NEW LONGITUDINAL AND INSTRUCTIONAL RESEARCH ON FRACTIONS

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Russell Gersten, Ph.D.
Director, Instructional Research Group
Professor Emeritus, University of Oregon

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CENTER FOR IMPROVING LEARNING OF FRACTIONS

1. **Strand 1 – Longitudinal Research: Students With Math Difficulties**: Improving At-Risk Learners' Understanding of Fractions (Nancy Jordan)

2. **Strand 2 –** The **Centrality of Fractions to Mathematics Learning** and the **Centrality of Magnitude Comparison** and the **Number Line** to Mathematics Proficiency (Bob Siegler)

3. **Strand 3 – Interventions for students at-risk for failure**: randomized controlled trials (Lynn Fuchs and Robin Schumacher)

4. **Strand 4 – Dissemination and Leadership**

(Note: slides adapted from those developed by Siegler, Jordan and Fuchs (and Schumacher)

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LONGITUDINAL RESEARCH

1. Highlight key findings in terms of:
   - ✓ The types fractions problems most students solve successfully
   - ✓ The types of fractions problems that are difficult for students
   - ✓ How students develop over time

Note: there are other facets of this research not covered today involving role of language and executive function and other aspects of cognitive psychology

Contemporary State State Standards and Common Core

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LONGITUDIINAL SAMPLE

Children placed into one of two groups:

- ✓ **Lower Math Achievement** (scored ≤ 35^{th} %ile in 3^{rd} grade). Includes at risk as well as very low performing. Mean 18.5%ile
- ✓ **Higher Math Achievement** (scored > 36^{th} %ile in 3^{rd} grade). Mean 69^{th} %ile

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PARTICIPANT CHARACTERISTICS

	Lower Math Achievement (n = 61)	Higher Math Achievement (n = 248)	Total Sample (n = 309)
% Female	**62.3%**	47.6%	50.5%
% ELL	**8.2%**	13.7%	12.6%
% Special Education	29.5%	6.5%	11.0%
% Low Income	77.0%	53.6%	58.3%
Mean (SD) math achievement score (%ile)	**18.5** (7.9)	**69.0** (20.1)	**59.1** (27.3)

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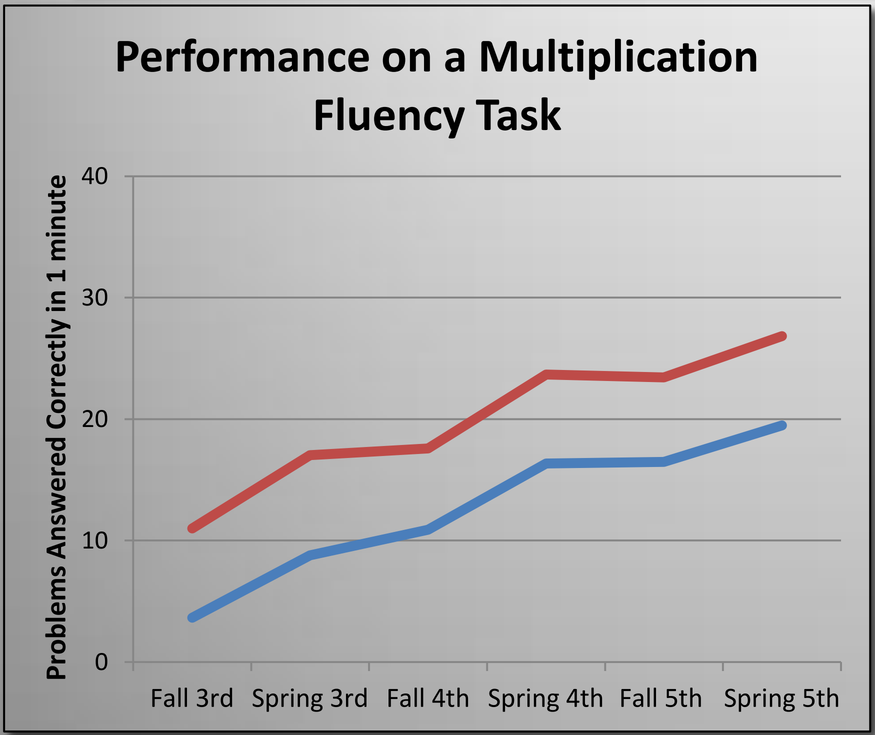
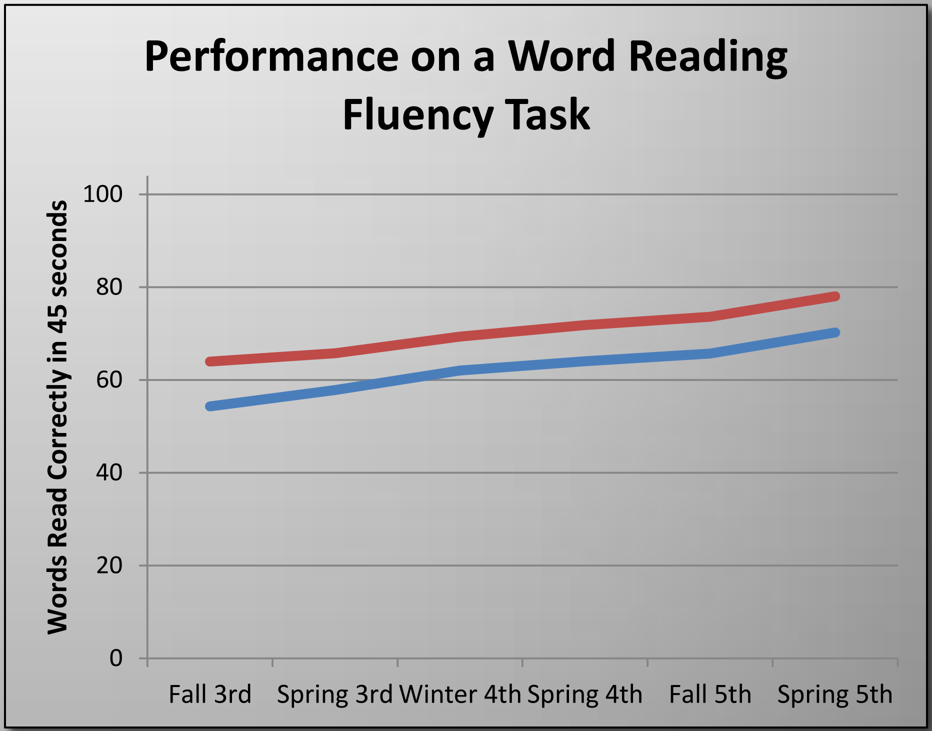
GROUPS ARE SIMILAR IN READING, BUT THE GAP DOES NOT NARROW IN ARITHMETIC OVER THE STUDY PERIOD





> 36th percentile in math achievement
≤ 35th percentile in math achievement

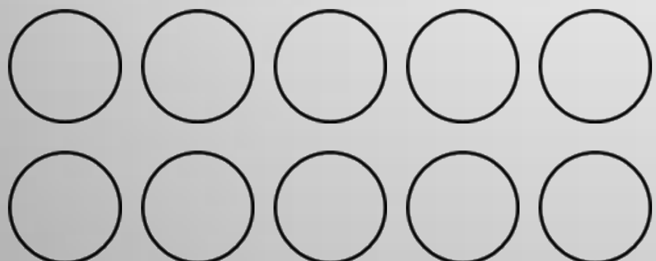
1. Children's prior knowledge of whole numbers interferes with learning when they are asked about more advanced fraction concepts problems.

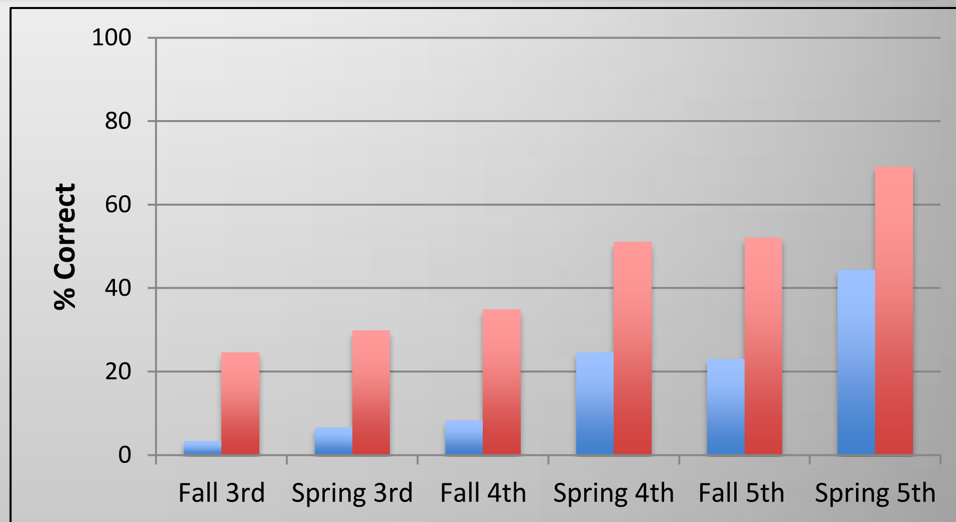
4th Grade CCSS

Extend understanding of fraction equivalence and ordering.

FRACTION CONCEPTS DEVELOPMENT SAMPLE

$\frac{4}{5} =$





> 36th percentile in math achievement
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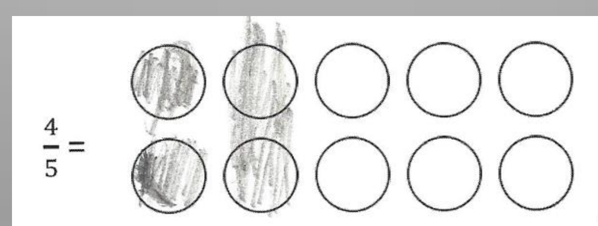
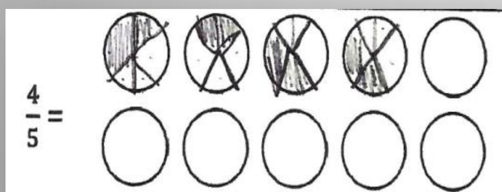
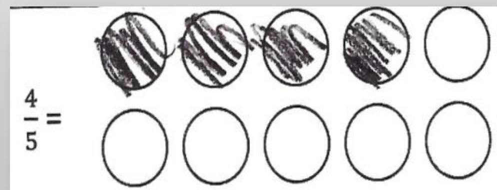
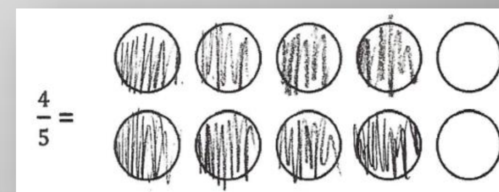
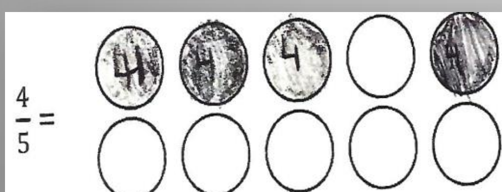
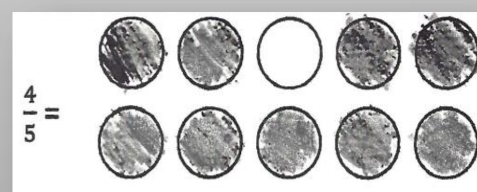
Most common incorrect response was to color 4 of the 10 circles, indicating that students were only attending to the numerator.



EXAMPLES OF STUDENT PROGRESS

3rd Grade	4th Grade	5th Grade
		
		

ORDERING FRACTIONS

Correct Response:

In which of the following are the three fractions arranged from least to greatest?

A. $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$ C. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$

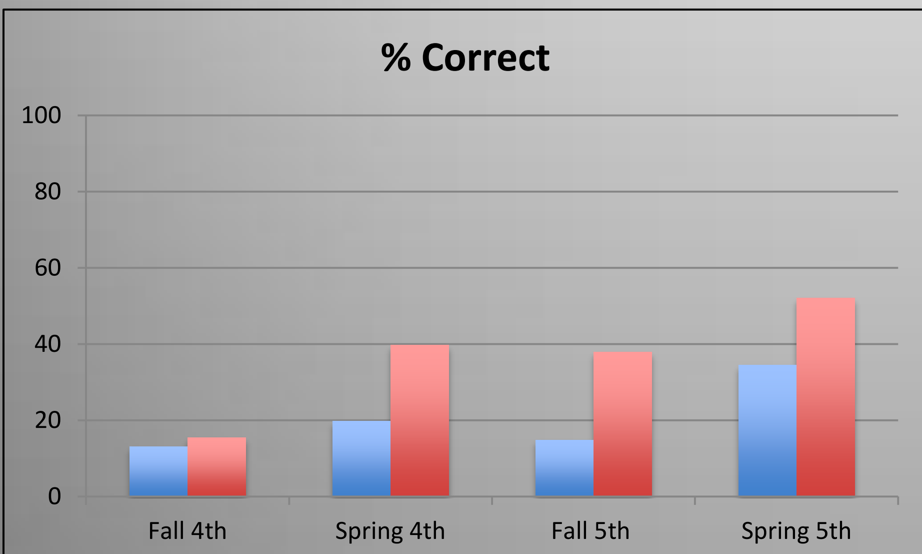
B. $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$ D. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

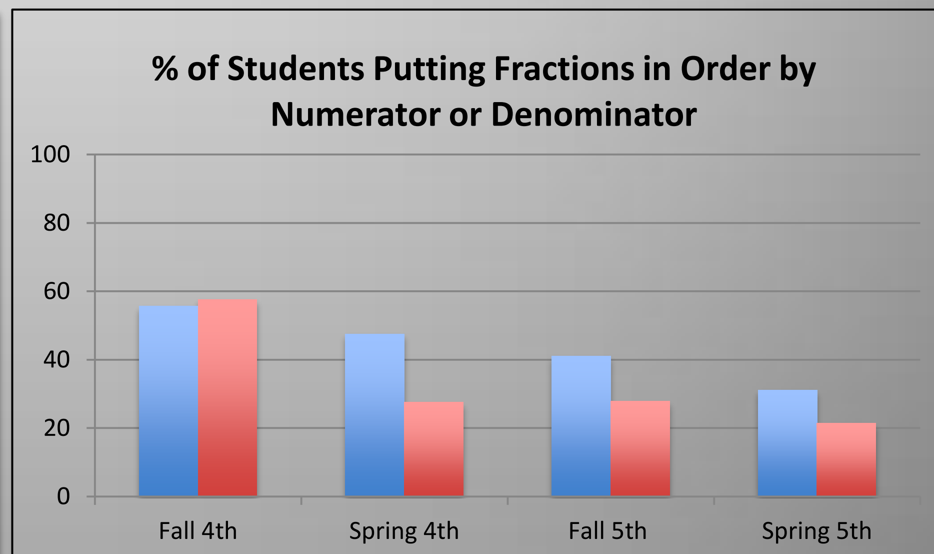
Incorrect Response:

In which of the following are the three fractions arranged from least to greatest?

A. $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$ C. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$

B. $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$ D. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$





> 36th percentile in math achievement
≤ 35th percentile in math achievement

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THAT WAS NAEP ITEM

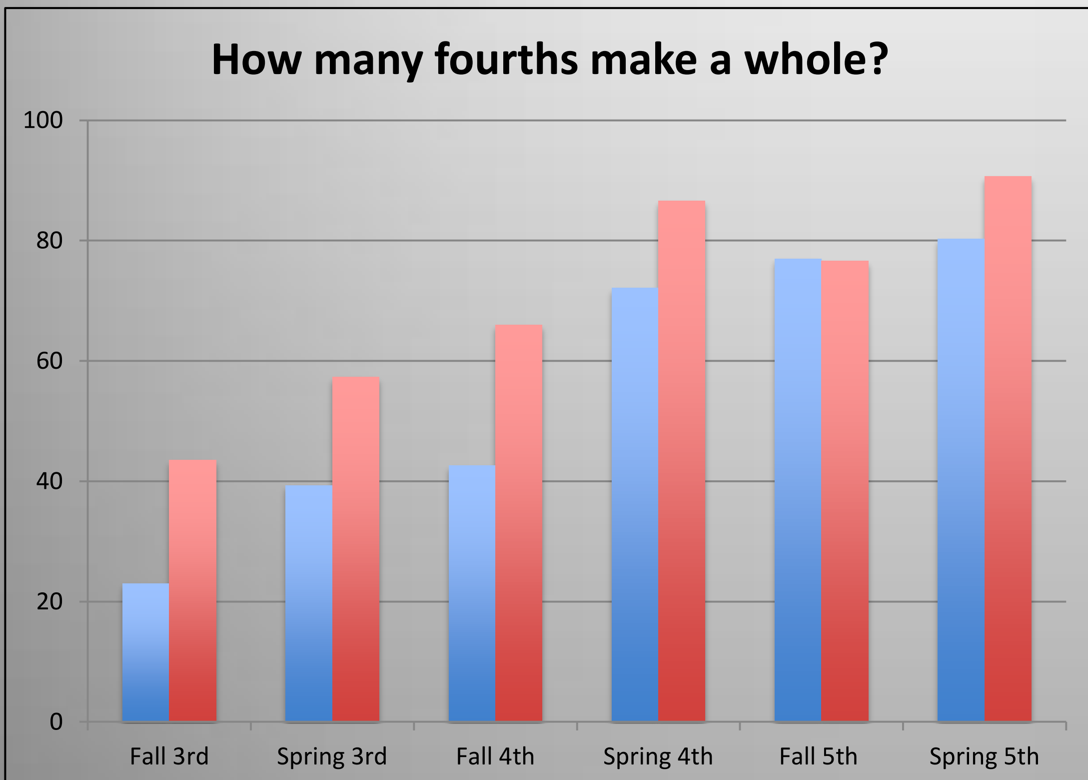
1. 8th graders in 2007 only correctly solved this problem 49% of the time!
2. Demonstrates critical importance of magnitude of fractions and how complex it is

2. Students do not have a strong understanding of the meaning of the denominator.

3rd Grade CCSS

Develop an understanding of fractions as numbers.

MEANING OF THE DENOMINATOR



> 36th percentile in math achievement
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Most students can state how many fourths make a whole by the end of 5th grade, BUT…

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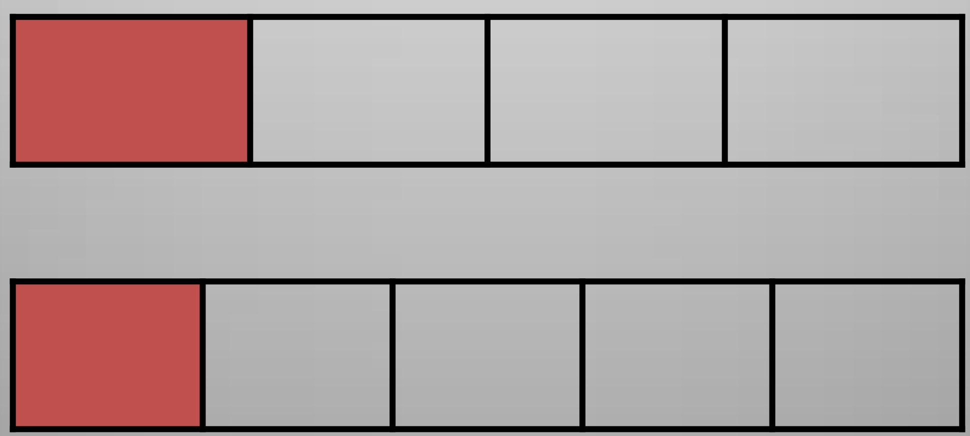
MEANING OF THE DENOMINATOR CREATES PROBLEMS WITH WORD PROBLEMS

When students do not understand that the **denominator** is the **size of one piece**, they have difficulty comparing unit fractions representing the same whole.



$\frac{1}{4}$

$\frac{1}{5}$

4. Students have difficulty identifying where fractions go on number lines.

3rd Grade CCSS

Understand a fraction as a number on a number line.

Even by the end of fifth grade, less than half of students in both groups can correctly identify fractions on the number line.

This points towards the need to further develop the understanding of a fraction as a location on a number line.

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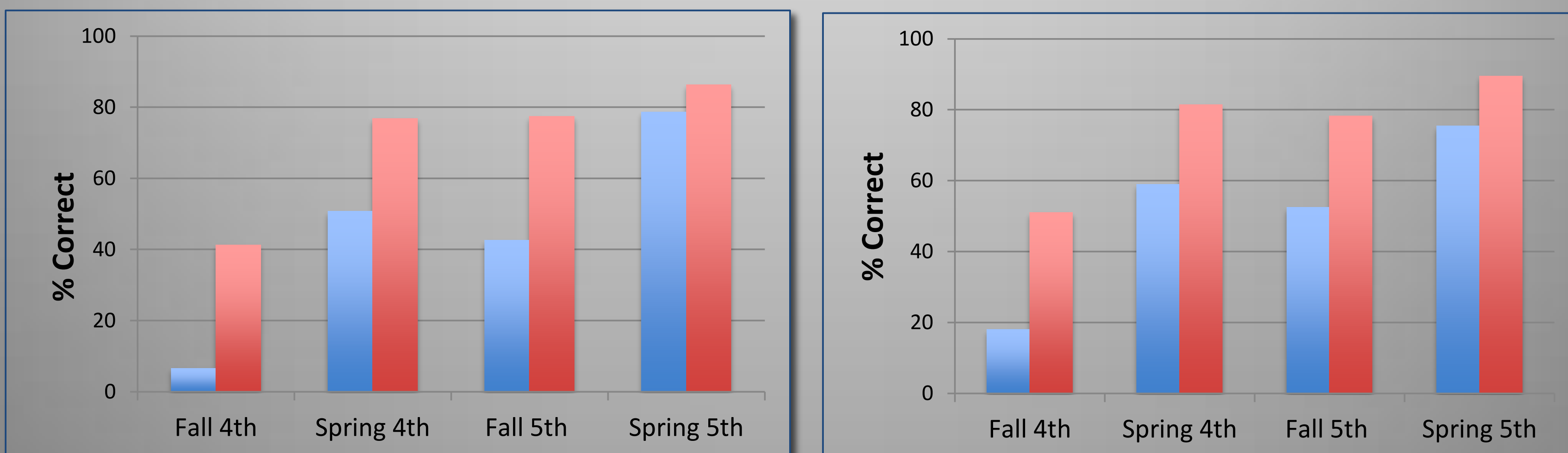
FRACTION PROCEDURES

1. By the end of 5th grade, most students successfully solved problems with like denominators.
2. But -- students in the lower math achievement group took an extra year to learn to solve these problems.

$$\frac{3}{6} + \frac{1}{6} = \qquad\qquad \frac{3}{4} - \frac{1}{4} =$$



> 36th percentile in math achievement
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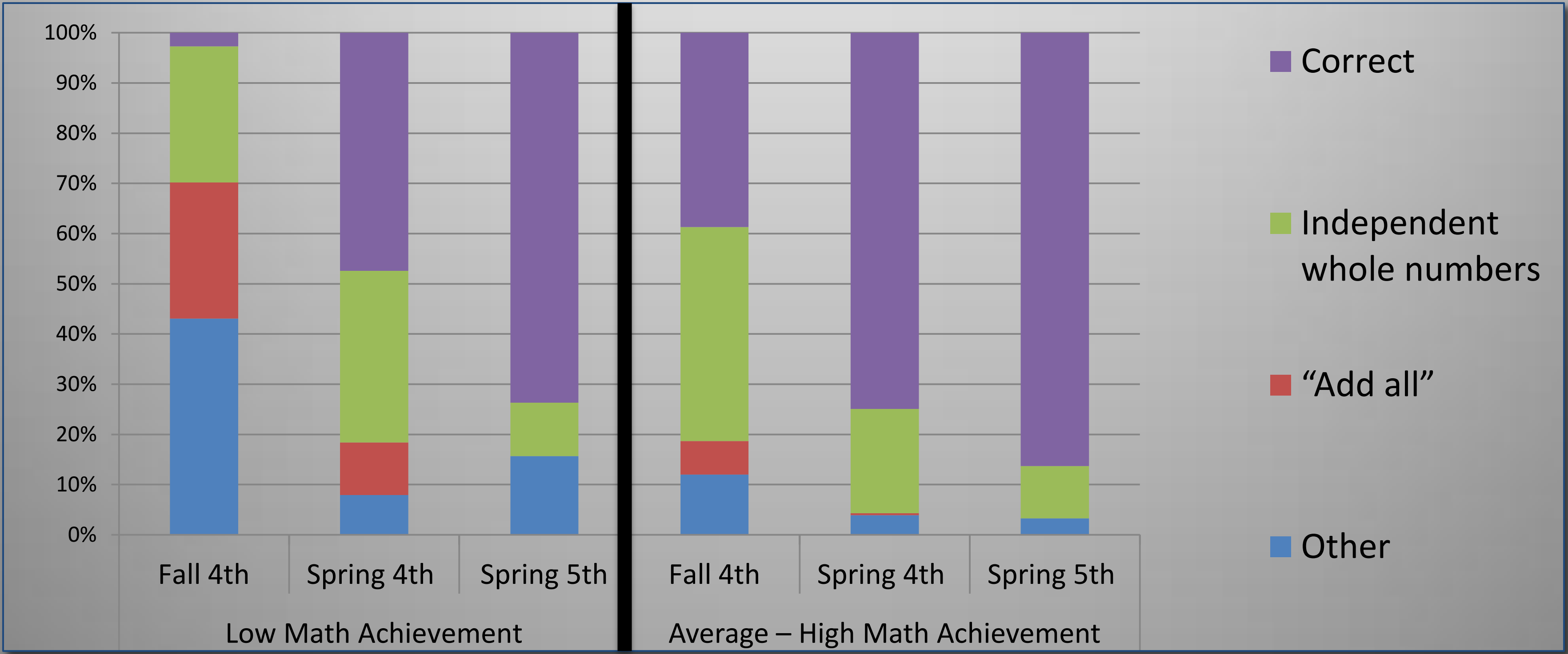
INCORRECT STRATEGIES USED

There were two main whole number strategies students used to solve problems with like denominators.

Strategy	Example
Independent whole numbers	$\frac{3}{6} + \frac{1}{6} = \frac{4}{12}$ $\frac{3}{4} - \frac{1}{4} = \frac{2}{0}$
“Add all”	$\frac{3}{6} + \frac{1}{6} = \underline{16}$

STRATEGY USED OVER TIME VARIED

The “whole number” strategy use varied according to group, and changed over time. $\frac{3}{6} + \frac{1}{6} =$



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WHAT TYPES OF **FRACTIONS PROCEDURES** PROBLEMS ARE DIFFICULT FOR STUDENTS?

5. Even after children overcome the "whole number bias" on addition and subtraction problems with like denominators, it resurfaces when they are confronted with problems with unlike denominators.

5th Grade CCSS

Add and subtract fractions with unlike denominators.

By the end of fifth grade, most students successfully solved problems with like denominators.

It took students in the lower-achieving group about a year to catch up to their higher achieving peers.

Although students successfully solve problems with like denominators by the end of 5^{th} grade, they still are not successful on problems with unlike denominators.

Students revert back to their whole number strategies when confronted with novel problems.

WHAT PREREQUISITE SKILLS DO STUDENTS NEED BEFORE THEY ENCOUNTER FRACTIONS?

It is important for students have fluent fact mastery so that they can execute fraction procedures correctly.

2nd and 3rd Grade CCCS

Fluently add and subtract within 20 by the end of 2nd grade.

Fluently multiply within 100 by the end of 3rd grade.

SUMMARY POINTS

1. Most students have a basic understanding of the part-whole meaning of a fraction but many struggle when it comes to items requiring knowledge of **equivalence and ordering**.
2. Struggling learners do not have a strong understanding of the meaning of a denominator.
3. Even by the end of fifth grade, less than half of students in both groups can correctly identify fractions on the number line.

SUMMARY POINTS (CONT.)

4. By the end of 5th grade, most students successfully solve addition and subtraction problems with like denominators but it took students with lower achievement more time to catch up to their higher achieving peers.
5. Most students struggle on addition and subtraction problems with unlike denominators.

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RESEARCH STRAND 2: THE CENTRALITY OF FRACTION MAGNITUDES

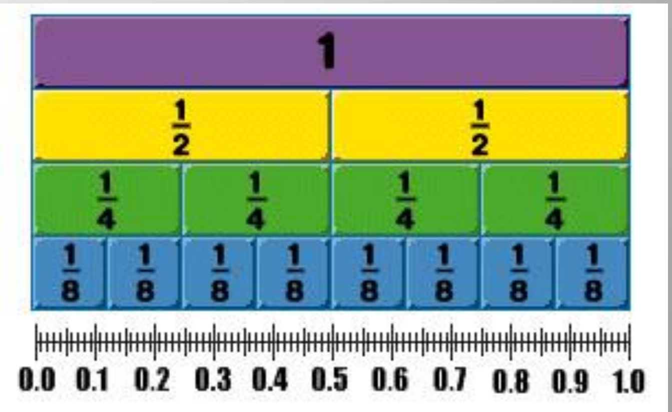
Adapted from Siegler, R. (2014). *An Integrated Theory of Numerical Development: Magnitude!!!!!*

and

Siegler, Thompson, & Schneider, (2011).

FRACTIONS ARE HARD!

1. Numbers of the same magnitude can look different (e.g., ¾ and 9/12)



2. Sometimes, when numerals get bigger, the fraction gets smaller (1/4, 1/6, 1/8)
3. Not always the case, however (2/4 < 6/7)
4. Infinite amount of numbers between 2 fractions.

U.S. CHILDREN AND ADULTS HAVE PARTICULARLY POOR FRACTIONS KNOWLEDGE

Siegler & Lortie-Forgues (in prep.):

"Is 13/15 x 12/17 > 13/15?"

- ✓ 6th & 8th graders: 30% correct
- ✓ **Pre-service teachers: 30% correct (real danger sign)**
- ✓ Carnegie Mellon Math/science students: 95% correct
- ✓ Only 50% of 8th graders correctly ordered 2/7, 1/12, and 5/9 (NAEP, 2007)
- ✓ Only 29% of 11th graders correctly translated .029 as 29/1000 (Kloosterman, 2010)

DEVELOPMENT OF NUMERICAL MAGNITUDE REPRESENTATIONS: A USEFUL UNIFYING THEME

1. **History:** Robbie Case indicated that mathematical development can be captured by the increased sophisticated of a person's **mental number line**.
2. **Contemporary research supports this position**
   - ✓ Linearity*(i.e. precision of estimates) of Whole Number Magnitude Representations Correlates Positively and Quite Strongly With:
     - Numerical magnitude comparison (Laski & Siegler, 2007)
     - Arithmetic proficiency (Gilmore, et al., 2007; Halberda, et al., 2008; Holloway & Ansari, 2008; Schneider et al., 2009)
     - Standardized achievement tests (Booth & Siegler, 2006; 2008; Geary et al., 2007; 2011)

BASIC TENETS OF THE INTEGRATED THEORY OF NUMERICAL DEVELOPMENT

1. Numerical development is in large part a process of generating increasingly precise representations of the magnitudes of an increasingly broad range of numbers.

2. Four main trends:

   - ✓ From non-symbolic to symbolic numbers
   - ✓ From small to large whole numbers
   - ✓ From whole numbers to rational numbers
   - ✓ From positives to positives and negatives

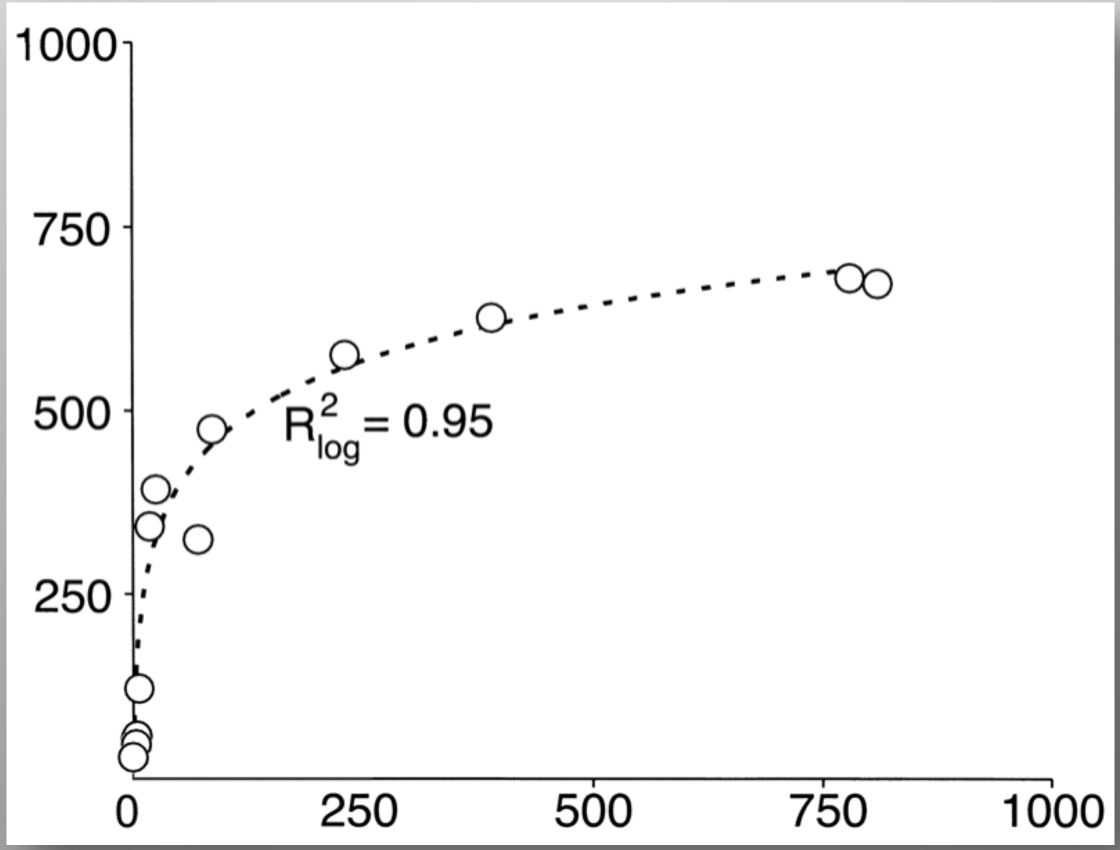
KEY MEASURE: NUMBER LINE ESTIMATION

The Number Line Task

"Where does 87 go?"

0 1000

SECOND GRADERS TEND TO USE A LOGARITHMIC NUMBER LINE



Median Estimate

Number Presented

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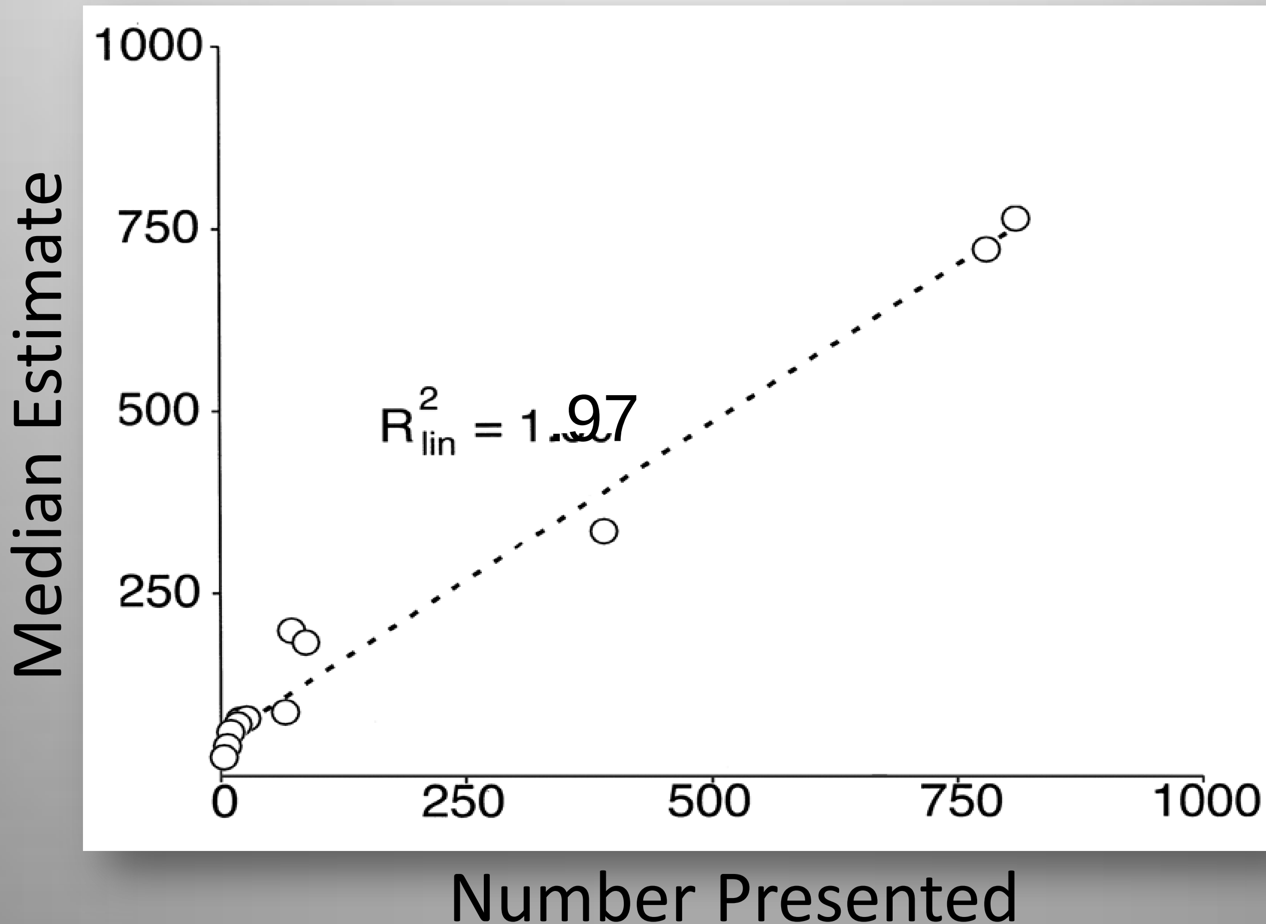
SIXTH GRADERS ON AVERAGE USE A LINEAR NUMBER LINE



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RELATIONS BETWEEN FRACTION MAGNITUDE REPRESENTATIONS AND MATH ACHIEVEMENT SCORES: 8TH GRADERS

Measure of Magnitude	Math Achievement
✧ Number line 0-1	**-.63****
✧ **Number line 0-5**	**-.86****
✧ Magnitude Comparison Accuracy	**.62****

**$p < .01$

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FRACTION NUMBER LINE

1. Same basic idea
2. Heavily correlated with whole number line precision

Think Pair Share: Ideas about why?

THE INTEGRATED THEORY PREDICTS THAT MAGNITUDE KNOWLEDGE PLAYS THE SAME CENTRAL ROLE WITH FRACTIONS AS WITH WHOLE NUMBERS

1. Knowledge of fraction magnitudes should correlate highly with facility with algebra and comprehension of algebra (on 3/4X = 6, X must be a little larger than 6).
2. Can also help with computation and problem solving as reasonable estimates can be generated to compare with answer computed.

IS EARLY FRACTION KNOWLEDGE UNIQUELY PREDICTIVE OF LATER, MORE ADVANCED, MATH ACHIEVEMENT? (SIEGLER, ET AL., 2012)

1. Fractions, including ratios and proportions, are heavily used in high school math – algebra, geometry, trigonometry, etc.
2. Individual differences in math knowledge are quite stable (Duncan et al., 2007; Stevenson & Newman, 1986).

U. S. DATA

	Algebra	Total Math Score
	Multiple Regression	Multiple Regression
Math		
Fractions	**.17**	**.18**
Addition		
Subtraction		
Multiplication		
Division	**.19**	**.26**
Other		
Digit Span Backward		
Passage Comprehension		.20
Household Income		
Parent Education	.19	

Standardized *β* coefficients, $p < .01$

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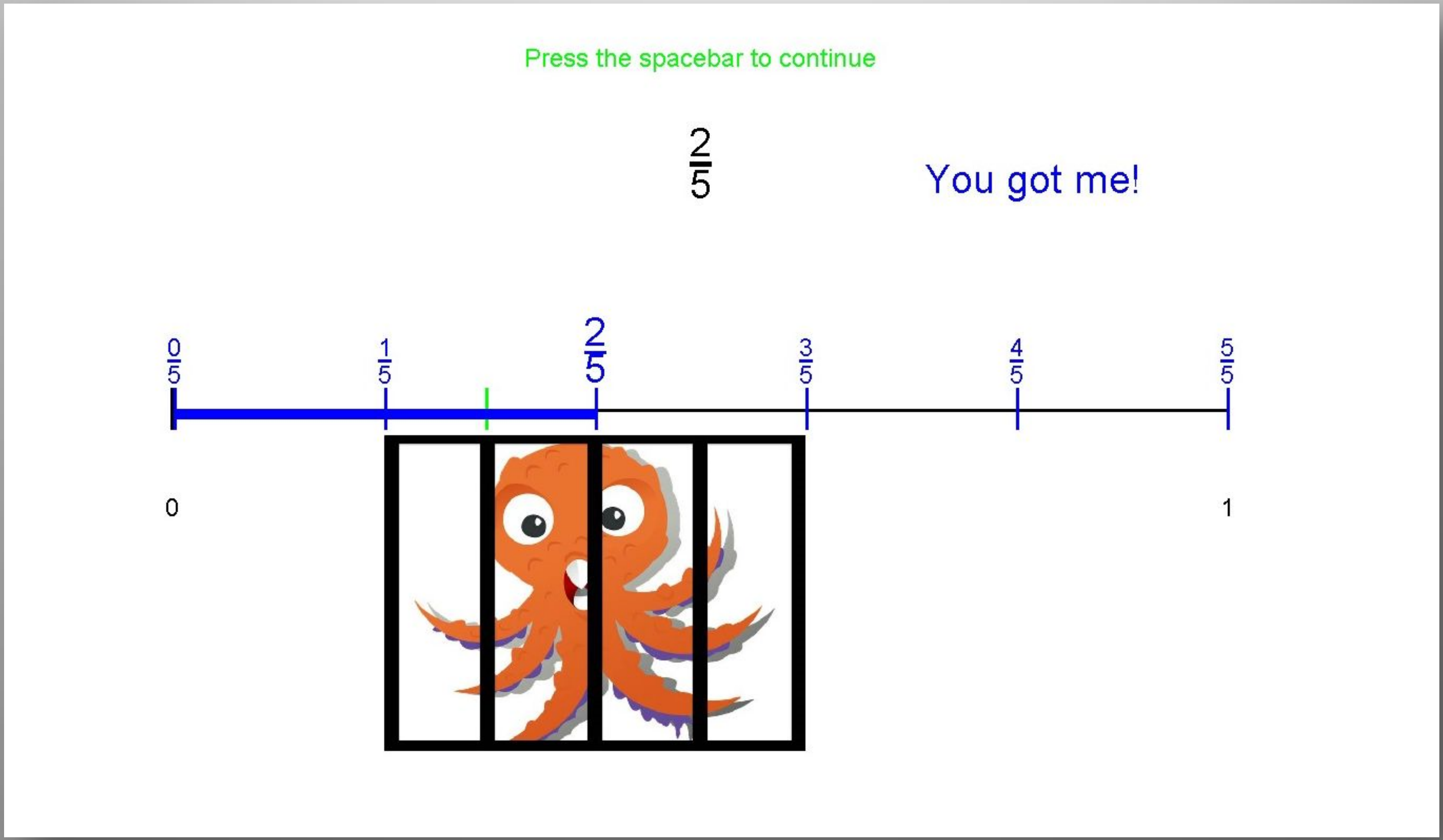
SCREENSHOT FROM *CATCH THE MONSTER*



CONCLUSIONS (CONT.)

3. Fractions are particularly important for distinguishing properties of whole numbers from those of all numbers
4. Precision of numerical magnitude representations is related to math achievement for up to 3-13 years.
5. However, it is only one factor, correlation over long haul of approx .34.

STRAND 3 – INTERVENTION FOR AT RISK 4TH GRADERS: THREE RANDOMIZED CONTROLLED TRIALS

Part 1: Link to ongoing research

Part 2: The three intervention studies

IMPLICATIONS FOR INSTRUCTION
MEASUREMENT INTERPRETATION OF A FRACTION

1. Build on students' part-whole understanding of a fraction and introduce them to the linear representation (aka measurement interpretation) of a fraction **expeditiously in intermediate grades.**

2. Example:

This measurement interpretation helps students **think of the fraction as one number**, instead of two separate whole numbers.

IMPLICATIONS FOR INSTRUCTION *EMPHASIZING THE MEANING OF THE DENOMINATOR*

1. Students need to understand the **meaning of** the denominator of a fraction:
   - ✓ **As the whole is divided into more pieces, each unit piece becomes smaller**
2. Without this understanding, students have difficulty comparing unit fractions representing the same whole.
3. **Interventions will be "fighting the tide of whole number memories, e.g. 5 is less than 6 so 3/5 is less than 3/6**

EMPHASIZING THE MEANING OF THE DENOMINATOR

Example of a classroom activity that might target this concept:

Fraction of the Day

$$\frac{3}{4}$$

$$\underline{\quad\frac{1}{4}\quad} + \underline{\quad\frac{1}{4}\quad} + \underline{\quad\frac{1}{4}\quad}$$

Students need to understand how to decompose (break apart) and put together fractions into unit fractions. This will require a lot of practice problems and activities

IMPLICATIONS FOR INSTRUCTION
OVERCOMING THE WHOLE NUMBER BIAS

Engaging in activities emphasizing the **measurement interpretation** and the **meaning of the denominator** might encourage students to think of a fraction as one number, rather than two independent whole numbers.

IMPLICATIONS FOR INSTRUCTION
BUILDING FACT FLUENCY

1. Important when solving fraction computation problems
2. Allows students to focus on concepts and more complex procedures
3. Must be emphasized in earlier grades and reinforced for struggling learners throughout elementary school.

FIVE RCTS

(one RCT each year; begin Year 5 this fall)

Lynn Fuchs, Robin Schumacher and colleagues, Vanderbilt University

1. In Year 1, contrasted core intervention program vs. instruction in 4th grade classroom(Business as Usual)
2. In Years 2-5,

✓ Replicating this effect for refined versions of core intervention program

✓ Isolating contribution of a series of additional program components (3 study conditions: 2 versions of intervention and BAU).

EACH YEAR, PARALLEL SET OF STUDY METHODS

1. Risk = low whole-number math skill (<35^{th} percentile) at start of 4^{th} grade
2. Sample same number of students with more vs. less severe risk (<15^{th} vs. 15^{th}-34^{th})
3. Assess understanding of fractions and procedural skill using experimental tasks and released NAEP fraction items.
4. Sample (after attrition) each year: ~250 At Risk (~35 moved) from ~50 classes in ~15 schools
5. 60% African American, 20% Hispanic

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WHAT WAS THE INTERVENTION

1. Primary focus: Linear representations (aka: Measurement Interpretation)
   - ✓ Number Lines
   - ✓ **Fraction tiles as a transition tool**
   - ✓ **Magnitude:** ability to reason about size and relative size
2. Secondary focus: Part-Whole Understanding
   - ✓ Shaded Regions of one or more Units (e.g. pizzas etc.)
   - ✓ **Tertiary** emphasis: computation
   - ✓ Grade level content (with some coverage of grade 3 material)
     - Exception: set of denominators more restricted than core curriculum (only up to 13 and no 9th or 11th for this set of lessons)
     - Theory: **temporary reduction of cognitive load**

MOTIVATORS (EXTRINSIC)

Students have three ways to earn fraction money:

1. On-Task Behavior
   - ✓ Unidentified intervals, group contingency
2. Solving problems correctly
   - ✓ Last activity of the day
3. Meeting or Beating Fluency Score
   - ✓ Tutors were instruction to give bonus money to increase focus as needed based on group needs
4. Denominations of $ include:
   - ✓ Whole dollars
   - ✓ Half dollars
   - ✓ Quarter dollars
5. The Fraction Store opens every 3 days with prizes at various price points; $1, $7, $13, $20
6. Students can choose to save or spend each time store opens

LOGISTICS OF INTERVENTION

Key components were intensive instruction in that

1. Small groups of 3
2. A good deal of practice with partners or individually or with interventionist
3. Frequent feedback when trouble began
4. Mastery assessments daily to see if more time needed
5. Grade level material covered……. With some backfilling
6. Duration:
   - ✓ **3 times per week,**
   - ✓ **30 min per session**
   - ✓ **For twelve weeks**
7. interventionists trained and monitored and fidelity of sessions audiotaped

ACTIVITIES PER LESSON

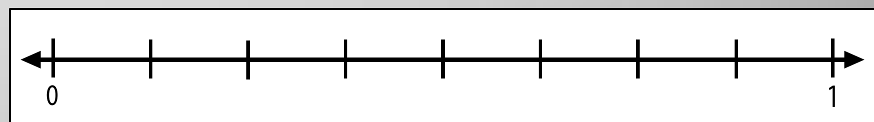
(VARIES BY YEAR, DEPENDING ON COMPONENT TESTED)

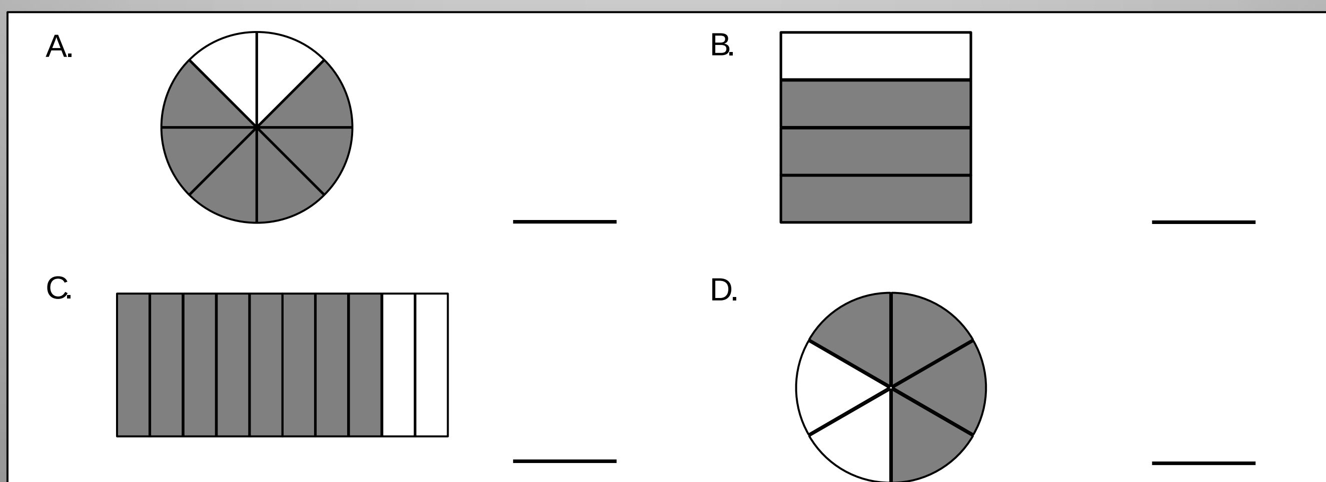
1. Training (new concepts/strategies introduced)
2. The Relay (group work on concepts and strategies taught that day)
3. The Sprint (speeded practice to develop procedural fluency on strategic measurement interpretation tasks, e.g., finding fractions equivalent to ½)
4. The Individual Contest (individual practice)

EARLY SKILLS:
UNIT FRACTIONS AND NAMING FRACTIONS

- ✓ Introduce unit fractions with Circles and Tiles
- ✓ Show fractions with shaded regions to show Unit fractions
- ✓ Show how *unit fractions* make larger fractions with manipulatives, number lines, and numbers
- ✓ Name fractions from shaded representational regions (see example below)

$$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{5}{8}$$





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USE OF STRATEGY REMINDERS AS TRANSITION TOOL

Most advanced card

Number Lines

Which number line?

0 — **1**

Find $\frac{1}{2}$

Compare to $\frac{1}{2}$ and write L or G

LL or GG?
Compare and write < or >

0 — **2**

Label:
Proper(P), Improper(I), or Mixed(M)
Change I to M

If Proper:
Look at 0-1
Compare to $\frac{1}{2}$ and write L or G

If Mixed:
Look at 1-2
Compare to $1\frac{1}{2}$

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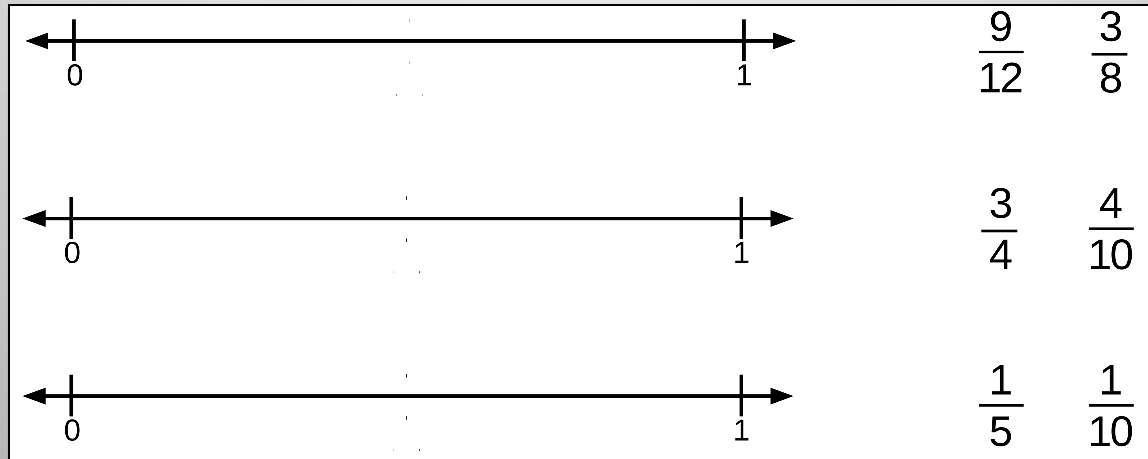
COMPARE THESE USING THE NUMBER LINES: 9/12, 3/4, 1/5



This was done first with ½ benchmark then without it.

FINDINGS TO DATE

Results on 3 core outcome measures, across 4 years for intervention vs. BAU

1. *Fractions Number Line*
   - ✓ Mean effect size for Intervention of 0.99
2. *Adding/Subtracting Fractions* mean effect size of 1.57, supporting idea of reciprocal relationship

NAEP ITEMS

1. Comparable emphasis on measurement and part-whole interpretation (comparably distal for both conditions)
2. Mean effect size for Intervention = 0.66
3. Post test achievement gap (ES for AR vs. not-at-risk classmates):
   - ✓Mean gap for intervention students = **-0.55**
   - ✓Mean gap for control group AR students = **-1.20**
   - ✓**i.e. some progress towards closing the gap**

RATIONALE FOR THE INTERVENTION: RECAP

1. Core of intervention is aligned with grade level mathematics instruction.
2. Content linked to contemporary state standards and current thinking about best practice in mathematics education.
3. Small group instruction targets an area that longitudinal research shows is particularly difficult for this group of students.
4. Intervention attacks an area that some elementary teachers are unsure how to teach.
5. Adequate practice provided.

QUESTIONS?