NCTM Regional Conference: Investigating Released AP Calculus Test Questions, Grades 6-12

2014

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Fill in the Vocabulary Knowledge Rating and survey individually. For the numbered columns, check 4 if you could explain the concept to the group; 3 if you know it but….; 2 if you have heard of it; and 1 if you do not know it at all.

Calculus Topic	4	3	2	1	^ _ ^ _	I preview these in my classes	The topic should be previewed in MS math
Vocabulary Knowledge Rating					^ _		X
Derivatives					^ _		X
Definite Integrals					^ _		X
End behavior of graphs					^ _		X
Extrema problems					^ _		X
Asymptotic behavior					^ _		X
Differentiable functions					^ _		X
Graph (function) analysis					^ _		X
Limits					^ _		X
Reimann sums					^ _		X
Mean Value Theorem					^ _		X
Inflection points					^ _		X
Average rate of change					^ _		X
# of checks					^ _		
Total score							

Speed vs. Time Problem 1

You are driving on the interstate and you travel at exactly 80 mph for the next three hours.

Sketch a graph of the problem.

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Questions and answers for Speed vs. Time Problem 1:

1. What are the units of the sides of the rectangle?

   Horizontal units are hours.

   Vertical Units are miles per hour.

2. What is the area of the rectangle, including units?

   The area is 240 miles.

3. What does that mean in terms of the problem?

   That is the distance you drove in the 3 hours.

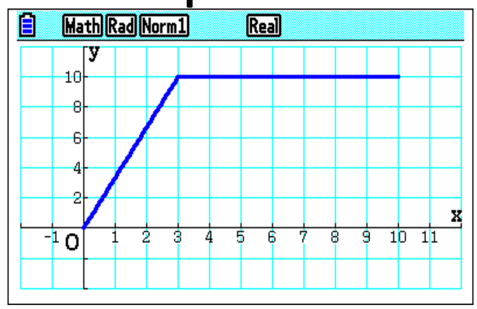
4. What did we just figure out?

   Actually, we just calculated $\int_0^3 80dt$, the area under the "curve"!!

If your MS students can graph the original problem, and answer the first three questions, *then they can do calculus!*

We used the graphs of the different function in order to investigate the released AP problems for middle school and high school students in classes well before Calculus.

Two runners – AP test question



Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = (24t)/(2t + 3)$.

- Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.
- Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.
- Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

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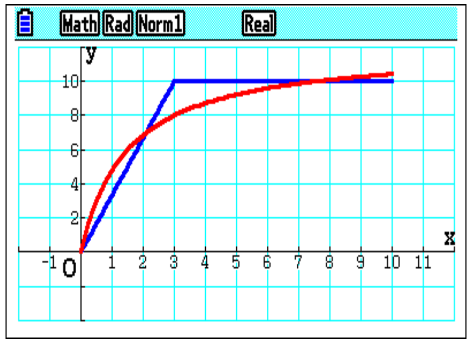
Will you have your students graph the two modeling functions or will they do the problem as a class investigation with the functions graphed?

Graphs for both runners:

Find the total distance run by Runner A and the total distance run by Runner B over the time interval 0 ≤ t ≤ 10 seconds. Indicate units of measure.

What is your answer?



Before we find the answer to the AP test questions, let’s look a few others:

What is happening when the graphs intersect?

Who is ahead when the graphs intersect the first time? The second time? After 10 seconds?

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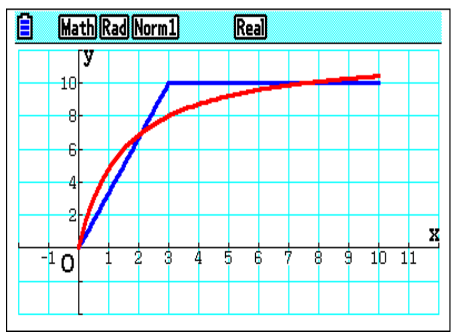
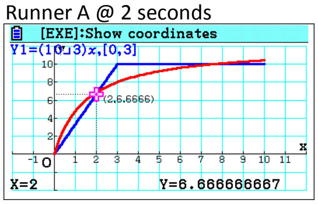
Graphs for both runners:



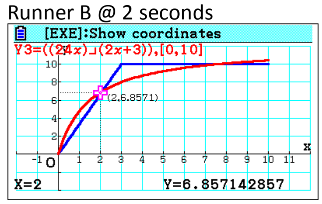
Find the velocity of Runner A and the velocity of Runner B at time t = 2 seconds. Indicate units of measure.

What is your answer?

Runner A @ 2 seconds



Runner B @ 2 seconds



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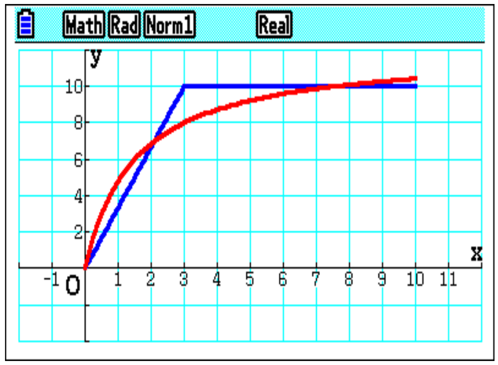
Graphs for both runners:

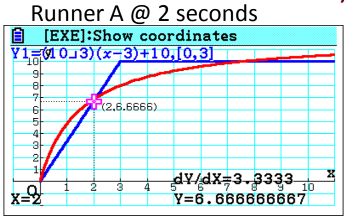


Find the acceleration of Runner A and the v acceleration of Runner B at time t = 2 seconds. Indicate units of measure.

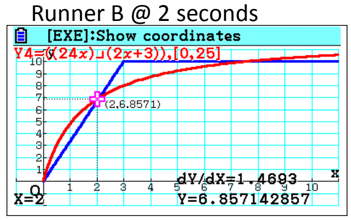
What is your answer?

(What about > 2 seconds?)

Runner A @ 2 seconds



Runner B @ 2 seconds



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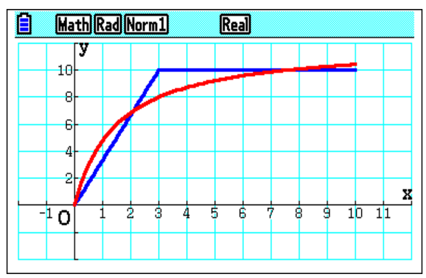
How far has each runner run after 4 seconds?

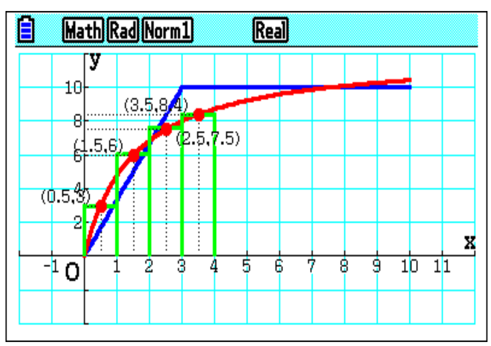
Runner A (blue segments) ?



To find the distance Runner B (red curve) has run, we are going to use Riemann Sums using rectangles with the center of the top of each rectangle on the red curve.

How does this help find the Distance for Runner B?

What is the area of the four rectangles?



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3rd Question:

How far has each runner gone after 10 seconds? (Ans.: A has run 85 meters and B has run 83.64 meters. To find Runner B's distance, take the area of the four rectangles we used in the question about what happens in the first four seconds, then trace the blue curve to see the Y-values at 4.5, 5.5, 6.5, 7.5, 8.5, and 9.5 seconds. Add these values to the answer how far B runs in four seconds to see how far B ran in 10 seconds

Additional questions:

Who wins a 100 meter race? How long does it take for the winner? (Ans.: Runner A wins a 100 meter race in 11.5 seconds. To find out har far B runs in 11.5 seconds, add the Y value of B's curve at 10.5 seconds and at 11.25 seconds (to approximate the distance B runs in the half second from 11 to 11.5. B runs 99.43 meters in 11.5 seconds.)

Who wins a 200 meter race? By how much? How long does it take for each runner to run 200 meters? (Ans.: Runner B wins a 200 meter race by nearly 9 meters in 20.682 seconds. To find this by Riemann Sums, the Excel spreadsheet in this handout lists the Y-values of B's curve at 11.5, 12.5,...19.5 which I added to the 83.64 meters B had run in 10 seconds. This shows that B ran 192.378 meters in 20 seconds. By trial and error with rectangles, I fould that in another 0.682 seconds (using 20.341 as the X value for the center of the last rectangle), B runs 7.622 meters, for a total of 200 meters in 20.682 seconds. The Power Point used 2 second intervals and found B took about 20.6132 seconds to run 200. It takes Runner A 21.5 seconds to run 200.0045 meters – 4.5 mm past the 200 meters. Finding the integrals using the emulator, it took B 20.7094 seconds to run 200.00189, or 1.89 mm past the finish line. The estimate of 20.6132 seconds is a percent error of 0.46+%, while the estimate of 20.682 is only a 0.13+% percent error. Runner A has run 191.132 meters in 20.6132 seconds and 191.82 meters in 20.682 seconds, so B wins by over 8 meters in a 200 meter race.

2009 AB CALCULUS #2

- The rate at which people enter an auditorium for a rock concert is modeled by $R(t) = 1380t^2 – 675t^3$ for $0 \le t \le 2$ hours; R(t) is measured in people per hour. No one is in the auditorium at time t = 0, when the doors open. The doors close and the concert begins at time t = 2.

  (a) How many people are in the auditorium when the concert begins.

  (b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

  (c) & (d) involve using a given derivative function to find the time people wait in their seats.

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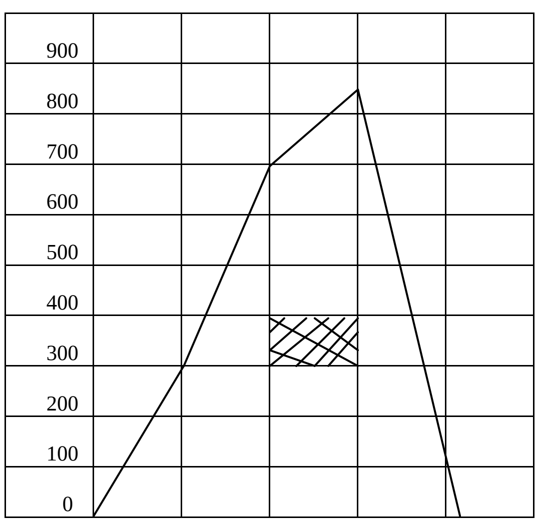
Middle School adaptations for the Auditorium Entry 2009 #2 AP Calculus Problem

This graph shows the number of people entering an auditorium for a rock concert. The vertical axis unit is the number of people entering per hour and the horizontal axis marks the time that has passed. We are showing what happens for each half hour. There is no one in the auditorium 2 hours before the concert starts, and the doors are closed when the concert starts at time = 2 hours.

Number of people entering the auditorium per hour



0 0.5 1.0 1.5 2.0 2.5

Hours (the concert begins at time = 2)

Questions and answers:

1. How many people do you think attended the concert?
    a. Whatever you write is fine with me, it's probably a guess

2. What is happening at 45 minutes?
    a. People are entering the auditorium at approximately 500 people per hour.

3. What geometric concept does the shaded part of the graph represent?
    a. The area of the rectangle that is shaded.

4. In the context of the problem and using the correct units, how would you interpret the meaning of the shaded region of the graph?
    a. The value of the area is (½ hour)*(100 people per hour) = 50 people. This is the number of people who entered the auditorium for that portion of the graph.

5. What does the area between the graph and the horizontal axis represent?
    a. The total area between the graph and the axis equals the total number of people who entered the concert, if you include the people who entered after 2.0.

6. Calculate the "areas" for the horizontal intervals

   People will get different answers if they estimate some values differently than I did:

    a. $0 \leq t \leq 0.5$; ½ (0.5)(280) = 70 people
    b. $0.5 \leq t \leq 1.0$; ½ (0.5)(280 + 700) = 245 people
    c. $1.0 \leq t \leq 1.5$; ½ (0.5)(700 + 840) = 385 people
    d. $1.5 \leq t \leq 2.0$. ½ (0.5)(840 + 110) = 238 people (round up 237.5)

   What does the sum of all of these values equal? What does this value represent?

   The sum equals 938 people, which is how many people entered the auditorium before the concert started.

7. What does the area after 2.0 tell you? What is its value?
    a. The area after 2.0 tells you how many people arrived late. 6 people arrived late. I rounded 5.5 up to 6. I estimated the time after 2.0 to be $\frac{1}{5}$th of the half hour from 2.0 to 2.5, which makes it $\frac{1}{10}$th of an hour. Number of people = $\frac{1}{2}\left(\frac{1}{10}\right)(110) = 5.5$.

8. Where is the maximum value of the graph? A student says this is when the auditorium is the most full. Is the student correct? Explain your reasoning.

a. The maximum value of the graph occurs at 1.5 hours, when the Y value is approximately 840 people per hour. This is not when the auditorium is most full, since 629 people entered after 1.5 on the graph.

9. When is the auditorium the most full? Justify your answer.
   a. The auditorium was most full when the graph gets back to the X-axis after the 2.0 mark, approximately 2 hours and 6 minutes. That is when no one else enters the auditorium.

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10. The total wait time for the wait time for all of the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The total wait time for all those who enter in the first hour is 398 hours. The total wait time for all those who enter during the second hour is 488 hours. What is the average wait time in minutes for the people in each of these two groups?
    a. (398 hr/315 people) = 1.26349+ hours per person = 1 hour15 minutes and 48.6 seconds
    b. (488 hr/623 people) = 0.7833+ hours per person = 46.998+ minutes= 46 min 59.9 secs.

11. Is it reasonable to add the two values determined in Question 10 together and divide by two in order to determine the average wait time for all of the people who enter the auditorium after the doors open? Why or why not?
    a. Since the number of people in the two parts of #10 are not the same, merely finding the average of the two averages would not be how to find the average wait time for everyone going to the auditorium.

12. On average, how long does a person who arrived before the start of the concert wait for the concert to begin?
    a. Average wait time = the total number of hours everyone waited divided by the total number of people who had to wait. I found this by finding (398 + 488)/1007 = 0.9445+ hours, which is 56 minutes and 40.4 seconds.

^^^^^^^^^^^^^^^ Additional questions you might consider ^^^^^^^^^^^^^^^^^^^^

13. What does the rate of change of the line segments tell you about the situation?

a. The rate of change tells you how fast the rate at which people are entering the auditorium is changing.

14. When would be the worst crowding at the doors to enter the auditorium?
    a. The worst time to enter the auditorium would be when the most people are trying to get in, which occurs at 1.5 hours.

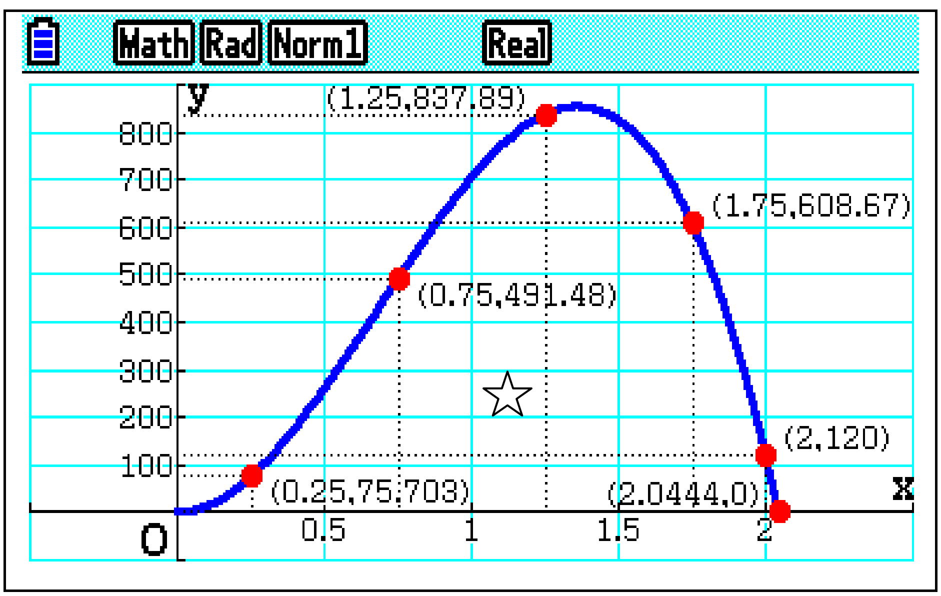
15. At 1 hour, is the number of people entering the auditorium increasing or decreasing?
    a. The number of people entering the auditorium at 1 hour is increasing because the segments before and after 1 hour both have a positive slope.

16. During what time period is the rate of the number of people entering the auditorium changing the fastest?
    a. The rate is changing the fastest after 1.5 hours, since the graph is steepest there.

Smooth curve adaptations for the Auditorium Entry 2009 #2 AP Calculus Problem

This graph shows the number of people entering an auditorium for a rock concert. The vertical axis unit is the number of people entering per hour and the horizontal axis marks the time that has passed. We are showing what happens for each half hour. There is no one in the auditorium 2 hours before the concert starts, and the doors are closed when the concert starts at time = 2 hours.

Number of people entering the auditorium per hour



Questions:

1. How many people do you think attended the concert?
   a. Whatever you write is fine with me, it's probably a guess.

2. What is happening at one hour (MS asks fat 45 minutes, since that number is on the graph above, I changed the time to one hour.)?
   a. People are entering the auditorium at approximately 700 people per hour.

3. What geometric concept does the starred rectangular of the graph represent?
   a. The area of the rectangle that is shaded.

4. In the context of the problem and using the correct units, how would you interpret the meaning of the starred rectangular region of the graph?
   a. The value of the area is (½ hour)*(100 people per hour) = 50 people. This is the number of people who entered the auditorium for that portion of the graph.

5. What does the area between the graph and the horizontal axis represent?
   a. The total area between the graph and the axis equals the total number of people who entered the concert, if you include the people who entered after 2.0.

6. Calculate the "areas" for the horizontal intervals, using Riemann Sums. What does the sum of all of these values equal? What does this value represent?
   a. $0 \leq t \leq 0.5$; (0.5)(76) = 38 people
   b. $0.5 \leq t \leq 1.0$; (0.5)(492) = 246 people
   c. $1.0 \leq t \leq 1.5$; (0.5)(838) = 419 people
   d. $1.5 \leq t \leq 2.0$. (0.5)(608) = 304 people

   What does the sum of all of these values equal? What does this value represent?
   The sum equals 1007 people, which is how many people entered the auditorium before the concert started.

7. What does the area after 2.0 tell you? What is its value?

a. The area after 2.0 tells you how many people arrived late. 6 people arrived late. I rounded 5.5 up to 6. I estimated the time after 2.0 to be $\frac{1}{5}$th of the half hour from 2.0 to 2.5, which makes it $\frac{1}{10}$th of an hour. Number of people = $\frac{1}{2}(\frac{1}{10})(120) = 6$.

8. Where is the maximum value of the graph? A student says this is when the auditorium is the most full. Is the student correct? Explain your reasoning.

   a. The maximum value of the graph occurs at 1.3629 hours, when the Y value is approximately 855 people per hour. This is not when the auditorium is most full, since around 310 people entered after 1.36 on the graph [the 310 who entered after 1.5 hours plus 120 people (0.14 × 855) who entered between 1.36 and 1.5 hours.]

9. When is the auditorium the most full? Justify your answer.

   a. The auditorium was most full when the graph gets back to the X-axis after the 2.0 mark, approximately 2 hours and 6 minutes. That is when no one else enters the auditorium.

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10. The total wait time for the wait time for all of the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The total wait time for all those who enter in the first hour is 398 hours. The total wait time for all those who enter during the second hour is 488 hours. What is the average wait time in minutes for the people in each of these two groups?

    a. (398 hr/284 people) = 1.4014+ hours per person = 1 hour 24 minutes and5.07 seconds
    b. (488 hr/623 people) = 0.674965+ hours per person = 40.4979+ minutes = 40 min 29.88 secs.

11. Is it reasonable to add the two values determined in Question 10 together and divide by two in order to determine the average wait time for all of the people who enter the auditorium after the doors open? Why or why not?

a. Since the number of people in the two parts of #10 are not the same, merely finding the average of the two averages would not be how to find the average wait time for everyone going to the auditorium.

12. On average, how long does a person who arrived before the start of the concert wait for the concert to begin?

    a. Average wait time = the total number of hours everyone waited divided by the total number of people who had to wait. I found this by finding (398 + 488)/1007 = 0.8798+ hours, which is 52 minutes and 47.4 seconds.

^^^^^^^^^^^^^^^^ Additional questions you might consider ^^^^^^^^^^^^^^^^^^^^^^

13. What does the rate of change of the line segments tell you about the situation?

    a. The rate of change tells you how fast the rate at which people are entering the auditorium is changing.

14. When would be the worst crowding at the doors to enter the auditorium?

    a. The worst time to enter the auditorium would be when the most people are trying to get in, which occurs at 1.36+ hours.

15. At 1 hour, is the number of people entering the auditorium increasing or decreasing?

    a. The number of people entering the auditorium at 1 hour is increasing because the curve has a positive slope.

16. During what time period is the rate of the number of people entering the auditorium changing the fastest?

    a. The rate is changing the fastest after just before the graph hits the axis after 2 hours, since the graph is steepest there.

These last questions can be investigated for the smooth curve version, but not for the line segment version of the problem:

17. Is the rate of change in how quickly the number of people are entering the auditorium ever 0? If so, when?

a. The rate of change equals 0 at the maximum point of the graph, which is 1.3629+ hours after the doors open, which is 1 hour 21 minute 46.44 seconds after the doors open. A more exact answer is 2760/2025 hours, which is when $R' = 2760t - 2025t^2 = 0$.

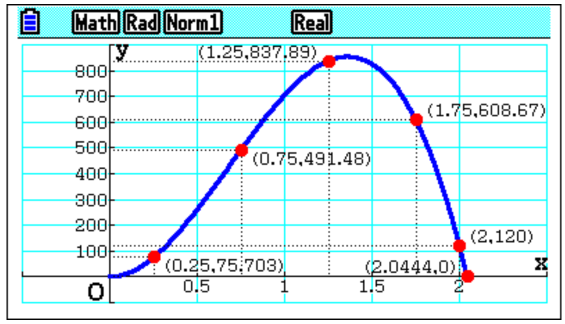
18. The rate of change in the number of people entering the auditorium is positive from the start of the graph until 1.36+ hours. At the start of that time range, the rate of change is also increasing, while at some point, the increase must start decreasing, while remaining positive, since the rate of change goes to zero at the maximum of the graph. When does the change from an increasing rate of change to a decreasing rate of change occur?
    a. Without doing the math (calculus), it looks like the tangent line slope is increasing from time 0 to somewhere after 30 minutes (0.5 hours) and before around 45 minutes (0.75 hours). By using a PRIZM (or Nspire) to see values of the slope of the tangent line, we can see that the slope keeps increasing until 0.666… hours. After the next point on the graph, 0.69047+ hours, the slope of the tangent line is still positive, but it is getting smaller as time progresses.
    If you do the calculus, you can find the exact time when the slope of the tangent line changes from increasing to decreasing. This happens when the second derivative equals zero. The first derivative is $R' = 2760t - 2025t^2$, so the second derivative is $R'' = 2760 - 4050t$. $R' = 0$ at 2760/4050, which is at 0.68148+ hours.
    NOTE: You can find the first or second derivatives of polynomial functions in order to find where the rate of change is zero (when the first derivative equals zero) or when the rate of change goes from increasing to decreasing or vice versa (when the second derivative equals zero). In more complicated functions, you could ask a Calculus teacher when the two derivatives equal zero, if they ever do.

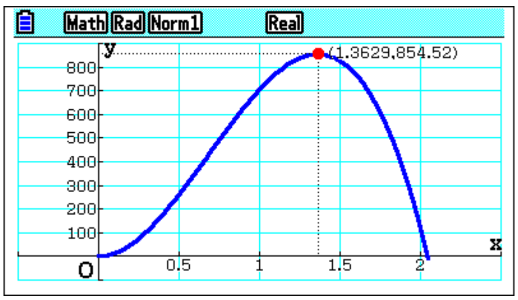
19. What is the Calculus concept that #19 previews?
    a. Number 19 previews the concept of the inflection point.

The slide on this page is used for answering some of the questions above.

2009 AB CALCULUS #2

- (a) How many people are in the auditorium when the concert begins?





(b) When are the most people entering the concert? Justify your answer.

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Differential Calculus problems

Distance vs. Time story problem:

John is standing at the classroom door. He decides to move the 4 yards to the water fountain. It takes him 3 seconds to do that. For the next 8 seconds, he drinks and stands at the fountain. He returns to the door in 2 more seconds. Draw a graph of the situation.

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Could your students draw the graph of this problem?

Stop and think of the questions your text would ask about the graph....

Questions for MS and/or early HS:

1. What are the units of the rate of change of the three segments?
2. What does this tell you about the situation?
3. When is John moving the fastest? Justify your answer without any calculations.
4. What is his speed at 1 second? At 4 seconds? At 11.5 seconds?
5. Is the average speed from 0 to 10 seconds greater or less than the average speed from 0 to 4 seconds? Use the graph to justify your answer. How else could you justify your answer without using value of the rate of change?
6. What is the average speed from 0 to 10 seconds? From 0 to 4 seconds?
7. Is there a time when John's instantaneous speed is equal to his average speed from 0 to 10 seconds? If so, when does that occur?
8. What is his speed at exactly 2 seconds? Justify your answer
9. What Calculus topics did we preview?

Solutions for the Distance vs. Time problems:

1. The units of the rate of change are yards per second.
2. Rate of change tells you the velocity, speed if you don't care about the direction of the movement.
3. He was moving fastest when he returned to the door, since the line is the steepest during the last second.
4. His velocity {speed, if all answers are positive} was $\frac{4}{3}$ yds./sec. (4 ft/sec.)@ 1 sec.; 0 yds./sec. (0 ft./sec.) @ 4 sec.; and -2 yds./sec. (-6 ft./sec.) at 11.5 seconds.
5. The average speed from 0 to 10 seconds is less than the average from 0 to 4 seconds since the slope of the line to 10 seconds is less steep.
6. The average speed from the start to 10 seconds is $\frac{4}{10}$ yd/sec. ($\frac{2}{5}$) yd/sec. or $\frac{6}{5}$ ft./sec.; from 0 to 4, seconds, average speed is 1 yd./sec. ($\frac{4}{4}$) or 3 ft./sec..
7. There is no place on the graph where the slope of the graph equals the slope of the segment from 0 to 10 seconds.
8. The speed from 0 to anything short of 3 seconds is 4ft./sec.. The speed any tiny amount after 2 seconds is 0 ft./sec.. You could justify *any* value from 0 to 4 ft./sec. for the speed as long as you had a good justification for it. You could say the speed was 2 ft./sec, since it is the average of the 2 values. Or, you could say the model is inaccurate, that you *cannot* go from 4 ft./sec. to a stop instantaneously, then say that the person was slowing down before 2 seconds, but did not come to a complete stop until after 2 seconds, so he was going 3 feet per second at exactly 2 seconds.
9. This previews: average rate of change; derivatives, differentiable and non-differentiable functions; graph analysis; limits; and delta epsilon proofs; and it sets up the Mean Value Theorem preview.

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NOTE: Do "enough" segment graphs for both differential and integral calculus problems with your students before you start investigating smooth curves.

Once you start working with curves, decide if you want to show your students why the tangent line is how you can find the rate of change of the curve at a given point. If you want to justify this to your students, have them find the slope of a line intersecting the curve in two points very near the given point (with the given point "between" the two points). You can select pairs of points, getting progressively closer to the point you want to explore (perhaps 0.1; 0.01; and 0.001 greater and less than the given point. Then, show that these slopes are very close if not equal to the slope at the point of tangency using dy/dx. Or, just tell the students that we will find the rate of change of the curve by using the slope of the tangent line.

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The slide below points out that you might want to have students explore a ball being thrown straight up in the air before exploring the human cannon ball problem, which we investigated in depth. For details about the ball toss problem, see my handout from my "Previewing Calculus Concepts in Grades 6 thru 12" on the AATM conference website.

Ball toss problem:

A student throws a ball vertically into the air and lets it hit the ground. When she releases the ball, it is 1.5 meters above the ground and it hits the ground a little more than 2 seconds after she releases it. The ball goes to a maximum height of almost 7 meters.

Sketch a graph of the ball's height vs. time

Traditional Text Questions:

1. How high is the ball at ½ second? At 1½ seconds? At 2 seconds? (Ans.: 5.2 m @ 0.5 sec.; 5.625 m (it is falling) @ 1.5 sec.; and 2.1 m @ 2 sec.)
2. When is the ball 4 meters above ground? 1 meter above ground? (Ans.: 4 meters above ground @ 0.2876738102 &1.77355068 secs.
   1 Meter above ground @ 2.109594365 &, possibly, some other time before she tosses the ball!!)
3. When does the ball get to its maximum height? (Ans.: 1.030612203 seconds.)
4. How high does it go up? (Ans.: 6.704591837 meters)
5. How long is it before the ball hits the ground? (Ans.: 2.200348988 sec.)

These can be answered using the graph, using a graphing calculator, or algebraically.

Here are some very much *non-traditional* questions:

6. What are the units of the rate of change of the graph? What can you use to represent the rate of change graphically? (Ans.: The units of the rate of change are vertical meters per second.)
7. What does the rate of change of the graph tell us about the problem situation? (Ans.: The rate tells how quickly the ball is rising or falling.)
8. How fast is the ball rising at ½ second? (Ans.: 5.2 m/sec @ 0.5 sec.)
9. How quickly is the ball moving at 1 second? at 2 seconds? (Ans.: 0,3 m/sec. @ 1 second; -9.5 m/sec. @ 2 seconds, it is *falling*.)
10. When is the ball moving the fastest? Justify your answer without finding its velocity at that time. (Ans.: It is moving the fastest just

before it hits the ground. Since it falls to the ground, it picks up speed after it falls through 1.5 meters which was the height when it was released.)

11. How fast was the ball moving when she threw it? (Ans.: 10.1 m/sec.)
12. What is its fastest velocity? (Ans.: At 2.2003489 seconds, the abll is falling at -11.46 m/sec. and it is 0.0000010095171 meters above the ground.)
13. When will the ball be falling at the same speed as it was rising when it left her hand? Justify your answer with the calculator. (Ans.: It will be falling at the same speed as when it left her hand when it is at 1.5 meters above ground on the way down. This happens at 2.06122449 seconds.)
14. What is the average rate of change from the time the ball is released until it hits the ground? Explain how you got the average rate of change. (Ans.: The average rate of change is the net drop divided by the time it took to hot the ground. This is 1.5 m/2.20034898 sec., which equals -0.68171+ m/sec.)
15. How can you represent the average rate of change graphically? (Ans.: Draw a segment from the starting point at (0, 1.5) to the ending point at (2.20034898, 0).)
16. Is there a time during the flight of the ball that its instantaneous velocity equals its average velocity? If so, when does this occur? (Ans.: At 1.10023 seconds, the slope of a line tangent to the graph equals -0.682 m/sec., which is the average slope rounded to the nearest thousandth. In the workshop, we drew the tangent line to the graph and moved the point of tangency several times. Other answers are acceptable. At 1.1 seconds, the slope of the tangent line is correct to the nearest hundredth at -0.68m/sec.. With a PRIZM or Nspire, you can draw the tangent line and move it until the dy/dx of the line equals -0.68 or -0.682. With the TI 83/84 family, you need to move the point in trace mode, then calculate dy/dx. This needs to be done for each point, so it becomes a bit cumbersome.)
17. What Calculus concept does this illustrate? (This is a question for teachers, not really for students) (Ans.: This illustrates the Mean Value Theorem of Calculus, which states that on a smooth curve, there is at least one point on the curve where the instantaneous rate of the (the derivative) equals the average rate of change between two given points..)

Next we looked at the line of best fit comparing the instantaneous rate of change versus to time. We collected data by finding the derivative at a number of points, recording those values, then creating a scatter plot. For the PRIZM, we graphed the scatter plot, stored the graph as a picture, then used the picture as background for the graph of Y=Mx + B. we used the "Modify" feature to find the value of M and B. For the TI 84 or 83 family, we created the scatter plot, then graphed it and the line Y=Ax + B after starting the "TRANSFRM" app on the calculator.

18. What are the slope and Y-intercept of the line of best fit, and what information do they tell us about the situation? (Ans.: The Y-intercept is 10.1 and the slope is -9.8 m/sec^2.. The Y-intercept tells us the upward velocity of the ball when it was released. The slope tells us the change in velocity pre second, which is the acceleration, shich is why it is -9.8 m/sec^2..)
19. Write the equation for the height of the ball toss using what you know about the situation and the general equation for something thrown vertically. (Ans.: The equation is $Y = -4.9X^2 + 10.X + 1.5$. Depending on the level of your students, you could either give them the quation, or at least show it to them at some point during the problem, or you could have them figure out the equation as we did.)

Note: If you wanted to do the same problem a second time, you could tell your class that the ball is thrown up on one of the planets or moons in our solar system, and they need to figure out which one.

3, 2, 1, ... Fire .... Cannon ball problem questions.

Traditional questions:

What is the maximum height? When does it occur? (Ans.: The maximum height is 78.6608+ feet, or nearly 78 feet and 8 inches -- 7.9296+inches. This occurs 1.5544+ seconds after the cannon baller leaves the cannon.)

How high is the person after 2 seconds? (Ans.: At 2 seconds, the person is 75.48 feet or 75 feet, 5¾ inches (5.76 inches).)

When is the cannon baller at a height of 50 feet? (Ans.: The cannon baller is 50 feet above the ground twice, once at 0.216+ seconds and at 2.8928+ seconds.)

How long until the person gets to the net? (Ans.: The person gets to the net after 3.625990+ seconds.)

Non-traditional questions:

What are the units of the rate of change of the graph and what does this tell us about the situation? (Ans.: The units of the rate of change are vertical feet per second, which tells us how fast the cannon baller is rising or falling.)

What is the initial vertical velocity? Is the initial velocity a reasonable velocity? (a research questions for students). (Ans.: The initial velocity is 49.742 ft/sec, which is 33.915 mph. Student could research human cannon ball shots to see if this is reasonable. From videos I have seen, it seems like the amount of time in flight is reasonable.)

What is not finished with the sketch (or my graph)? (Ans.: I do not show the net bouncing a little once the person hits the net. Also, the person never jumps out of the net and goes to the ground.)

When is the person moving the fastest? (Ans.: The person is moving the fastest just before hitting the net, nearly 3.626 seconds after launch.)

What is the biggest velocity? Is that a reasonable velocity? (a research questions for students) (Ans.: The greatest velocity is –66.28 ft/sec, or –45.2 mph.)
When is the person going up at a speed of approximately 20 feet per second? Down at approximately 20 feet per second? (Ans.: At 0.62 seconds, the person is ggoing up at 20.002 ft/sec. After 2.18 seconds, they are falling at –20.01 ft/sec..)

Can you find a place where the downward vertical velocity is equal to the upward vertical velocity when the person exited the cannon? When does that happen? (Ans.: The downward velocity is the opposite of the upward velocity at the start when the person is falling through the 40 foot height. This occurs at 3.108890897 seconds.)

What is the average speed from start to the net? (Ans.: The average velocity is net distance divided by the time it takes. This is –30/3.62599, which equals –8.2736+ ft/sec.

Does the person ever go exactly that vertical velocity? (Ans.: Yes, at 1.813008 seconds, dy/dx of the graph equals –8.274 ft/sec.)

Find some values for the rate of change and create a scatter diagram of them.
What kind of relationship seems to be true? (Ans.: It looks linear.)
Find the line of best fit using Modify on PRIZM (or Transfrm App on TI).
(Ans.: By using Modify on a PRIZM or the TRANSFRM app on the TI-83/84 calculator, you can find the line of best fit is Y = –32X +49.7 (or 49.742)
What do the slope and Y-intercept of the line of best fit tell you about the situation? (Ans.: The slope of the line is its rate of change. Since the points tell us the vertical velocity at certain times, the rate of change units are ft/sec^2, which is the acceleration of the cannon baller. While you may have been a little surprised when we found that the slope was –32, we then said that this is the force of gravity when you use feet. The Y-intercept of 49.742 was the initial *vertical* velocity of the cannon baller. This value is the initial velocity coming out of the cannon times the angle of elevation of the cannon (60×sin56°).

Can your Algebra 1 students find the graph of the derivative of a parabolic graph? We just did!!

Ferris wheel problem: We only covered a few of these questions

This summer you are planning to go to the County Fair. You know it has a new Ferris wheel, and you plan to take several rides on it.

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The Ferris wheel has a diameter of 40 feet and it takes 30 seconds to make one revolution. When you sit down on the ride, you are 4 feet above the ground.

Sketch a graph of your height vs. time.

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Ferris wheel questions and answers for MS and HS:

1. What does the Y-intercept tell you about the ride? Start ht.
2. When do you get to the top of the Ferris wheel? 15 & 45 sec
3. When are you going up and when are you going down?
   a. Up 0 to 15 & 30 to 45 secs
   b. Down 15 to 30 secs and 45 to 60 secs
4. What are the units of the rate of change of the graph? Ft/sec
5. Where are you on the ride when the vertical rate of change the largest? (at what times during the ride does this occur?)
   a. You would be even with the center hub of the ride (9 and 3 o'clock) at 7.5, 22.5, 37.5, & 52.5 seconds
6. Where are you on the ride when the vertical rate of change is equal to zero? (at what time during the ride does this occur?)
   a. Top or bottom of the ride, at 0; 15; 30; 45; & 60 seconds
7. When is the vertical rate of change increasing?
   a. Same as 3a
8. Are there times when the vertical rate of change is positive, but the changes are getting smaller? Yes, 7.5 to 15, 37.5 to 45 secs (larger?) Yes (positive and getting larger) , from 0 to 7.5 and 30 to 37.5 secs
9. What is the average vertical speed from when you are at the "bottom" of the ride until you are at the "top" of the ride?
   a. 40/15=8/3 ft/sec… the vertical distance is 40 and it takes 15 seconds.
10. What was your height above ground 5 seconds before the ride?
    a. Probably 0 feet, you were most likely on the ground 5 seconds before you got on the ride… students need to think about this one, and be willing to make a conjecture. I accept other heights if they explain their answer well. :-}

11. How many people got on the ride after you did? (I am assuming one thing about the graph most of you have drawn.)
    a. None, if someone had gotten on the ride after you, it would have stopped, assuming it is not one of the wheels that keeps going around and never stopping like at Navy Pier in Chicago.
12. Does your sketch show that you got off the ride or that you are still on it? (What is the end behavior of the graph of your ride?)
    a. No you have not gotten off, you would need to go back to 0 for height above ground… If you were the first one off, it would just show your height going to 0… if two people got off b4 you, the sinusoidal would continue, but would have two flat spots for when the ride stopped to let people off ;-}

These next questions require a graphing calculator.

13. When are you 26 feet above the ground?
    a. With TI family calculators: Graph y=26 and find the intersection of y=26 and the sine graph.
    b. With Casio PRIZM: Press G-Solv (F5); then F6, to move to the next screen; then select X-CAL (F2); select the sine graph; enter 26 in the prompt "Y:" then press EXE; Press EXE if you want to label the point on the screen; Press the right arrow to move to the next value where Y=26 and repeat the last two steps.
14. When are you at the same height as the center of the Ferris wheel?
    a. 7.5, 22.5, 37.5, & 52.5 secs
15. How high are you off the ground 3 seconds after your ride began?
    a. 7.81966+ feet (trace to 3)
16. What is your vertical speed 3 seconds after you got on the ride?
    a. 2.4621 ft/sec… its dy/dx of the graph!
17. When is your vertical velocity –3 ft/sec?
    a. By trial and error and symmetry, find dy/dx = -3 at 18.813; but also at 26.187; 48.813; and 56.187 seconds
18. What is the value of the greatest vertical velocity? Where does this occur?
    a. ±4.1887 ft/sec, whenever you are even with the center (if you include negative speeds). Times are same as in 13a
19. Find the time where your increasing vertical velocity starts to decrease. (reword the question if you use it)
    a. @ 7.5 and 37.5 seconds… when you are above the center headed to the top of the wheel, velocity decreases, but is still positive
20. What is your average vertical velocity between:

a. 5 and 20 seconds?
   i. 1.333.. ft/sec – the slope of the segment connecting those two points { (34 – 14)/(20 – 5) = 20/15 = 4/3=1.333…}
b. Find where the instantaneous velocity (dy/dx) on the graph equals the average velocity from 5 to 20 seconds.
   i. At 13.4533 seconds, dy/dx = 1.3333. This also occurs at 1.5467 seconds (Can you find the other two times when dy/dx = 1.3333? What about all four places where you were on the ride when dy/dx = -1.3333 ? Two hints: use the symmetry of the graph….and the fact that the answer to question 13b occurs 1.5467 seconds before the first maximum height on the height vs. time graph! I am not giving you the answer here… if you need help finding it, ask your HS 's Trig or Pre-Calc teacher)
   ii. This question and answer perview the Mean Value theorem in calculus. In the problem we did earlier of Joe going to the water fountain, you **cannot** find a place on the distance vs. time graph where the instantaneous velocity equals the average velocity between 1 and 7 seconds, since Joe's distance graph is not a smooth curve like the Ferris Wheel graph is. The problem you had finding a velocity at exactly 2 seconds (since it was a "corner" of the trapezoid) precludes using the Mean value Theorem. Joe's graph is non-differentiable, while the Ferris wheel graph is differentiable.
c. What is your average vertical velocity between 10 and 20 seconds?
   i. 0 ft/sec, your height at 10 and 20 seconds are the same, so the average vertical change is 0/2 = 0 ft/sec

These next questions would be for Algebra 2/ Trigonometry classes or Pre-Calculus classes, after students had learned about trigonometry function graphs and transformations of the graphs.

21. Find the equation of your ride on the Ferris wheel from the time you got on the last open car for two complete revolutions. Graph this equation.
    a. The equation of your height vs. time for this Ferris Wheel is: $Y = 20\sin\left(\frac{\pi}{15}(x - 7.5)\right) + 24$

b. Graph Y = 24 if you want to have the center line of the Ferris wheel graphed. You may want to make this a thin or dotted line.

22. Sketch the tangent line using your PRIZM and find where the slope of the tangent line is positive, negative, and zero.
23. Are there places on the graph where your vertical speed is positive (or negative) and the change in your speed is also positive?
24. What about places where the vertical speed is positive (or negative) and the change in your speed is negative?
25. Find enough values of your vertical velocity in order to find a regression equation for your velocity. Graph your vertical velocity on the same axes as the graph.
    a. The equation for the graph of the derivative of the original graph is: $y = 4.1887\sin(0.20943951x)$
26. Move the graph up so that the center line of the velocity graph is on the same Y= value as the center line of the height graph.
    a. Add 24 to the velocity graph from question 24 above.
27. Compare the two graphs:
    a. Periods
    b. Amplitudes
    c. Trig function
        i. Periods are the same and the amplitude of the velocity graph is smaller.
        ii. When you look at the original graph and the velocity graph as a pair, one is a cosine graph while the other is a sine graph. There is a phase shift to get from one to the other and there is a sign change to consider as well. This fits the Calculus facts that dy/dx of sine equals cosine, while dy/dx of cosine equals the opposite of the sine. Both of these can be previewed using the two graphs in Ferris wheel problem.

Here is a way to alter the question in order to have students analyze the situation in a different way, especially if your students have worked on the Ferris wheel idea before, or if you have a strong Honors or gifted class or once they get to a class where they have seen the Ferris wheel several times already:

28. Sketch a graph of your **horizontal distance** from the start of the ride.
    a. How does this change your graph and the questions?
        i. The graph will have both positive and negative values, since you are going back and forth past your starting

point. The shape of the graph will be the same as the vertical distance graph, but it will be shifted vertically and possibly horizontally, depending on your answer to question 27b. (Whether you start out going to positive values or negative values depends on your answer to question 27b). The questions now ask about the horizontal distance and all the answers are shifted in time.

b. You have a choice of how your graph is drawn, which did you choose?
    i. When the ride starts, did you decide to make the change in X positive since you are going forward (or because a friend not on the ride sees you moving to the right (+)). Or, did you make your graph start out going into negative numbers because the friend not on the ride sees you moving to the left?

Note, if you have students work this problem in Geometry, there are many other questions you could have them explore:

29. How far do you go on your ride?
    a. In two revolutions, you travel 251.3274123 feet .... (2* (40π)ft.). NOTE: if your students use 22/7 not they will get a slightly different answer for questions 28 and 29.
30. What is your **circular velocity** on the Ferris wheel?
    a. 4.18879 ft/sec (the circumference divided by 30 seconds)
31. If there is a bug halfway between you and the center of the Ferris wheel, is the bug's circular velocity half of yours? How far would the bug travel? What is the bug's circular velocity? Justify your answers.
    a. Your circular velocities would be the same, since you each complete one circle in the same 30 seconds.
    b. The bug's distance travelled and circular velocity would both be half of yours, half the answers for 28 & 29.

In Calculus or Pre-Calc, ask students to sketch the vertical velocity vs. time graph from the sketch of height vs. time, before they do the analysis or write the equation for height vs. time.

Research Projects: I made up the information about the size and speed of the Ferris wheel. You could assign students these research projects:

20. How fast is the speediest Ferris wheel in the world? Model a ride on it for two revolutions.
21. How large is the largest Ferris wheel in the world? Model a ride on it for two revolutions.
22. The World's Fair in Chicago had a very large Ferris wheel in which people stood in large cars. Model a ride on it for two revolutions.

What other questions or projects could you ask your students?

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Here are the Solids of Revolution slides:

Solids of Revolution

A third main idea in Calculus is finding the volumes of solids of revolution. These problems involve rotating the region between a curve and an axis or between tow graphs around a line.

As I did with derivatives, I will put a released AP Calculus question into a real world context.

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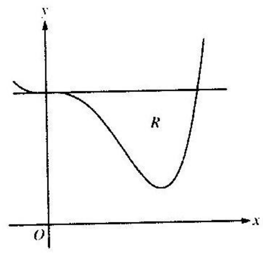
If we have time:

2014 AP® CALCULUS AB FREE-RESPONSE TEST

2014 AP® CALCULUS AB FREE-RESPONSE QUESTIONS



2. Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure above.

(a) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(b) Region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.

(c) The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

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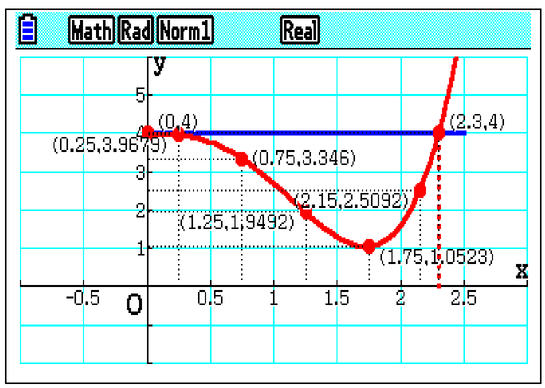
Adaptation of 2014 AP® CALCULUS AB FREE-RESPONSE #2



1. A cylindrical glass is made by rotating the area between the two graphs around the X-axis. What is the volume of the Inside of the glass (consider the volume of the curve rotated around the X-axis). NOTE: the units on each axis are inches.
2. How much water does the glass hold? (1 cu, in. = 0.016387L)
3. How much glass is needed to create the glass? (The bottom of the glass is the volume between X=2.3 and 2.5, from the X-axis to the line Y = 4.)

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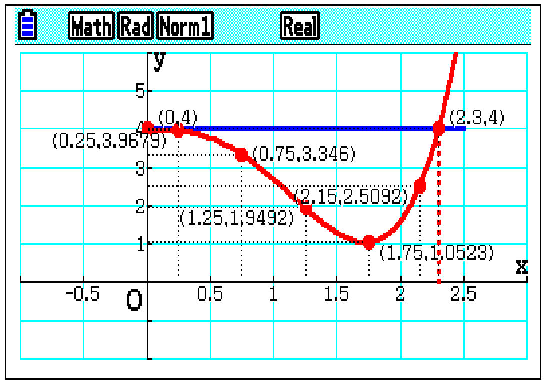
Adaptation of

2014 AP® CALCULUS AB FREE-RESPONSE #2

4 Region R (the area between the horizontal line and the curve) is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in R. Find the volume of the solid.



5. How would your calculations for these questions if the graph had been rotated around the line Y = -2 instead of the X-axis?

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2013 AP Calculus AB Free Response #2:

This problem (slides on the next couple pages) has one derivatve question and three integral questions. The derivative question is simple using a calculator to find dy/dx at 5, but the three integral problems are more involved than the integral problems we started with. The slide show how I adapted the calculus vocabulary for all four problems.

2013 AP Calc AB/BC #1

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, , where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday (t=0), the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at the rate of 100 tons per hour.

(a) Find G'(5). Using the correct units, interpret your answer in the context of the problem.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time t = 5 hours? Show the work that leads to your answer.

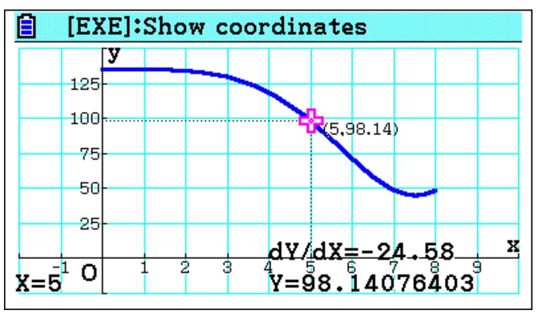
(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation? Justify your answer. 32

My adaptations:

(a) Find G'(5). Using the correct units, interpret your answer in the context of the problem. (How do we do this?)

- What is the rate of change of the graph at t=5? Use the correct units, interpret your answer in the context of the problem?

Y1=$90+45\cos\frac{x^2}{18}$, [0,8]



Ans: At t=5, the rate of change is -24.58, which means that the amount of unprocessed gravel arriving at the plant per hour

Note: In a Trig course, you could have students enter the function using radians.

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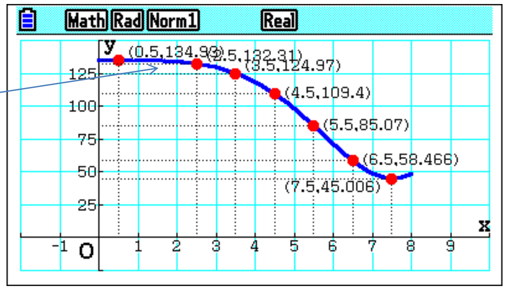
(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday. (How do we find the answer?)

- By tracing the graph to 0.5, 1.5, 2.5,..., 7.5 we can find an approximation to the area under the curve:

- (1.5, 134.6) is not shown



- These values (including 134.6 from t = 1.5) add to 824.8, which means 824.8 tons of unprocessed gravel arrived at the plant. (The integral from 0 to 8 equals 825.551 tons. Our answer is a mere 0.09097% below the Calculus answer!

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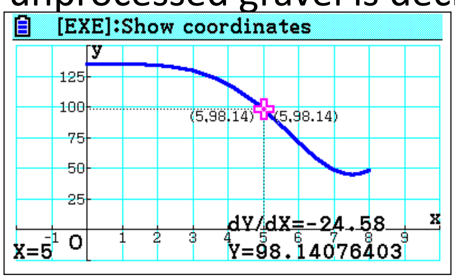
(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time t = 5 hours? Show the work that leads to your answer. (How do we find the ans.?)

- From the graph, G(5) = 98.140764. That is how many tons of unprocessed gravel are arriving at the plant at t = 5 hours. Since the plant is processing 100 tons per hour, the amount of unprocessed gravel is decreasing.



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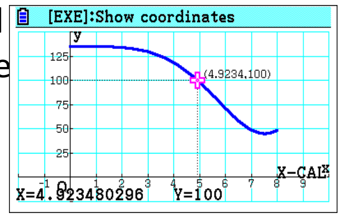
(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation? Justify your answer. (How do we find the max?)

- The plant processes 100 tons per hour. As long as the amount of unprocessed gravel being received is over 100 tons per hour, the amount of unprocessed gravel is increasing. If you find where the blue curve dips below 100, you will know the time when the maximum amount of unprocessed gravel is in the plant.



<How

- How do you the *amount* of unprocessed gravel?
- See next slide…

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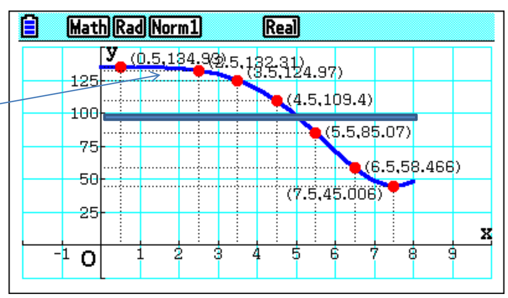
The amount of unprocessed gravel that has accumulated before 4.92348 hours is the area under the curve but above the line Y = 100.

- By using the traces from earlier, (at 0.5, 1.5, 2.5,…, 7.5) we can find an approximation to the area under the curve:

- (1.5, 134.6) is not shown



- We only need the areas to Y = 100, so subtract 100 from each value. Adding 500+34.995+34.649+32.314+24.975+29.4 yields 636.323 tons. (I did not use 29.403) since that goes to 5 and we should stop at 4.9234. Our answer is a mere 0.149% over the Calculus answer!

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Here is how we found the line of beat fit for the Ball Toss problem:

For Casio PRIZM:
We traced to several points on the height versus time graph with the Derivative feature On. We entered 8 different dy/dx values and the corresponding times into a table (on paper first, then in the PRIZM statistics menu) and made a scatter plot of the Derivatives (which tell us the vertical speed) versus time. After saving the scatter plot as a Picture, we used it as background for the graph of Y = MX + B. Using the MODIFY feature, we found the Y-intercept, which was the initial vertical velocity, then found the slope of the line of best fit. The slope of the line of best fit is the value of the force of gravity, $-9.8 m/sec^2$ in the metric system. Next, we graphed the height vs. time graph and the velocity vs. time graph on the same axes and noticed the relationship between the two graphs.

For TI-83/84 family calculators:
You need to "Calculate" dy/dx for each point by tracing to it then going into calculate (2nd/Trace) and wrote down the time and the value of dy/dx. Make a scatter plot and leave the Stat Plot on. Use the TRANSFRM app to figure out the line of best fit. NOTE: you can not use M for your equation in the app, you need to use Y = AX + B.

For the Ferris wheel, we used the Sinusoidal regression of the calculator to get the equation of the vertical velocity vs. time after finding enough values of dy/dx on the height vs. time graph. By graphing both the height and the dy/dx graph on the same axes, you can see the relationship between the two. Also, your students have just graphed the derivative function of a sine graph without using any Calculus other than the dy/dx values from the calculator.

The data for the 2 Runner and Human Cannon ball problems:

Human Cannonball

time (sec.)	height	dy/dt (vert. ft/sec)			
0	40	49.742	**=33.915 mph**		
0.25	51.4	41.742			
0.5	60.87	33.742			
0.75	68.3	25.742			
1	73.74	17.742			
1.25	77.178	9.742	**MAX height @**	**time**	**height**
1.5	78.61338	1.742		**1.554445539**	**78.66081044**
1.75	78.04	-6.257		**dy/dt= 0**	
2	75.48	-14.25			
2.25	70.92	-22.25		dy/dt=	
2.5	64.355	-30.25	Avg Vel 0 to3.625+	-8.2736004+	
2.75	55.79	-38.25	-32		
3	45.227	-46.25	@1.813007	-8.273	
3.25	32.66	-54.25	@1.813008	-8.274	
3.5	18.098	-62.25			
~~3.75~~	~~1.533~~	~~-70.25~~			
3.6	11.712	-65.45			
3.62	10.396	-66.09			
~~3.63~~	~~9.733~~				
3.625	10.0656	-66.25			
3.625990923	**10**	-66.28	**= - 45.190909…mph**		
3.625990923	X=121 feet				
0.9294375		20			
2.1794375		-20			

Two runners

time (sec.)	Runner A	total Runner A	rect area for B	total rec areas	Integral for B	W
0 to 1	1.666...	1.666...	3	3	2.805	
1 to 2	5	6.666...	6	9	8.7486	
2 to 3	8.333...	15	7.5	16.5	16.22498	
3 to 4	10	25	8.4	24.9	24.61291	
4 to 5	10	35	9	33.9	33.60593	
5 to 6	10	45	9.43	43.33	43.0301	
6 to 7	10	55	9.75	53.08	52.77718	
7 to 8	10	65	10	63.08	62.7751195	
8 to 9	10	75	10.2	73.28	72.9736172	
9 to 10	10	85	10.36	83.64	83.3361253	
10 to 11	10	95	10.5	94.14	93.8352563	
11 to 12	10	105	10.615	104.755	104.449957	
12 to 13	10	115	10.714	115.469	115.163696	
13 to 14	10	125	10.8	126.269	125.963251	
14 to 15	10	135	10.875	137.144		
15 to 16	10	145	10.941	148.085		
16 to 17	10	155	11	159.085		
17 to 18	10	165	11.053	170.138		
18 to 19	10	175	11.1	181.238		
19 to 20	10	185	11.14	192.378		
20 to 20.6132	{6.132}	191.132				
20 to 20.682	{6.82}	191.82	7.622	200	199.695	
21 seconds	10	195		at 20.71 sec-->	200.0086	
21.5	5	200				

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I hope these problems have piqued your interest. If you do start using the ideas I talked about, make sure the teachers in grades after you and Calculus teachers know you are doing this so they can continue to develop and preview these ideas.

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"As teachers your job is to perturb your students, but not so much that they shut down." Prof. Bob Horton, Clemson University.