A Visual and Contextual Approach to Key Fraction Concepts

Matt Jones, California State University Dominguez Hills

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Goals:

- Learn about multiple ways to compare fractions, including same number of parts, same-size parts, compare to reference point, and ratio addition. Learn when particular approaches to fraction comparison might be more appropriate.
- Learn how to use the number line and an area model as ways to represent multiplication of fractions.
- Learn how to divide fractions without invert-and-multiply.

Guidelines:

The general outline of events will be: After a brief introduction, I will ask you to begin working on the problems. You should work alone first, and then compare your ideas with others seated near you. As you work, I will observe and provide assistance or clarification as needed. Then, some of you will share your solutions at the document camera. Please ask questions or offer your comments or your alternative solution ideas. After we have shared, we will return to work, and then we will proceed to share again, repeating this sequence until we have completed most of the handout. If you work fast, and you have compared your solution with others, you are welcome to move on to additional problems while you wait for the whole group discussion to begin.

Problems:

1. Five people are going to share 3 identical sandwiches equally. What is each person's share?
2. Juan has a submarine sandwich cut into 5 equal sections, and he decides to eat 3 of these sections for lunch. How much of the sandwich did Juan eat?
3. Compare $\frac{4}{7}$ and $\frac{5}{11}$ using any method you know.

Comparing Fractions:

Problems **??** to **??** are about comparing fractions. Rather than immediately using your own favorite method for comparing fractions, try to use the situation given to help you compare the fractions. Then, if you like, solve the problem another way.

4. Four identical candy bars are going to be shared equally by the 7 kids sitting at Lunch Table A, while 5 of the same kind of candy bars are going to be shared equally by the 11 kids sitting at Lunch Table B. Who gets more candy, a kid at Table A or a kid at Table B?
5. Brandi is on the basketball team. In the first half, she makes 4 of her 7 free throws. In the second half, she made 1 of her 4 free throws. Was her shooting percentage higher at the end of the first half or after both halves?
6. Mrs. Day keeps a jar of pencils on her desk. Currently, there are 77 pencils in the jar. Of these pencils, $\frac{4}{7}$ are mechanical pencils, while $\frac{5}{11}$ of the pencils are yellow. Are there more pencils that are mechanical, or more that are yellow?
7. Kurt has some coins in each of his two pockets. In his left pocket, 4 out of 7 of his coins are dimes. In his right pocket, 5 out of 11 of his coins are dimes. In which pocket are more than half of Kurt's coins dimes?
8. The school chess club has 9 boys and 11 girls. Five of the boys have brown hair, and five of the girls have brown hair. Is the fraction of brown-haired boys on the chess club more or less than the fraction of brown-haired girls on the chess club?
9. Kenji and Lin each bought a 12-pack of soda. Kenji took $\frac{2}{3}$ of his soda to work, while Lin took $\frac{3}{4}$ of her soda to her work. Who took more soda to work?

10. One day, the school math club orders 5 identical pizzas. At Table A, 5 people share 3 of the pizzas, while at the Table B, 3 people share 2 pizzas.
    (a) Which table gets more pizza per person? Explain using a diagram.
    (b) Rebecca solved the problem this way: At Table A, 5 people sharing 3 pizzas is the same as 10 people sharing 6 pizzas. At Table B, 3 people sharing 2 pizzas is the same as 9 people sharing 6 pizzas. Since more people have to share the same number of pizzas at Table A, Table B gets more pizza per person. Explain what Rebecca is thinking.
    (c) Use Rebecca's method to compare Table A with 2 pizzas shared by 7 people to Table B with 4 pizzas shared by 13 people.
11. Andrea earned 8 out of a possible 11 points on her first math assignment. She earned 5 out of 5 points on her second assignment, so that she now has 13 out of 16 points overall. Was Andrea's grade higher immediately after her first assignment, or after her second assignment?
12. Camilla bought 10 apples, of which 7 were green apples. Damon bought 14 apples, of which 11 were green apples.
    (a) For which person are more than $\frac{3}{4}$ of the apples green?
    (b) Compare Camilla and Damon's apples using addition of ratios.
    (c) If Camilla bought 70 apples and the same fraction of her apples were green as when she bought 10 apples, and Damon bought 70 apples, and the same fraction of his apples were green as his earlier purchase, how many green apples would they each have?

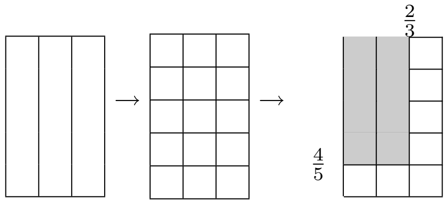
    For each of problems **??** to **??**, try several different ways of comparing the fractions, and see which one(s) work best for which kinds of fractions.
13. Compare $\frac{5}{9}$ and $\frac{15}{29}$.
14. Compare $\frac{7}{8}$ and $\frac{5}{6}$.
15. Compare $\frac{13}{15}$ and $\frac{13}{19}$.
16. Compare $\frac{11}{15}$ and $\frac{22}{28}$.
17. Compare $\frac{9}{19}$ and $\frac{11}{21}$.
18. Describe what kinds of fractions lend themselves to being compared with each of the strategies: same number of parts, same-size parts, ratio addition, and compare to a reference point.

Multiplying Fractions:

1. A square mile is a unit measurement of area equal to a square with sides that measure one mile. For example, the Greater Los Angeles Region has an physical area of 4,850 square miles of land. The shape of the region is of course not a square. The area of the region is equivalent to 4,850 one mile by one mile squares. Suppose a farm is $\frac{3}{4}$ mile by $\frac{1}{5}$ of a mile. How big is the farm, measured in square miles?
2. Kara has a small garden $1\frac{2}{3}$ meters by $\frac{4}{5}$ meter. Find the area of the garden in square meters. Justify your answer with a diagram.
3. Alejandro is running a race that is $\frac{4}{5}$ of a kilometer. He has just gotten to the halfway mark in the race. How far has Alejandro run? Justify your answer with a diagram.
4. Phuong has a ribbon that is $\frac{5}{8}$ of a meter long. She wants to use $\frac{1}{3}$ of the ribbon to tie a birthday present for her friend. How much ribbon will she use? Justify your answer with a diagram.

As with whole number multiplication, the area model is a way to find the product of two fractions so that there is a visual way to interpret the product. For instance, to find the product $\frac{2}{3} \times \frac{4}{5}$, we will draw a rectangle. The rectangle is then cut vertically into thirds, and horizontally into fifths, and finally, the rectangle is shaded so that what is shaded has a width of $\frac{2}{3}$ of the width of the rectangle, and the height is $\frac{4}{5}$ of the height of the rectangle, as shown in Figure **??**. Notice that the resulting rectangle has the whole cut into fifteenths, and eight of them are shaded, so that the answer is $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

Figure 1: A rectangle cut vertically into thirds, then horizontally into fifths, and finally shaded



Show how to find each product in problems **??** to **??** using an area model.

5. $\frac{1}{3} \times \frac{1}{4}$
6. $\frac{3}{5} \times 4$
7. $\frac{5}{9} \times \frac{3}{4}$
8. $\frac{7}{8} \times \frac{2}{3}$
9. $\frac{5}{7} \times \frac{3}{2}$
10. Bonus: Use a diagram to explain how an area model can be used to show $.3 \times .7$.

Dividing Fractions:

For each of Problems **??** to **??**,

- Represent and solve the problem using either a diagram or a number line.
- Clearly label the unit of measure (one cup, one tablespoon, etc.) in your diagram of the problem.

1. Enrique has $2\frac{1}{4}$ cups of orange juice. He drinks $\frac{3}{4}$ of a cup of orange juice every morning. How many days will Enrique's orange juice last?
2. Tran has a recipe calling for $\frac{2}{3}$ tablespoon of vanilla. She has $3\frac{1}{3}$ tablespoons of vanilla. How many times can she make the recipe?
3. Delia has $7\frac{1}{2}$ pounds of concrete mix. If it takes $1\frac{1}{2}$ pounds of concrete mix to anchor a fence post, how many posts can she anchor?
4. Norma has $3\frac{2}{3}$ ounces of frosting. It takes $\frac{1}{6}$ ounce of frosting to make a baby rose. How many baby roses can Norma put on her cake?
5. Review your answers so far. Look for a relationship between the fractions used and the answers to each problem. Describe what you observe. You are looking for a new way of dividing fractions without "invert-and-multiply."
6. Crystal plans to run 4 miles. Each day, she runs $\frac{1}{2}$ of a mile. How many days will it take to run 4 miles?
7. Faith has a hot tub 3m wide. If she wants to place new tiles along the width of the pool (just one side), and the tiles are $\frac{3}{5}$ of a meter long, how many tiles will she need? How many of the same kind of tiles would she need if she were to put them along one length of her pool, which is 15m?

8. Miguel has $4\frac{1}{2}$ gallons of gasoline for his chainsaw. If the chainsaw uses $\frac{1}{3}$ of a gallon per hour, for how many hours can Miguel use the chainsaw?
9. Maria has $5\frac{1}{6}$ ounces of perfume. She has a supply of bottles of which each holds $\frac{5}{6}$ of an ounce. How many bottles can she fill? How much perfume goes in the last bottle? What fraction of the last bottle will be filled?
10. Dina notices that it takes $\frac{2}{3}$ of a minute to fill a water bottle. If she spends 5 minutes filling water bottles, how many bottles will she fill? How full is the last water bottle?
11. Alberto solved Problem **??** and got an answer of $7\frac{1}{2}$. Bianca solved the same problem and got an answer of 7 remainder $\frac{1}{3}$. How would you respond to each student?
12. Represent and solve the following problem using a diagram or a number line: Ariel has a piece of fabric $2\frac{1}{4}$ meters long. She needs to cut the fabric into sections $\frac{3}{8}$ of a meter long.
    (a) How many sections can she get from her fabric?
    (b) How much cloth will be left over?
13. Represent and solve the following problem using a diagram or a number line: Magda has $2\frac{1}{3}$ gallons of lemonade. She needs to pour all of her lemonade into pitchers that hold $\frac{1}{2}$ gallon each.
    (a) How many pitchers can Magda fill completely?
    (b) How much lemonade will go in the last pitcher, in gallons?
    (c) How full is the last pitcher?
14. It takes $1\frac{2}{3}$ pizzas to feed a junior league basketball team. If the league gets a donation of $5\frac{2}{3}$ pizzas, how many teams can be fed?
15. Reflect on the problems in this section by answering the following: When dividing fractions, how do you find the remainder? How do you deal with remainders? How do you get your answer as a mixed number?

Use the common denominator method to divide the fractions shown in problems **??** to **??**.

16. $\frac{8}{9} \div \frac{2}{3}$
17. $\frac{9}{20} \div \frac{3}{5}$
18. $\frac{12}{55} \div \frac{2}{11}$
19. $\frac{10}{9} \div \frac{5}{6}$
20. $\frac{30}{7} \div \frac{5}{7}$
21. Besides invert-and-multiply and common denominator approaches, there is a third way to divide fractions. However, it is sometimes inconvenient. This method is "dividing straight across," meaning that you divide the numerators, and divide the denominators. Review problems **??** to **??** and see if you can see how to apply this idea.

Further Reading: I highly recommend *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*, Third Edition, by Susan J. Lamon.