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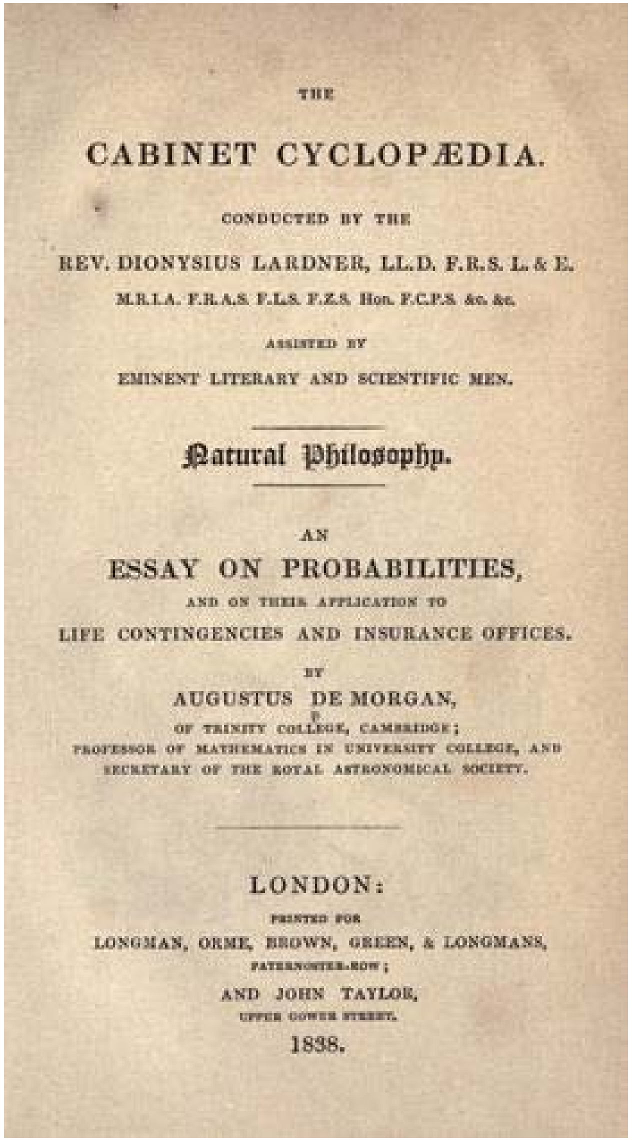
Triple Play: From De Morgan to Stirling to Euler to Maclaurin to Stirling

Recent Historical Research

- I recently obtained an1838 copy of Augustus De Morgan's *Essay on Probabilities and on their Application to Life Contingencies and Insurance Offices* to add to my extensive antiquarian math book collection.
- My wife regularly threatens me that the collection, accumulated over 45 years, could be sold to pay off our current mortgage.

Augustus DeMorgan
1806-1871
DeMorgan's Laws, Mathematical Induction



THE

CABINET CYCLOPÆDIA.

CONDUCTED BY THE

REV. DIONYSIUS LARDNER, LL.D. F.R.S. L. & E.

M.R.I.A. F.R.A.S. F.L.S. F.Z.S. Hon. F.C.P.S. &c. &c.

ASSISTED BY

EMINENT LITERARY AND SCIENTIFIC MEN.

Natural Philosophy.

AN

ESSAY ON PROBABILITIES,

AND ON THEIR APPLICATION TO

LIFE CONTINGENCIES AND INSURANCE OFFICES.

BY

AUGUSTUS DE MORGAN,

OF TRINITY COLLEGE, CAMBRIDGE;

PROFESSOR OF MATHEMATICS IN UNIVERSITY COLLEGE, AND

SECRETARY OF THE ROYAL ASTRONOMICAL SOCIETY.

LONDON:

PRINTED FOR

LONGMAN, ORME, BROWN, GREEN, & LONGMANS,

PATERNOSTER-ROW;

AND JOHN TAYLOR,

UPPER GOWER STREET.

1838.

Augustus DeMorgan

- Considered the BEST math teacher of the 19^{th} Century
- Known primarily for his work on Symbolic Logic and Mathematics History
- Did significant work in Calculus.

Historical Research

- Factorials of whole numbers play an important role in Probability Theory, and were difficult to compute in the 19th Century for large whole numbers.
- Therefore, De Morgan introduces the following algorithm (pp. 15-16) to approximate n!, where n is a whole number.
- Note that [n] in De Morgan’s notation means n factorial (n!).

RULE.—To find very nearly the value of [a given number], from the logarithm of that number, subtract ·4342945, and multiply the difference by the given number, for a first step. Again, to the logarithm of the given number add ·7981799, and take half the sum, for a second step. Add together the results of the first and second steps, and the sum is nearly the logarithm of the product of all numbers up to the given number inclusive. For still greater exactness, add to the final result its aliquot part, whose divisor is 12 times the given number.

EXAMPLE.—What is [30] or $1 \times 2 \times 3 \times \ldots\ldots \times 29 \times 30$?

log. 30	1·4771213		1·4771213	
	·4342945		·7981799	
Subtract	1·0428268		2)2·2753012	Add.
	30			
			1·1376506	Second step.
Multiply	31·284804	First step.		
	1·137651	Result of second step.		
	32·422455	log. of result.		

The result has, therefore, 33 places of figures, of which the first six are (nearly) 264,518; or, if this be increased by its 360th (12 times 30) part, or about 735, the result is 265,253, followed by 27 ciphers; or

the approximate result is —

265253000000000000000000000000000

The true result is —

265252859812191058636308480000000

and the error is not so much of the whole, as one part out of 500,000.

In this way, we are able to do with more than sufficient nearness, and in a few minutes, what it would take days to arrive at by the common method, and with much greater risk of error.

Historical Research

- I wondered why De Morgan’s algorithm for n! worked, and was so accurate.
- That led me to an Intermediate Algebra investigation of the algorithm, which led to Stirling’s Formula.
- I also wondered why Stirling’s Formula so closely approximated n!.
- That led to the Gamma Function (n!), a proof by Mathematical Induction of the Gamma Function for whole numbers, & a proof of Stirling’s Formula using Multivariable Calculus, and Maclaurin Series.
- In the process, I learned some Mathematics History I can use in the classroom and I discovered some interesting enrichment exercises for secondary and lower division college mathematics.

Historical Research

- Stirling’s approximation of n! (n a whole number) was discovered by James Stirling (1692-1770), a Scottish Mathematician. Stirling’s approximation states that

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^{n}$$

De Morgan’s Algorithm

Referring to De Morgan’s algorithm for n!, and noting that:

.4342945 is an approximation of log e,

.7981799 is an approximation of $\log 2\pi$,

we have:

De Morgan’s Algorithm

1. Take the log of the number, and subtract .4342945: log n – log e = $\log\left(\frac{n}{e}\right)$
2. Multiply by n: n log$\left(\frac{n}{e}\right)$ = $\log\left(\frac{n}{e}\right)^n$
3. Take log n and add .7981799 = log n + log 2π =log 2nπ
4. Take half of the latter result: $\frac{1}{2}\log 2n\pi = \log\sqrt{2n\pi}$

De Morgan's Algorithm

5. Add the results of the 2nd and 4th steps:
6. $\log\left(\frac{n}{e}\right)^n + \log\sqrt{2n\pi}$ = log $\sqrt{2n\pi}\left(\frac{n}{e}\right)^n$

   But according to DeMorgan, this an approximation of the log n!, so $n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n$

   which is Stirling's Formula!

De Morgan to Stirling

- This would be a nice enrichment exercise for Intermediate Algebra or College Algebra students.
- Next, I wondered why Stirling’s Formula approximated n!

From Stirling to Euler and the Gamma Function

- The Gamma function (valid for complex numbers with positive real parts, partially due to Euler) states for n a whole number:

$$n! = \int_0^\infty x^n e^{-x} dx$$

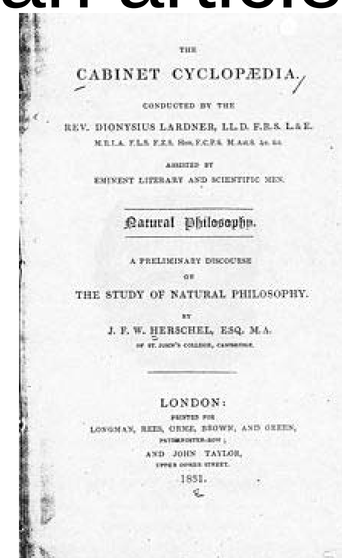
Leonhard Euler
1707-1783

- Most prolific Mathematician of all time

Proving the Gamma Function= n! for whole numbers

I wanted to see if I could prove that the Gamma Function resulted in n! for whole numbers using mathematical induction.

Mathematical induction was introduced by Augustus DeMorgan an article in the Cabinet Cyclopedia.



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EMINENT LITERARY AND SCIENTIFIC MEN.

Natural Philosophy.

A PRELIMINARY DISCOURSE

ON

THE STUDY OF NATURAL PHILOSOPHY.

BY

J. F. W. HERSCHEL, ESQ. M.A.

OF ST. JOHN'S COLLEGE, CAMBRIDGE.

LONDON:

PRINTED FOR

LONGMAN, REES, ORME, BROWN, AND GREEN,

PATERNOSTER-ROW;

AND JOHN TAYLOR,

UPPER GOWER STREET.

1831.

Inductive Proof

- Verify it is true for n=0. 0! =1 and

$$\int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = 1$$

A nice exercise for a Calculus 2 class

Inductive Proof

- Assume k! = $\int_0^\infty x^k e^{-x} dx$

Inductive Proof

- Then (k+1)! should equal

$$\int_0^{\infty} x^{k+1} e^{-x} dx$$

- Using Integration by Parts with u= x^{k+1}

  and dv = $e^{-x}dx$ we get

Inductive Proof

.

$$\int_0^\infty x^{k+1}e^{-x}dx = -x^{k+1}e^{-x}\Big|_0^\infty + (k+1)\int_0^\infty x^k e^{-x}dx = 0 + (k+1)k! = (k+1)!$$

where $-x^{k+1}e^{-x}\Big|_0^\infty$ = 0 by L'Hospital's Rule

Stirling’s Formula to the Gamma Function

- I wanted to prove that Stirling’s formula was an approximation of the Gamma Function, and thus an approximation of n! for whole numbers.
- That led me to establish a lemma using Fubini’s Theorem and polar substitution (good Multivariable Calculus enrichment) that I needed before I established that Stirling’s formula approximated the Gamma Function.

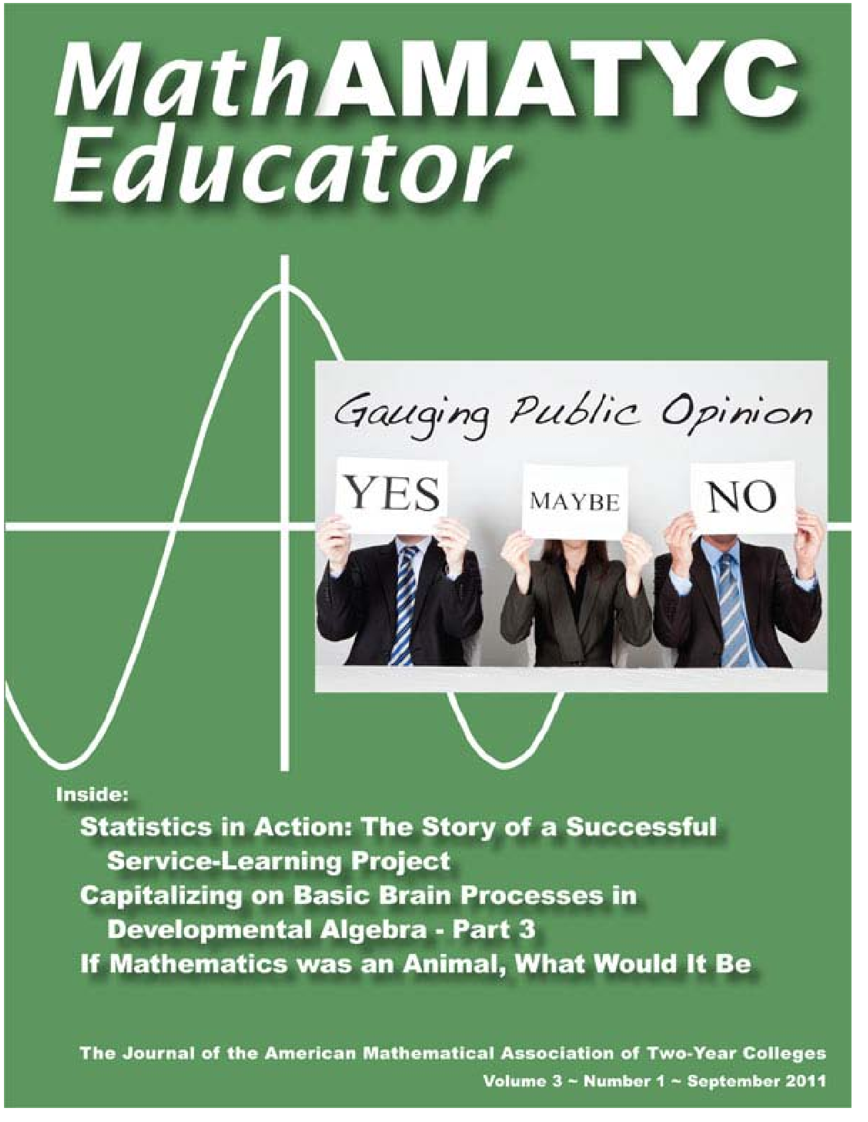
Lemma proved with Multivariable Calculus Techniques

$$\int_{-\infty}^{\infty} e^{\frac{-x^2}{2n}}\,dx = \sqrt{2n\pi}$$

Using symmetry, Fubini's Theorem, and Polar Substitution

Note: Email me at skolpas@dccc.edu and I'll send you the complete article.

Sept. 2011 MathAMATYC Educator



Triple Play: From De Morgan to Stirling to Euler to Maclaurin to Stirling

Sid Kolpas, Delaware County CC

From the Gamma Function (Euler) to Maclaurin (which leads us back to Stirling)

- My goal was now to show that Stirling’s Formula is an approximation of n!, and therefore an approximation of the Gamma Function. To establish this, I did the following:

The Gamma Function

- n! = $\int_0^\infty x^n e^{-x} dx$

by the Gamma Function where n is a whole number. Since I want an approximation of the Gamma Function which will lead to Stirling's Formula, I tried a Maclaurin Series Approximation.

Proof that Stirling's Formula approximates the Gamma Function

- Let u = (x-n) or x= (u+n). Also du = dx. Substituting, we get

$$\int_{-n}^{\infty}\left(n\left[\frac{u}{n}+1\right]\right)^n e^{-(u+n)}du = \int_{-n}^{\infty} n^n\left(\frac{u}{n}+1\right)^n e^{-u}e^{-n}du = \left(\frac{n}{e}\right)^n \int_{-n}^{\infty}\left(\frac{u}{n}+1\right)^n e^{-u}du$$

Look at $\ln\left(\frac{u}{n}+1\right)^n$ = $n\ln\left(\frac{u}{n}+1\right)$

The first two terms of the Maclaurin Series for

$\ln\left(\frac{u}{n}+1\right)$ = $\left(\frac{u}{n}-\frac{u^2}{2n^2}\right)(-n<u\le n)$ and n$\left(\frac{u}{n}-\frac{u^2}{2n^2}\right)$ = $\left(u-\frac{u^2}{2n}\right)$

Therefore

$$\ln\left(\frac{u}{n}+1\right)^{n} = n\ln\left(\frac{u}{n}+1\right)$$

$$\sim n\left(\frac{u}{n}-\frac{u^{2}}{2n^{2}}\right) = \left(u-\frac{u^{2}}{2n}\right) \text{ and}$$

$$e^{\ln\left(\frac{u}{n}+1\right)^{n}} = \left(\frac{u}{n}+1\right)^{n} \sim e^{\left(u-\frac{u^{2}}{2n}\right)}$$

Proof that Stirling’s Formula approximates the Gamma Function

- Substituting the Maclaurin Series approximation back into

$$\int_{-n}^{\infty}\left(n\left[\frac{u}{n}+1\right]\right)^{n} e^{-(u+n)}du = \int_{-n}^{\infty} n^{n}\left(\frac{u}{n}+1\right)^{n} e^{-u}e^{-n}du = \left(\frac{n}{e}\right)^{n}\int_{-n}^{\infty}\left(\frac{u}{n}+1\right)^{n} e^{-u}du$$

as $n \to \infty$

$$n! \sim \left(\frac{n}{e}\right)^{n}\int_{-n}^{\infty} e^{\left(u-\frac{u^2}{2n}\right)} e^{-u}du = \left(\frac{n}{e}\right)^{n}\int_{-\infty}^{\infty} e^{\left(-\frac{u^2}{2n}\right)}du$$

Proof that Stirling's Formula approximates the Gamma Function

which by the lemma is

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n$$

which takes us back to Stirling's Formula.

This would be a good enrichment exercise for Calculus II students.

How good of an estimate of n! is Stirling's Formula?

- As $n \to \infty$

how closely does Stirling's Formula approximate n!?

One way to look at this is to look at

$$\lim_{n\to\infty} \frac{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n}{n!}$$

How good of an estimate of n! is Stirling's Formula?

- Because of n!, it would be impossible to apply L'Hospital's Rule. However, using Maple based software,

$$\lim_{n\to\infty} \frac{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n}{n!} = 1$$

indicating that as n increases, Stirling's Formula becomes n!.

- This would be a good enrichment exercise for Calculus 2 students when learning L’Hospital’s Rule.

Conclusion of my historical research

- I was interested in reading De Morgan's work on probability (a first edition presentation copy I just added **without my wife's knowledge** to my antiquarian math book collection) to get an historical perspective on the subject.
- It was gratifying that I ended up developing enrichment exercises for classes from Intermediate Algebra through Multivariable Calculus, and learned more Mathematics History along the way that can be used in the secondary and college classroom.

James Stirling

- **Born: May 1692 in Garden (near Stirling), Scotland**
  Died: 5 Dec 1770 in Edinburgh, Scotland
- Published his most important work *Methodus Differentialis* in 1730. This book is a treatise on infinite series, summation, interpolation and quadrature.