Using Origami to Teach Proportional Reasoning

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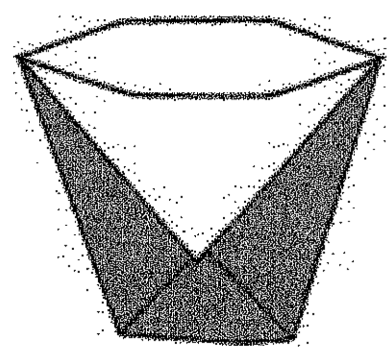

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The success of middle grades students in higher mathematics is decidedly dependent on developing a deep understanding of proportionality, which, Lamon (2005) states, "opens the door to high school mathematics and science, and eventually, to careers in the mathematical sciences." Proportional reasoning is the ability to ***recognize*** situations in which there exists a linear relationship between two quantities which vary together. Additionally, proportional reasoning implies students can identify situations that are not proportional, but reflect a more sophisticated relationship.

All too often teaching proportions consists of focusing on a mechanistic process such as "cross multiplication" where students are supplied three numbers and asked to find a fourth. The notion of scale factor, such as 1 cm. represents 10 km. on a map, occasionally arises, though usually in isolation. While this form of instruction may prepare students for multiple-choice-type tasks, it does little to develop proportional reasoning. The National Council of Teachers of Mathematics (2000) suggests instruction in proportional reasoning should have a "strong intuitive basis" in order that students may construct such reasoning.

Recognition of proportional situations requires an intuitive basis; experiences with familiar and unfamiliar settings often allow students to "see" linear and non-linear relationships. The authors have found these scenarios take two forms—verbal and sensory. In verbal situations the questions are already written and students need to be able to interpret information to determine that a proportional relationship exists. Problems involving chemical mixtures, percentages, and distance-rate-time are common "textbook" examples. By contrast, sensory scenarios involve identifying the question and then recognizing a proportion exists as a result of physically interacting with materials, visually or tactilely. Construction of scale models, varying recipes, and enlarging containers are exemplars of sensory problems. While, these problems are less likely to be found in written form, they are often encountered in daily living. It is critical that students develop the ability to recognize proportionality in applications rather than mimic a technique taught with little or no attention to conceptual understanding.

In an effort to provide instruction on proportional reasoning in sensory situations that is lacking in many curriculums, the authors developed and field tested a number of investigations with pre-algebra students using origami that combine the geometric idea of similarity with measurement.

The ancient art of origami lends itself well to investigating proportional reasoning related to scaling. Origami has the advantages of being both creative and tactile. It also provides students with the additional motivation of wanting to apply what they learn about origami to projects outside the mathematics classroom. It is assumed that students have some pre-knowledge of squares, square numbers, the notion of ratios, and scaling.

Doing investigations that engage students' senses allows them to be more open to the generalizations inherently found in proportional reasoning. Too often, teachers tell students what they are to learn during an investigation rather than allowing students to grapple with the mathematics associated with the activity and make sense of what they discover. To emphasize a discovery-oriented approach, each activity begins with two questions, one that is explicit, concrete in nature focusing on a specific object. The implicit question is one that is not initially stated, but can be "discovered" by generalizing the problem to a broader context. It may be asked by students, but even then might only be answered by another question, like "what do you think?"

Investigation 1: Linear Scaling

The Explicit Question: If we know the size of the paper, can we figure out the width of the cup before we fold it?

The Implicit Question: Given proportional figures, what is the relationship between corresponding lengths?

Distribute four-inch and six-inch squares of paper and rulers to students. Begin with a comparison of the squares themselves. "How are the squares alike? How are the squares different?" Each student makes an origami cup using the six-inch square. Folding instructions for the cup are found in Figure 1. However, directions are also available in most origami books and on the internet. The cup was chosen because it requires very few steps and mea-

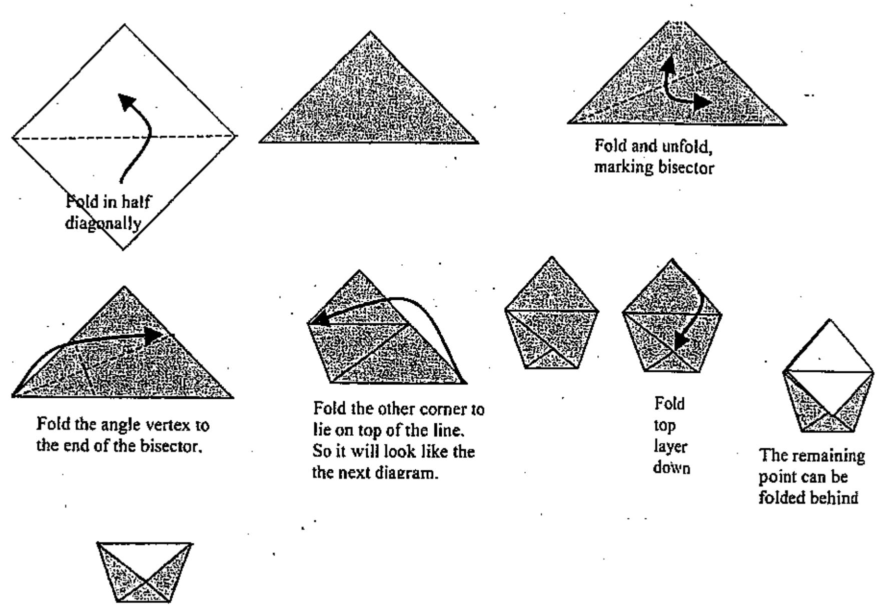


Figure 1

surements taken tend to be more consistent than other origami figures.

Have students measure and record the width of the top and bottom as well as the height of the six-inch cup. A discussion question could be: "What is the ratio of the side length of the square to the width of the top of the cup?" Based on this information, students predict the width of a cup made with the four- inch square. They discuss their predictions and reasoning and then justify their prediction in writing. When performing this activity, some students may predict that you subtract 2 inches since the edge of the paper is 2 inches shorter than the edge length of the 6-inch square. Other students may make a prediction based on physical comparison of the papers, while a few students may use proportions. The students test their conjectures by folding and measuring the four-inch cup. This is followed by a discussion of accurate and inaccurate predictions. This process usually leads to a discussion of precise measurement since students will measure with different degrees of accuracy. However, this topic is outside the purview of this paper. Testing and talking about results creates an experience for all students to consider proportions or scale factors as producing not only an accurate result, but a better method for finding an answer. Next, have students repeat the process for predicting and calculating the height and width of the bottom of the four-inch cup. At this point most students will have shifted to a successful strategy of using proportion or scale factors and had opportunity to practice the strategy. In an effort to generalize thinking, present students with several questions. "Given a twelve-inch square, can we predict the width of the cup? The height? The length of the diagonals?" In comparing cup models, students commented that lengths are proportional and angles are congruent which led to a discussion about those characteristics that vary and those that remain invariant in scaling problems. At this point in the activity, most students will have developed an intuitive idea related to the unasked "implicit" question: Given proportional figures, what is the relationship between corresponding lengths? Investigation 2 addresses finding a scale factor that relates proportional objects.

Investigation 2: Scaling Up a Cup to be a Hat

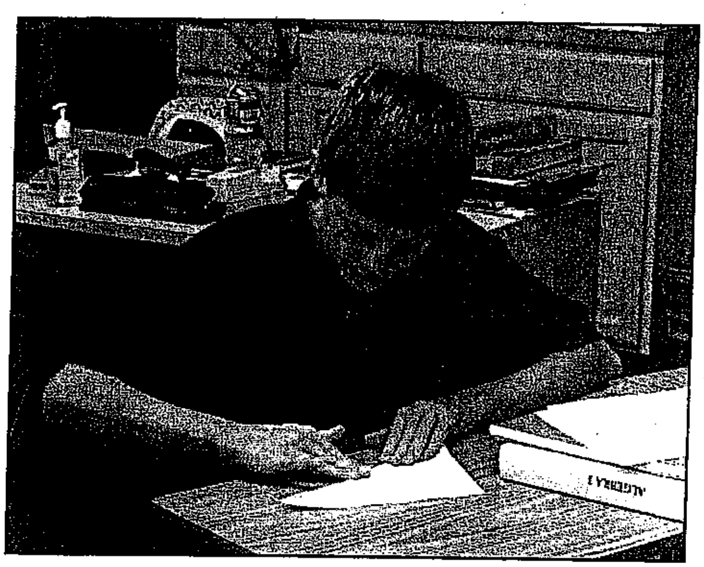
The Explicit Question: How large a sheet of paper is needed to make a hat that will fit my head?

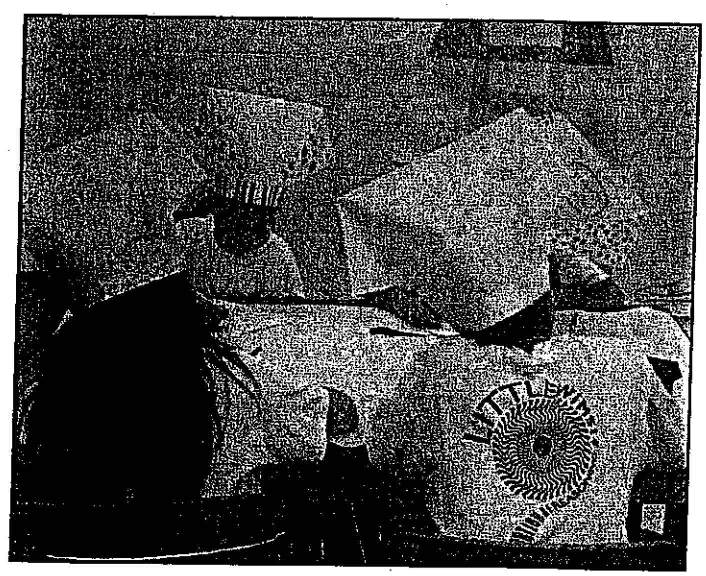
The Implicit Question: How does one find and use a scale factor?

Using a 12 inch square of paper, students fold a cup. Turning the cup upside down creates a hat, albeit a hat too small to fit the head of pre-algebra students. Students are asked, "How large a sheet of paper is needed to make a hat that will fit your head?" Discussion amongst classmates will typically focus on measurements of the head that might be relevant to determining the scale factor (these numbers are often the circumference, the front-to-back measurement, or side-to-side measurement). Note multiple methods will work and stimulate good discussion on what measurements are relevant to the problem. Using the information from investigation 1 and the head measurements, students are able to use proportions or scale factors to determine a correct paper size. Before students cut the paper to test their predictions, they should explain their reasoning and computation in writing.

Occasionally some hats will not fit. These "mistakes" are opportunities to clarify and deepen students' proportional reasoning. For example, some of the students switch units (using inches for one set of measurements and centimeters for another) that create ill-fitting hats, though not all students that switched units get a bad hat. This can lead to a lively discussion about units and their role in proportions and the notion that the scale factor is a constant of proportionality and does not have units. Another common error that can lead to rich discussion is an inverted scale factor, meaning that students calculate using an incorrect scale factor, a/b, rather than b/a. This occurs when students fail to compare corresponding parts.

Bulletin board paper was used to test solutions because enlarged hats require large squares of paper. The authors suggest allowing time for decorating hats, modeling hats, and using photos or video to preserve the moment!





	Linear Scale Factor	Ratio of Lengths	Ratio of Volumes
6 inch to 12 inch	2	1:2	1:8
4 inch to 12 inch	3	1:3	1:27
4 inch to 6 inch	1.5	2:3	8:27

Figure 2

Students are typically surprised to see the size square needed to construct a "cup" hat. The notion of "size" has moved from the length of the side to the area of the paper. The use of a tactile/visual model introduces the students to a non-linear relationship in a natural way. Investigation 3 explores volume, another non-linear relationship using the paper cup.

Investigation 3: Comparing Volumes of Similar Cups

The Explicit Question: How many little origami cups are needed to fill a big origami cup?

The Implicit Question: What effect does linear scaling have on volume? Are all relationships proportional?

Each student makes a six-inch and twelve-inch origami cup. "How many small cups are needed to fill the large cup?" is posed. Responses and rationale tend to involve the use of one of two strategies. Students use a pattern seeking strategy and conjecture that 2, 4, 6, or 8 small cups would be required to fill the large cup. Based on prior knowledge gleaned in Investigation 1 and Investigation 2, many students respond 2 cups assuming the scale factor is 2 to 1. Other students employ an estimation strategy. They visually study the two-sized cups and make a prediction such as "I think 6.2 little cups are needed to make the big cup."

While the cups will hold water, the authors recommend using popcorn kernels for testing conjectures. The students then answer the question by determining the number of small cups of popcorn needed to fill the large cup. The individual results are entered in a table and the numbers will be close to 8. Students are asked, "How the ratio of 8:1 relates to the original ratio of 2:1?" And "Do you see any connections between the two ratios?" " How many 4 inch cups are needed to fill a 12-inch cup?" Students find that twenty-seven 4-inch cups are contained in a 12-inch cup producing a ratio of 27:1.

Note, to do comparisons of the volumes of the 4-inch and 6-inch cups can lead to some valuable information, but may frustrate or lead students astray (see figure 2) since the volume conjecture is difficult to physically test and calculation produces the less than "nice" ratio, 8:27. Since we advocate no explicit teaching in these investigations, we recommend conjecturing this volume ratio with the most curious and insightful students.

The authors have found that the pre-algebra and algebra students who engage in these discovery-based investigations gain a rich understanding of proportionality. We have used the investigations successfully during a week-long unit focusing on proportional reasoning as well as doing the investigations spread over several weeks. We recommend providing plenty of time between investigations so that students may reflect on newly constructed ideas. Introducing writing to help students bring their thoughts and ideas together can be helpful in any of the investigations and allows students to communicate more effectively with peers and their teacher in the process.

Knowing something intellectually and knowing the same thing tactilely are very different experiences, as anyone who has tried home repair can attest. We agree and suggest that when students are given the opportunity to grapple with proportional situations both tactilely and intellectually, students develop a deep and rich understanding of proportionality.

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