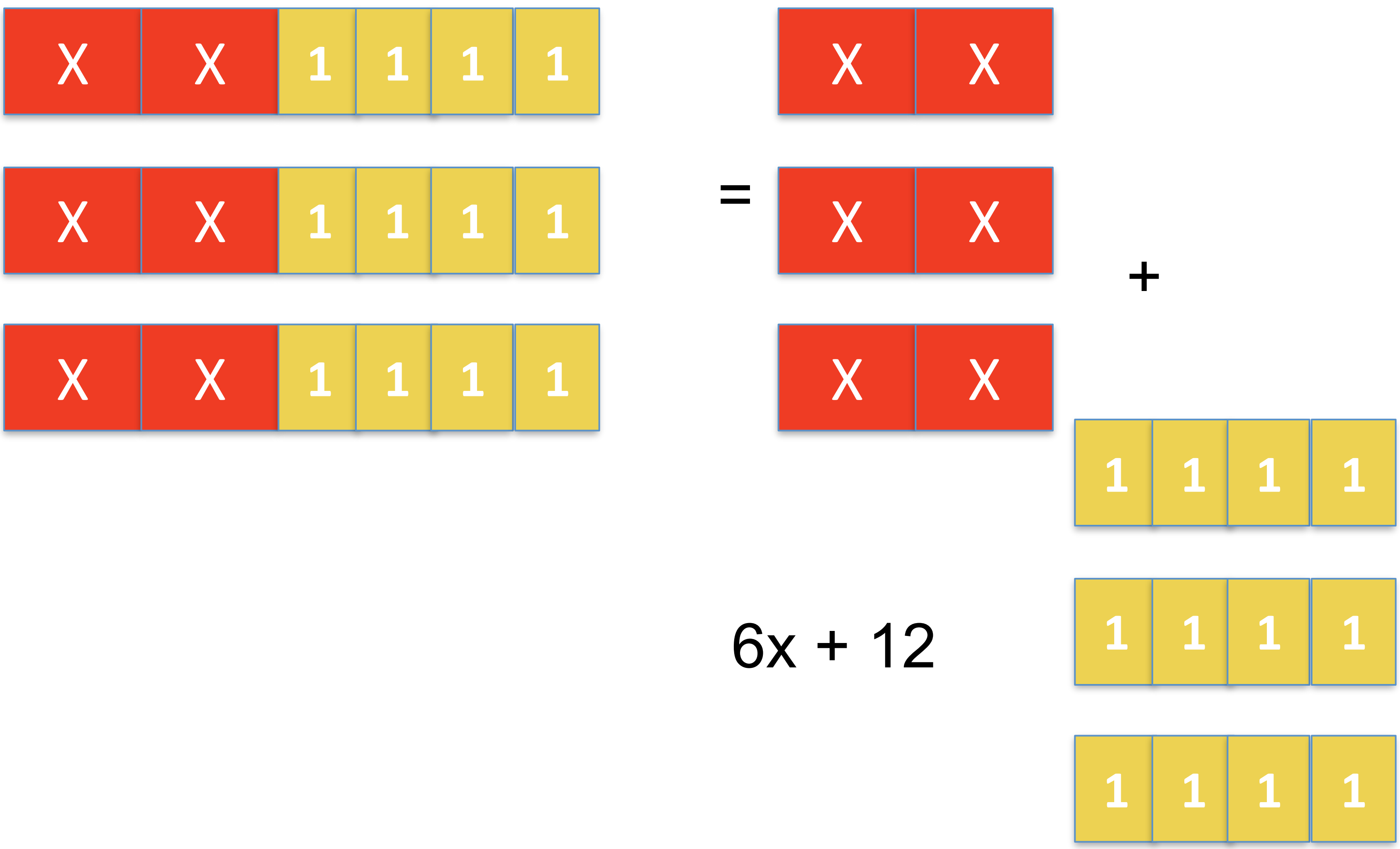
$y = x + 10$

Visualizing variable expressions



Visualize: Distributive property

3 (2x + 4)



6x + 12

- Visualize word problems

Guided Practice

Solve. Copy and complete.

1. Mark wrote a riddle: A negative number is $\frac{2}{5}$ of another negative number. If th sum of the two negative numbers is −35, find the two negative numbers.

- Teach to mastery and proficiency

Guided Practice

Solve. Copy and complete.

1. Mark wrote a riddle: A negative number is $\frac{2}{5}$ of another negative number. If th sum of the two negative numbers is −35, find the two negative numbers.

x + 2/5x = -35

7/5 x = -35

divide both sides by 7/5

x = -25   2/5 (- 25) = -10

Help

John doe

Grade 7 Practice Items (2009 Math SOL)

X Exit

Beatrice has 18 pencils. Beatrice has 2 more than 4 times the number of pencils Rick has. Exactly how many pencils does Rick have?

- **A** 11 pencils
- **B** 7 pencils
- **C** 5 pencils
- **D** 4 pencils

Flag for Review

Question 5 of 31
Section 1

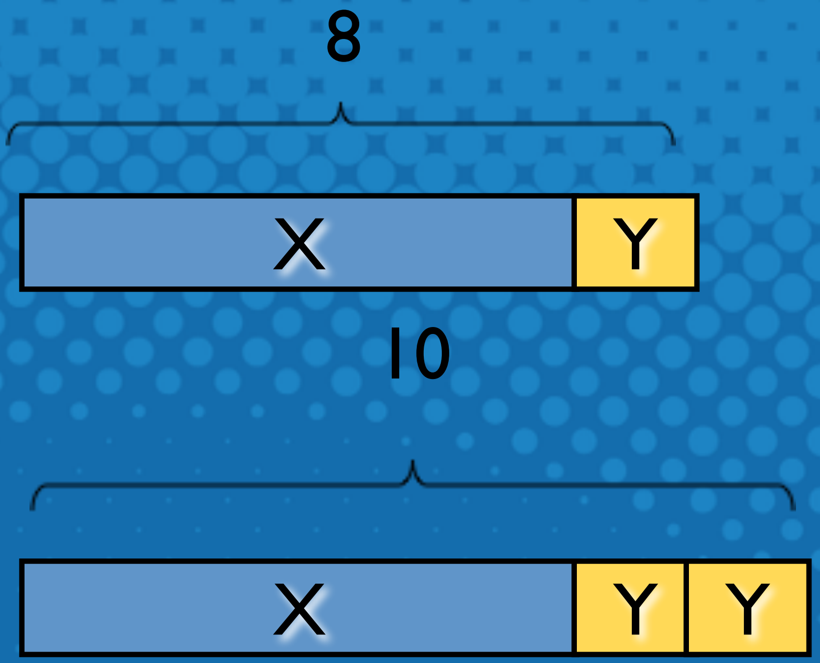
Section Review

Previous

Next

$x + y = 8$

$x + 2y = 10$



Solve algebraically, first with elimination, then with substitution

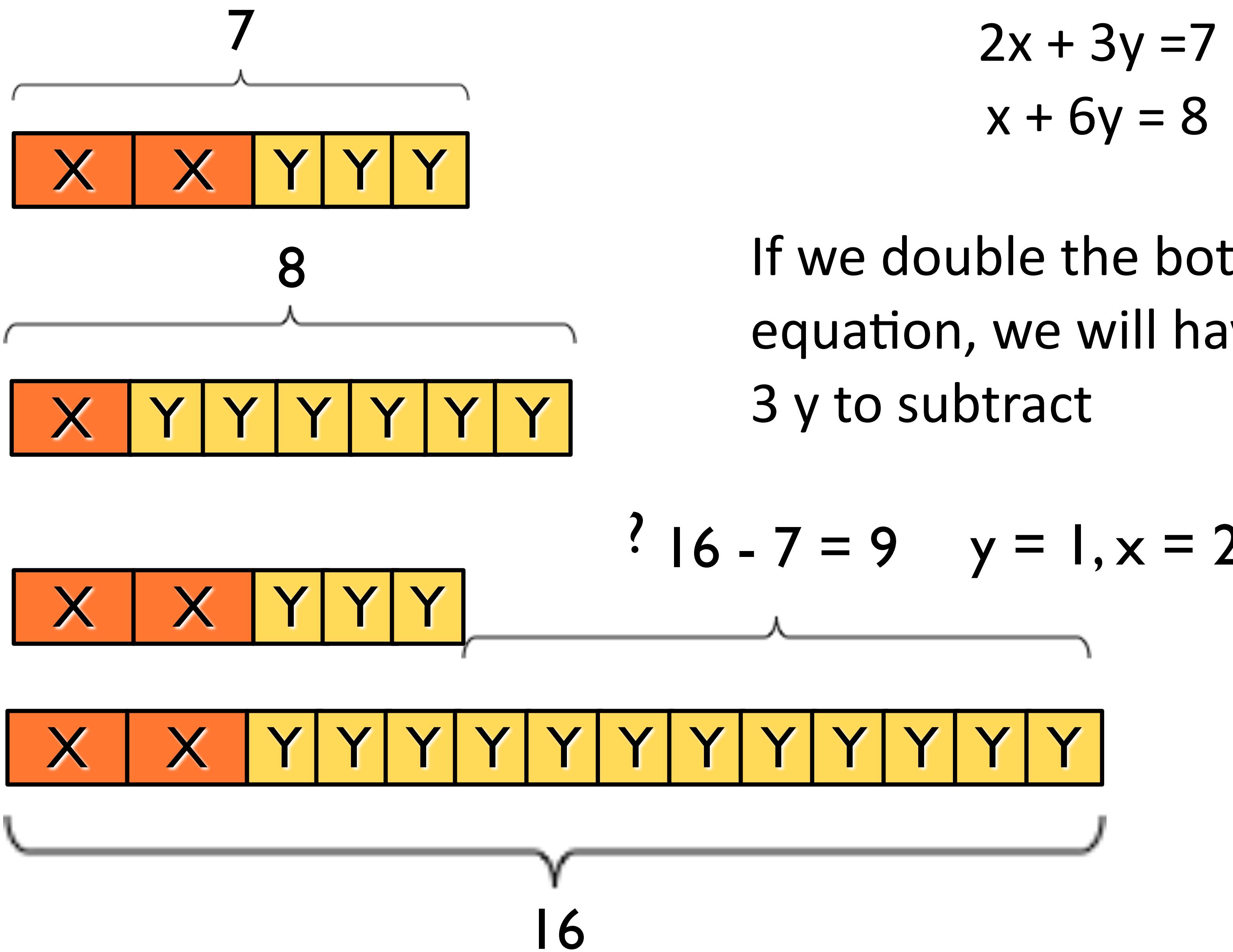
$(x + 2y) - (x + y) = 10 - 8 = 2$

Trajectory Algebra 8th Grade

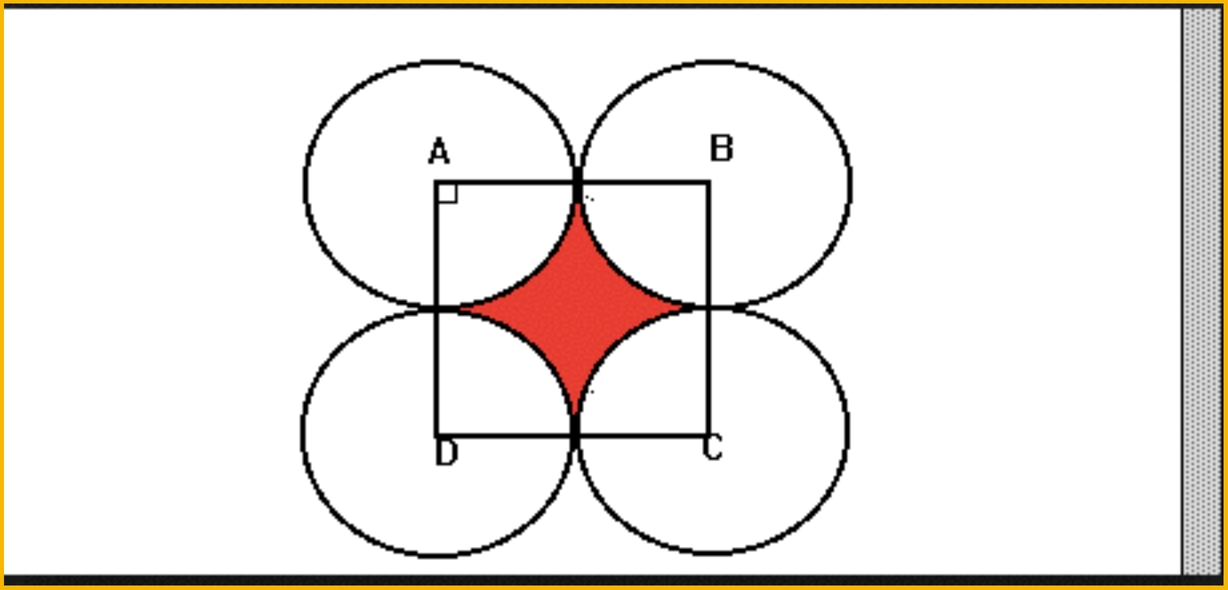
$2x + 3y = 7$

$x + 6y = 8$

If we double the bottom equation, we will have 2 x a
3 y to subtract

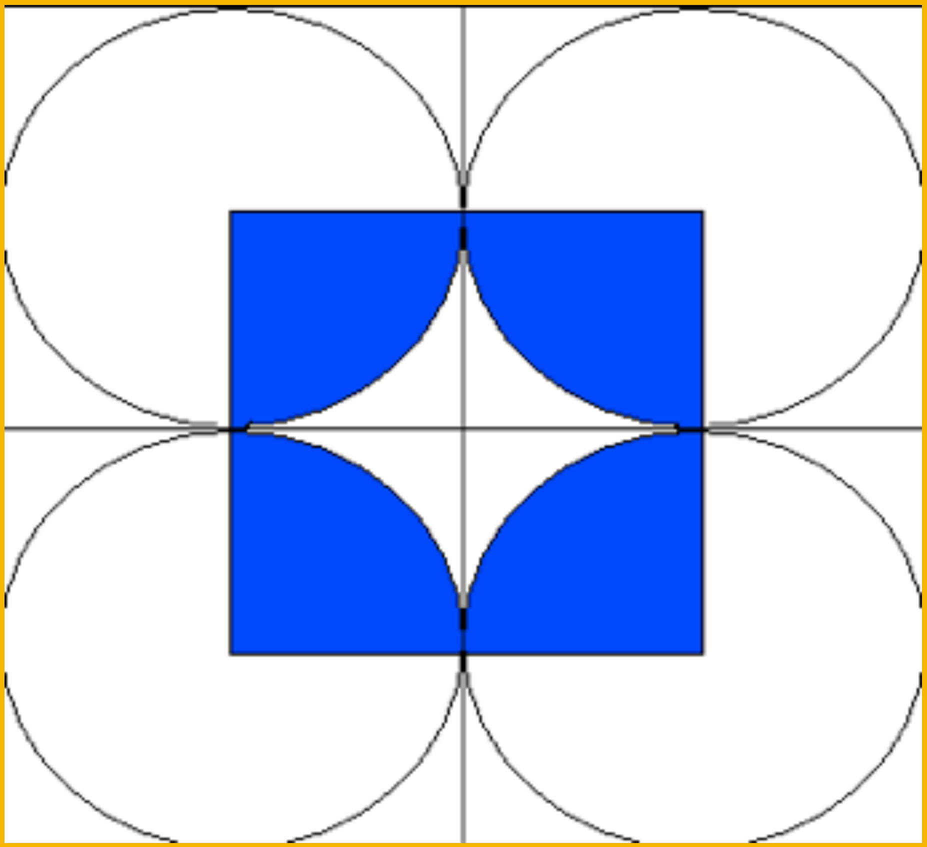


Visualization: spatial relationships



Area of shaded area

Visualization: Spatial relationships



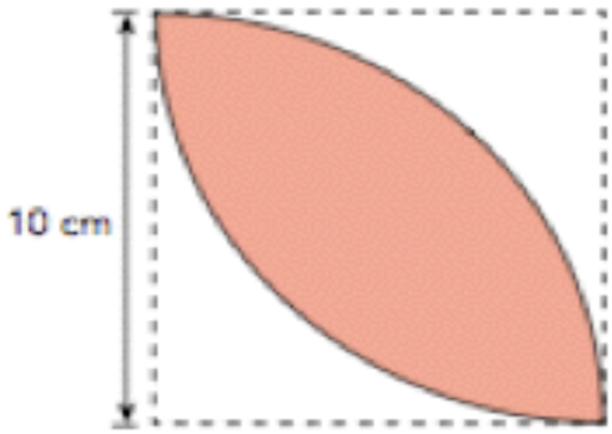
Area of shaded area

Visualization: spatial relationships

10 The petal of a paper flower is created by cutting along the outlines of two overlapping quadrants within a square. Use 3.14 as an approximation for π.

a) Find the distance around the shaded part.

b) Find the area of the shaded part.

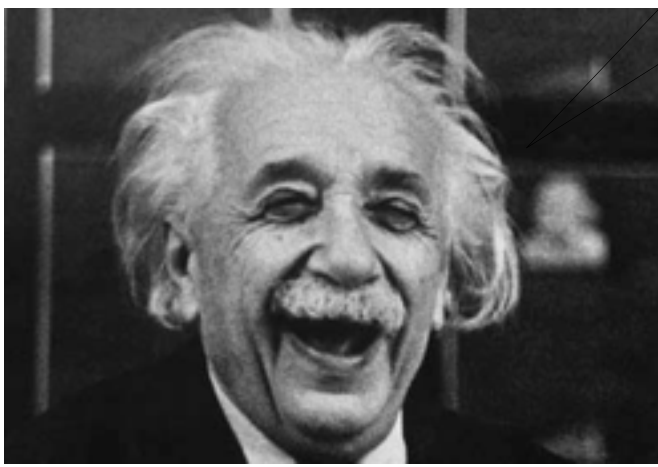


The Power of visualization

- To understand the problem: see relationships
- To simplify the problem
- To see connections to a related problem
- To cater to individual learning styles
- As a substitute for computation
- As a tool to check the solution
- To transform the problem into a mathematical form

Visualization: Enabling students to persevere & have entry points

Do you think
a bar model
will help?



2014 NCTM Regional Conference: Houston, TX

Andy Clark
andyclark@qwest.net

How Singapore's Visual Models (and Visualization) Enable Algebraic Thinking