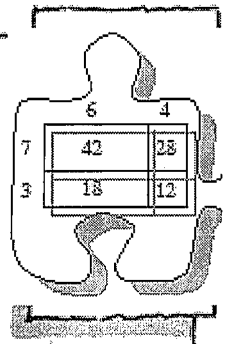
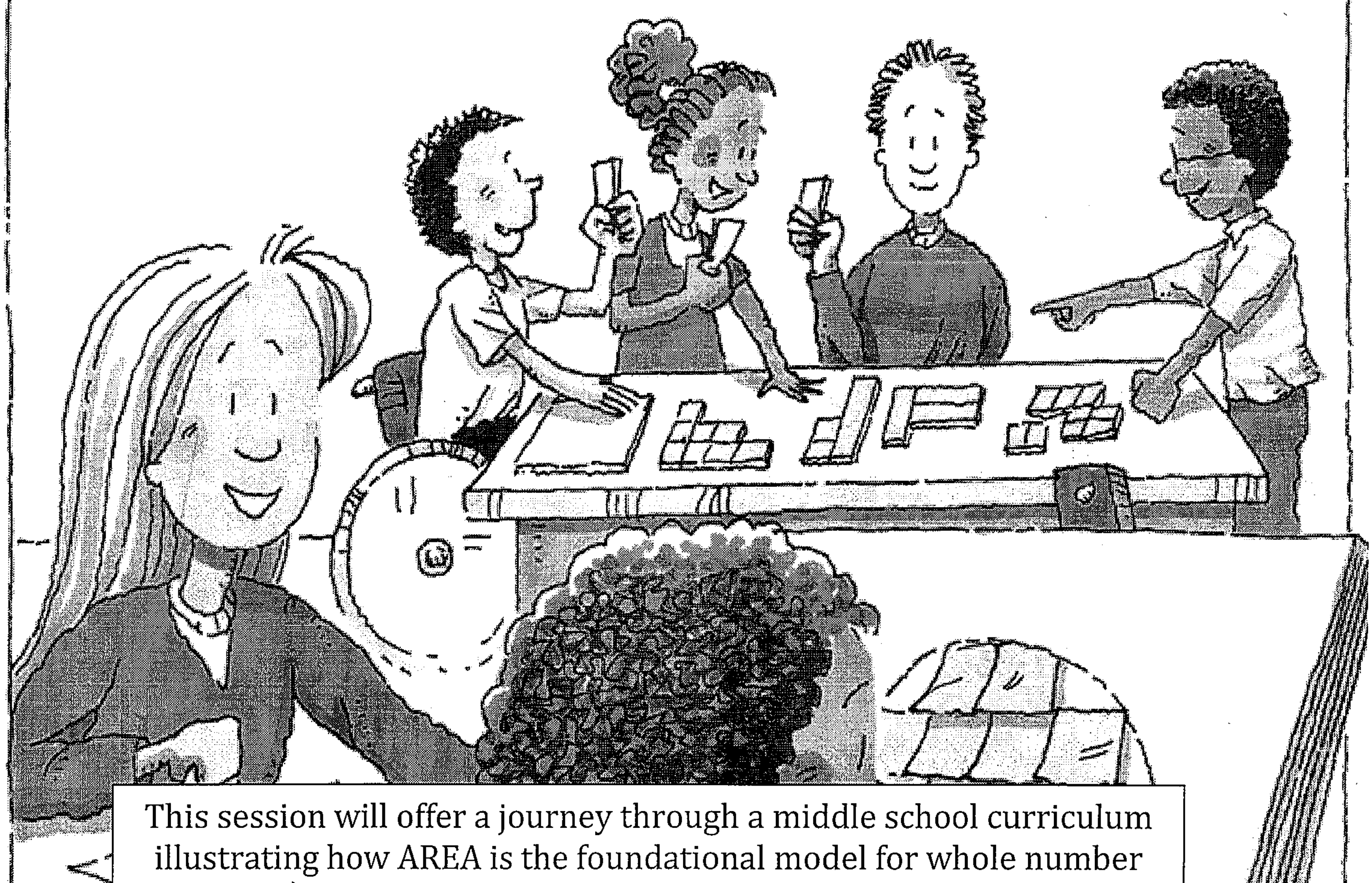


Come Take a Modeling Journey with Area

presented by
Erica Warren



This session will offer a journey through a middle school curriculum illustrating how AREA is the foundational model for whole number multiplication, operations with fractions, distributive property and probability. Participants will investigate problems that frequently offer a challenge to students using manipulatives as visual models of

Materials from Making Connections: Foundations for Algebra Year 1 and 2, and Algebra Connections Courses

CPM

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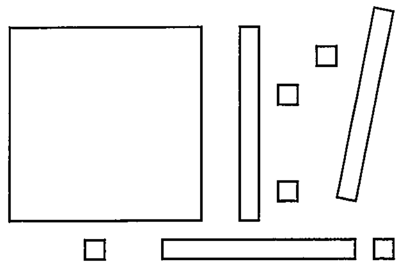
Operations with Whole Numbers

Addition and Subtraction of Whole Numbers

3-11. The Base 10 blocks represent the main grouping strategies for *our* counting system. In this activity, you will work with your team to investigate the blocks, their relationships to each other, and how they can help you to add numbers correctly. Be sure to use the blocks in a way that all team members can see and touch them.

a. **Compare** the sizes of the blocks. If the smallest block represents the number 1, what do the other blocks represent? Using words or pictures, **explain** how you know what the other blocks represent.

b. The set of blocks can be combined to form new numbers. For example, the blocks drawn at right represent the number 135.

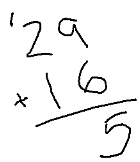


How could blocks be used to represent the number 127? With your team, find **at least two ways** to build 127.

3-12. Can Base Ten Blocks also help us add? Can they help us subtract?

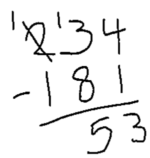
a. How would you use the blocks to add 16 and 29? What is the least number of blocks you can use to represent the result? Do you need to exchange or regroup any of the blocks in order to use fewer blocks? Build, sketch, and record your work.

b. Cecilia started part (a) by adding 16 and 29 without the Base Ten Blocks but was called to the principal's office before she was finished. Her work is at right. This method is called the **standard addition algorithm**. Finish the problem and **explain** how her work uses the idea of Base Ten Blocks.



c. How would you use the blocks to subtract $234 - 181$? What is the least number of blocks you can use to represent the result? Again, how have you regrouped or exchanged blocks in order to accomplish this task? Build, sketch, and record your work.

d. Again Cecilia wanted to subtract $234 - 181$ without her Base Ten Blocks. Her work is shown at right. This method is called the **standard subtraction algorithm. Explain** how this method uses the idea of Base Ten Blocks.



3-14. BLOCK BOGGLE

Work with your team to use the clues below to figure out how many Base Ten Blocks each set of students must have had.

a. Carol had three more blocks than Leslie. Leslie's blocks represent 26. Together they had 19 blocks and the total number of blocks represented the number 82. What blocks could each girl have? What is the value? What number does each set of blocks represent?

b. Van and Jaime had 24 blocks altogether. The total number represented by their blocks was 411. Van had 8 more blocks than Jaime. Jaime's blocks represented 350. What blocks could each boy have? What number does each set of blocks represent?

Multiplication of Whole Numbers

3-45. Take 12 one blocks and make as many different rectangles as you can. What is the area of each rectangle? What are the dimensions of each one? What is the relationship between the area of the rectangle and its dimensions.

3-59. Alan is working with 1 one-hundred block, 5 ten-blocks, and 6 one-blocks, and he wants to make another rectangle.

a. Obtain blocks from your teacher and work with your team to help Alan arrange his blocks into a rectangle. Is there **more than one way** to do this? Be prepared to share your ideas with the class.

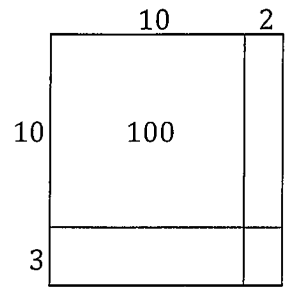
b. Sketch two of your rectangles on your paper and label their **dimensions** (length and width). Are all of the dimensions the same or are some of them different?

c. Alan labeled the dimensions of his rectangle "$10+2$" and "$10+3$." Why might these labels **make sense**?

d. Which of the possible arrangements makes it easiest to see the dimensions and area of the rectangle? Contribute your ideas to the class discussion and then sketch the rectangle that you chose.

e. How are the total value of the blocks and the dimensions of the rectangle related? If the 1-block has one square unit of area, what is the area of Alan's rectangle? How did you determine your answer?

3-60. Alan has another idea! This time, he is trying to make it easier to multiply $12 \cdot 13$ and get an exact answer. Instead of building the product with blocks as he did in problem 3-59, Alan drew the diagram at right.



Examine Alan's diagram and discuss with your team how it relates to the shape he built with blocks. Why did he label the lengths on the sides 10, 3, 10, and 2?

a. The shape that Alan drew is called a **generic rectangle**, because it can be used to represent rectangles of different dimensions without having to draw each individual block.

   Find the areas represented by the four smaller rectangles; the first one is partially done for you. In the box, show how Alan got "100." Then fill in the other three smaller rectangles the same way.

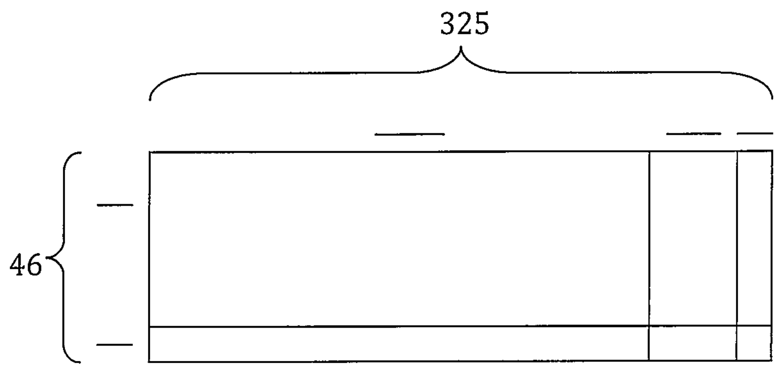
a. How can you find the total area represented by the entire rectangle? Work with your team to find at least two ways to do this.

b. Alan would like to use this strategy to do other multiplication problems that are more complex. Work with your team to help Alan draw a generic rectangle to multiply $59 \cdot 46$ and find the product.

3-61. Alan wanted to use his new idea of generic rectangles to help him multiply 325 and 46. He started the diagram below.

a. Fill in the blanks with numbers that represent the dimensions of the rectangle as broken apart (or **decomposed**) into their place value components. (Note that Alan estimated the different lengths in the rectangle, so they may not be the right size to match with Base Ten Blocks.)

c. Find the product by calculating and adding the areas represented by each of the six small rectangles.



d. Write your answer as a numerical multiplication sentence.

3-62. Draw generic rectangles to help you multiply the following numbers without a calculator. Show the place value of each part of the rectangle, the area of each part, and the area of the whole rectangle. Make sure your solution is written in a numerical sentence.

a. $25 \cdot 18$

b. $153 \cdot 25$

c. $472 \cdot 57$

d. $289 \cdot 77$

3-68. WHICH PARTS MATTER?

Alan is still thinking about multiplication. Now he wants to know how much the ones digits really affect most products. He drew the generic rectangle at right.

a. What multiplication problem is represented by Alan's generic rectangle?

b. Which of the four sections of the area is the largest? Does this area represent a multiplication problem you can do in your head? What is that product?

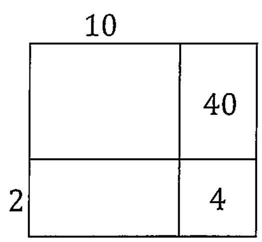
c. Now calculate the exact product for Alan's generic rectangle. How do the exact answer and the calculation that you did in your head in part (b) **compare**?

d. Can you estimate Alan's product in your head and get a more accurate answer than 3000? Discuss this with your team and be prepared to **describe** your ideas to the class.

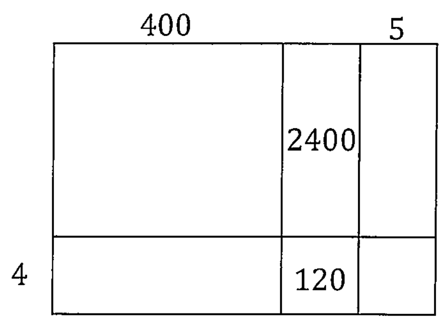
3-70. GENERIC RECTANGLE PUZZLES

Work with your team to fill in the missing parts of each of the generic rectangles below. Then for each rectangle, write the product as a numerical sentence in the form: (total length)(total width) = area part one + area part two + area part three + area part four = total area. Can you find **more than one** possibility for any of these rectangles?

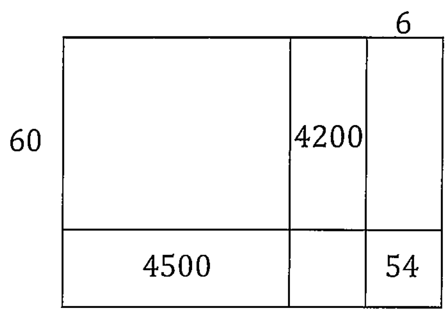
a.



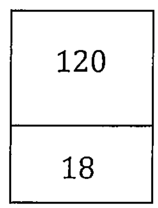
b.



c.



d.



3-71. Ethan was trying to show that $5(10+3)$ is the same as $50+15$. Is he right? Draw a diagram to demonstrate Ethan's idea.

3-80. Tammy has always multiplied two-digit numbers using the **standard multiplication algorithm** and she wonders how her new knowledge of generic rectangles might connect to that. Ingrid complicates matters by showing a third way to multiply.

For the product $67 \cdot 53$, Tammy's method, Ingrid's method, and a generic rectangle are shown below. Work with your team to **compare** these methods and **explain** how they are alike and how they are different.

Tammy's Method

$$\begin{array}{r} {}^{3}_{2} \\ 67 \\ \times\, 53 \\ \hline 201 \\ +\,3350 \\ \hline 3551 \end{array}$$

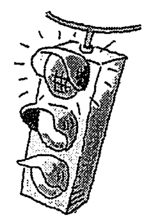
	60	7
50	3000	350
3	180	21

$53 \cdot 67$
$= 3000 + 350 + 180 + 21$
$= 3551$

Ingrid's method

$$\begin{array}{r} 67 \\ \times\, 53 \\ \hline 21 \\ 180 \\ 350 \\ +\,3000 \\ \hline 3551 \end{array}$$

3-81. Martin thinks he has found a shortcut for multiplying two-digit numbers! When he multiplied $22 \cdot 35$, he thought of 22 as $20 + 2$ and 35 as $30 + 5$. Then he multiplied the tens together ($20 \cdot 30 = 600$) and multiplied the ones together ($2 \cdot 5 = 10$) and added the results ($600 + 10 = 610$).



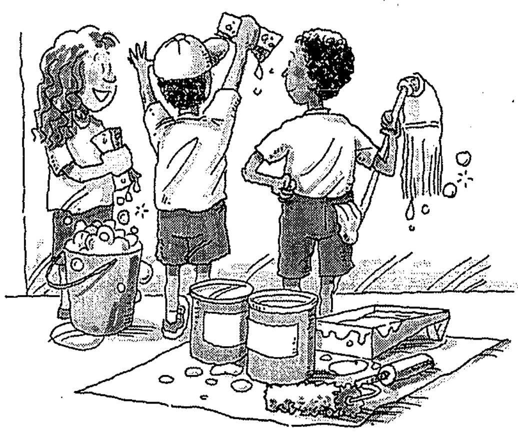
Then Martin checked his work by multiplying $22 \cdot 35$ on his calculator and, to his dismay, he got 770!

Work with your team to **explain** to Martin why his method will not give exact products.

Operations with Fractions

Multiplication of Fractions

6-44. MURAL MADNESS



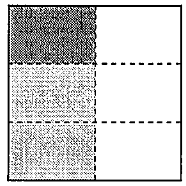
Riley, Morgan, and Reggie were planning a mural on the side of a community center and they needed to clean and seal the wall before painting it. Riley agreed to prepare $\frac{1}{2}$ of the area, Morgan agreed to do $\frac{1}{3}$ of the area, and Reggie agreed to do the remaining $\frac{1}{6}$ of the area.

A few days later, none of them had completed all of his or her section. Riley had completed $\frac{1}{3}$ of his part. Morgan had completed $\frac{5}{6}$ of her part. Reggie had completed $\frac{2}{3}$ of his part.

Your task: Work with your team to decide:

- Who has completed the least of the total mural area?
- Who has completed the most of the total mural area?

6-46. Janine drew the square diagram at right as she was working on Riley's portion of the previous problem. Her brother James looked over her shoulder and asked, *"Oh, you're learning about area?"*

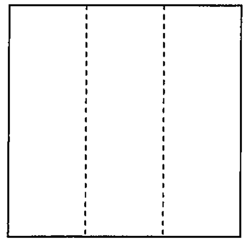


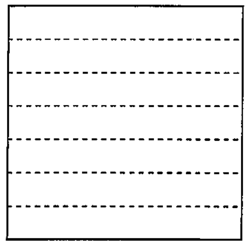
"Why do you say that?" Janine asked.

He answered, *"It looks like you have shaded a rectangle with a length of* $\frac{1}{3}$ *unit and a width of* $\frac{1}{2}$ *unit, and you have shaded its area."*

Is James right? What is the area of the darkly shaded rectangle in Janine's diagram? Write the area as a product of length and width.

6-47. For each product below, choose one of the diagrams at right that might be useful. Copy the diagram on your own paper and complete it to find the product.





a. $\frac{3}{4} \cdot \frac{1}{3}$

b. $\frac{1}{5} \cdot \frac{1}{7}$

c. $\frac{3}{7} \cdot \frac{2}{5}$

d. $\frac{4}{4} \cdot \frac{2}{3}$

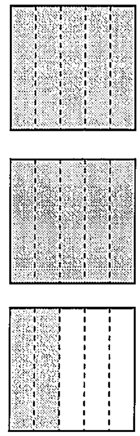
6-56. Grace and William were wondering if *one half of a third* would be the same as *one third of a half.*

a. Draw a picture that shows one half of one third.

b. Draw a picture that shows one third of one half.

c. Write a note to Grace and William **explaining** how these two values **compare** and why the result **makes sense**.

6-58. George drew the diagram at right to represent the number $2\frac{2}{5}$. *"Look,"* said Helena, *"This is the same thing as* $\frac{12}{5}$*."*



a. Is Helena correct? If so, **explain** how she can tell that the diagram represents $\frac{12}{5}$. If she is not correct, **explain** why not.

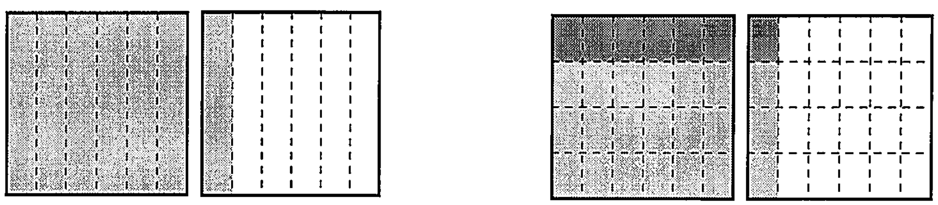
b. Draw a diagram to represent the mixed number $3\frac{2}{3}$. How can you write this as a single fraction greater than one?

b. How can you write $\frac{7}{4}$ as a mixed number? Be sure to include a diagram in your answer.

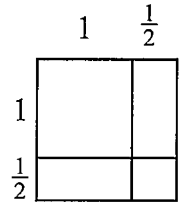
6-64. Each of the pairs of diagrams shows a first and second step that could be used to represent a multiplication problem. For each pair, write a multiplication problem and its solution. Be prepared to share your ideas with the class.

b.

d.



6-76. Mrs. McElveen plans to devote a $1\frac{1}{2}$-meter by $1\frac{1}{2}$-meter section of the school garden to tomato plants. She is wondering how much area would be devoted to tomatoes. Owen made the sketch below right to help determine the area. With your team, answer the following questions.



a. **Explain** how this sketch shows a $1\frac{1}{2}$-meter by $1\frac{1}{2}$-meter area.

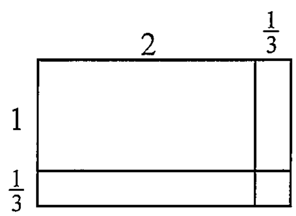
b. Complete the generic rectangle at right.

c. How much area in the school garden is devoted to the tomato plants?

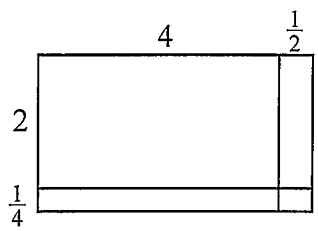
6-77. In the previous problem, you multiplied two mixed numbers using a generic rectangle. For each of the generic rectangles below:

- Write the two numbers that are being multiplied, that is, the length and width of the rectangle.
- *Predict* the approximate size of the product and be ready to **explain** your thinking.
- Copy the rectangle on your paper and use it to multiply the given numbers.
- Write each answer as a complete multiplication sentence. (An example of a multiplication sentence from problem 6-76 would be $1\frac{1}{2} \cdot 1\frac{1}{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 2\frac{1}{4}$.)
- **Compare** the exact answer with your prediction. How close did you get?

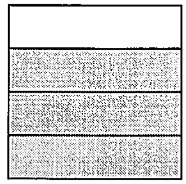
a.



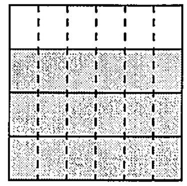
b.



7-77. Murph and Hanson were working on a problem with adding fractions, when Murph had an idea. *"I wonder,"* he said, *"if we can use an area model to figure out the sizes of pieces that we need for these fractions. How could we try it for $\frac{3}{4}+\frac{5}{6}$?"*

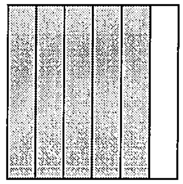


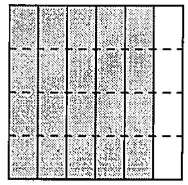
"Let's start by drawing $\frac{3}{4}$," said Hanson as he drew the diagram at right. *"Now how can we figure out what size pieces will work also with sixths?"*



"I have an idea." said Murph. *"What if we split the rectangle into sixths in the other direction like this? Then we can see that $\frac{3}{4}$ is equal to $\frac{18}{24}$."*

"Okay," said Hanson. *"Let's see if this works for $\frac{5}{6}$ also. We can start by drawing $\frac{5}{6}$ like this and then break them into fourths. Then we can see that $\frac{5}{6}$ is equal to $\frac{20}{24}$."*





a. Work with your team to finish Murph and Hanson's work to add $\frac{3}{4}+\frac{5}{6}$. Does the result agree with the one you got in part (b) of problem 7-76? **Explain.**

b. Draw an area model to help you add $\frac{1}{3}+\frac{4}{7}$.

7-78. As you have discovered, any fraction can be rewritten in many equivalent ways. When choosing a denominator that will work to add two fractions, there is no single correct choice. Often people find it convenient to use the smallest whole number that all denominators divide into evenly. This number is called the **lowest common denominator**.

For example, when adding the fractions $\frac{2}{3}+\frac{5}{6}+\frac{3}{8}$, you could choose to rewrite each fraction as some number of 48^{th}s or 96^{th}s, but the numbers will stay smaller if you choose to rewrite each fraction as some number of 24^{th}s, since 24 is the lowest number that 3, 6, and 8 divide into evenly (that is, without a remainder).

For each of the following sums, rewrite each fraction using the **lowest common denominator** and then add.

a. $\frac{5}{12}+\frac{1}{3}$

b. $\frac{4}{5}+\frac{3}{4}$

7-79. Earlier in this course, you used generic rectangles to multiply mixed numbers, but you were only able to estimate their product. Can your new knowledge about adding fractions help you to use generic rectangles to multiply mixed numbers? Think about the product $4\frac{2}{5} \cdot 3\frac{1}{3}$.

a. Draw a generic rectangle to find this product. Write your answer as the sum of all of the parts of the rectangle.

b. With your team, discuss how you can add the four areas to get an answer that is one number. Then find the sum.

c. Another way to find the product $4\frac{2}{5} \cdot 3\frac{1}{3}$ is to rewrite both mixed numbers as fractions greater than one and then to multiply them. Use this method to find the answer.

c. Which method do you prefer? Why?

7-87. WHO GOT MORE? HOW MUCH MORE?

In a previous lesson, you found that the "fair shares" were not truly fair. For example, each of the students in Team W received $\frac{5}{3}$ of a piece of licorice, while each of the students in Team X received $\frac{9}{5}$ of a piece.

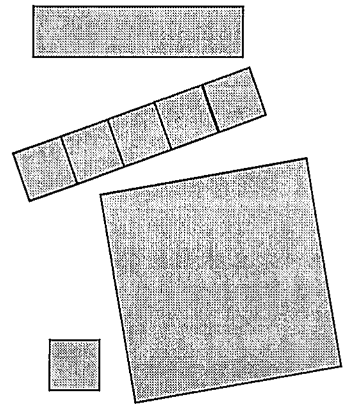
a. Which students received more licorice? How can you tell?

b. When you want the difference between $\frac{5}{3}$ and $\frac{9}{5}$, should you write $\frac{5}{3} - \frac{9}{5}$ or $\frac{9}{5} - \frac{5}{3}$? Why

c. Work with your team to find the exact difference between $\frac{5}{3}$ and $\frac{9}{5}$. Be prepared to share your ideas with the class.

Distributive Property

3-2. AREA OF ALGEBRA TILES

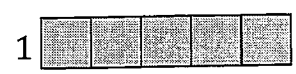


Your teacher will provide your team with a set of algebra tiles. Remove one of each shape from the bag and put it on your desk. Trace around each shape on your paper. Look at the different sides of the shapes.

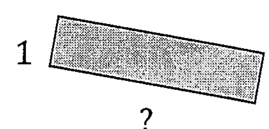
a. With your team, discuss which shapes have the same side lengths and which ones have different side lengths. Be prepared to share your ideas with the class. On your traced drawings, color code lengths that are the same.

b. Each type of tile is named for its area. In this course we will say that the smallest square has a side length of 1 unit, so its area is 1 square unit. We will call this tile "one" or the "unit tile." Can you use the unit tile to find the side lengths of the other rectangles? Why or why not?

c. If the side lengths of a tile can be measured exactly, then the area of the tile can be calculated by multiplying these two lengths together. The area is measured in square units. For example, the tile at right measures 1 unit by 5 units, so it has an area of 5 square units.



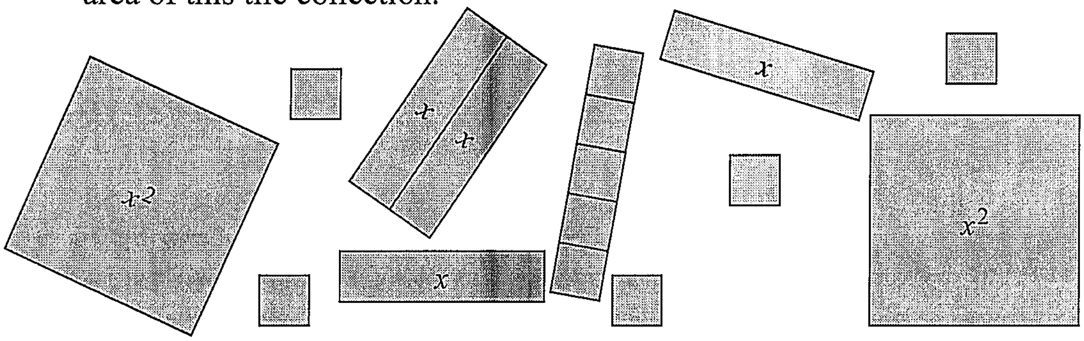
The next tile at right has one side length that is exactly one unit long. If we cannot give a numerical value to the other side length, what can we call it?



d. If we agree to call the unknown length "x," label the side lengths of each of the four algebra tiles you traced. Find each area and use it to name each tile. Be sure to include the name of the type of units it represents.

3-4. When a collection of algebra tiles is described with mathematical symbols it is called a **variable expression**. Take out the tiles shown in the picture below and put them on your table.

- Use mathematical symbols (numbers, variables, and operations) to record the area of this collection of tiles.
- Write at least three different variable expressions that represent the area of this tile collection.



6-83. Chen's sister was making a riddle for him to solve, *"I am thinking of a number. If you add two to the number then triple it, you get 9."*

a. Build the equation on an Equation Mat. What are *two* ways that Chen could write this equation?

b. Solve the equation and show your work by writing the equation on your paper after each legal move.

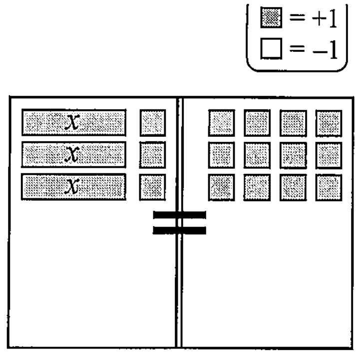
c. When Chen told his sister the mystery number in the riddle she said he was wrong. Chen was sure that he had figured out the correct number. Find a way to **justify** that you have the correct solution in part (c).

6-84. Now solve the equation $4(x+3)=8$. Remember to:

- Build the equation on your Equation Mat with algebra tiles.
- Simplify the equation using your legal moves.
- Record your work on your paper.

Solve for x. That is, find the value of x that makes the equation true.

6-89. Consider the Equation Mat at right.



a. Write the original equation represented.

b. Simplify the tiles on the mat as much as possible. What value of x will make the two expressions equal?

6-90. When Lakeesha solved the equation $3(x+1)=12$ from problem 6-89, she **reasoned** this way:

"*Since 3 groups of* $(x+1)$ *equals 3 groups of 4, then I know that each group of* $(x+1)$ *must equal 4.*"

a. Do you agree with her **reasoning**? Explain.

b. How can the result of Lakeesha's **reasoning** be written?

c. Verify that your answer from problem 6-89 will make the equation you wrote in part (b) true.

5-21. Use the Distributive Property to find each product below.

a. $6(-3x+2)$

b. $x(4x-2)$

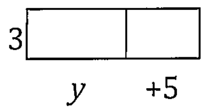
c. $5t(10-3t)$

d. $-4(8-6k+y)$

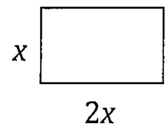
5-24. Diagrams like the one in problem 5-23 are referred to as **generic rectangles.** Generic rectangles allow you to use an area model to multiply expressions without using the algebra tiles. Using this model, you can multiply with values that are difficult to represent with tiles.

Draw each of the following generic rectangles on your paper. Then find the area of each part and write the area of the whole rectangle as a **product** and as a **sum.**

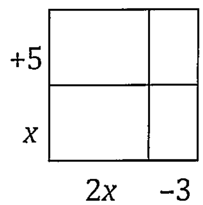
a.



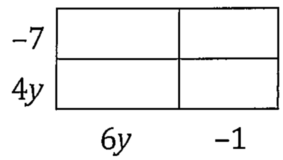
b.



c.



d.



e. How did you find the area of the individual parts of each generic rectangle?

5-25. Multiply and simplify the following expressions using either a generic rectangle or the Distributive Property. For part (a), verify that your solution is correct by building a rectangle with algebra tiles.

a. $(x+5)(3x+2)$

b. $(2y-5)(5y+7)$

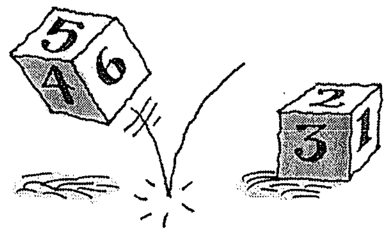
c. $3x(6y-11)$

d. $(5w-2p)(3w+p-4)$

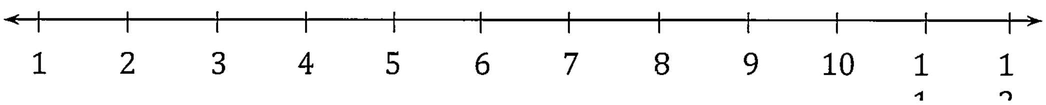
Probability

10-28. TEN O's

In this game, you will create a strategy to play a board game based on your predictions of likely outcomes. Your teacher will roll two number cubes and add the resulting numbers. As your teacher rolls the number cubes and calls out each sum, you will cross out an O over the number called. The goal of the game is to be the first person to cross out all ten of your O's.



Talk with your team about the possible outcomes of this game and then draw a number line like the one below on your own paper. Place a total of ten O's on your number line. Each O should be placed above a number, and you should distribute them based on what results you think your teacher will get. More than one O can be placed above a number.

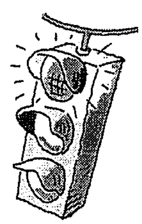


Follow your teacher's instructions to play the game.

10-29. Gerald's strategy for the Ten O's game was to place an O on each number from 1 to 10. He was frustrated that his strategy of placing his ten O's was not working, so he decided to analyze the game.

Gerald began by creating the table at right to list all of the possible combinations of rolls.

a. Did he list them all? If so, how can you be sure that they are all there? If not, give examples of a few that he has missed.



b. Does Gerald's table include information about the sums for each possible roll of the two number cubes? What could be added to help analyze this game more easily?

Cube 1	Cube 2
1	1
2	2
3	3
4	4
5	5
6	6
1	2
2	3
3	4
4	5
5	6
1	3
2	4
3	5
4	6
1	4
2	5
3	6
1	5
2	6
1	6

10-30. Gerald decided that this method was taking too long and was too confusing. Even if he listed all of the combinations, he still had to find the sums and then find the theoretical probabilities for each one. Inspired by multiplication tables, he decided to try to **make sense** of the problem by organizing the possibilities in a table like the one shown at right.

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3				
2	3	4				
3	4					
4						
5						
6						

a. How does Gerald's table represent the two events in this situation? What should go in each of the empty cells? Discuss this with your team and then complete Gerald's table on your own paper.

b. How many total possible number combinations are there for rolling the two cubes? Is each combination equally likely? That is, is the probability of getting two 1's the same as that of getting two 2's or a 3 and a 1?

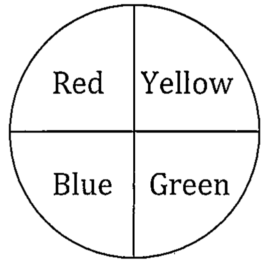
c. How many ways are there to get each sum? Are there any numbers on the game board that are not possible to achieve?

d. What is the theoretical probability for getting each sum listed on the Ten O's game board?

e. Now work with your team to determine a better strategy for Gerald to place his ten O's on the game board that you think will help him to win this game. **Explain** your strategy and your reasoning.

10-31. Gloria and Jenny each have only one O left on their game board. Gloria's O is at 6, and Jenny's is at 8. Which student is more likely to win on the next roll? **Explain**.

10-40. Imagine spinning the spinner at right two times.



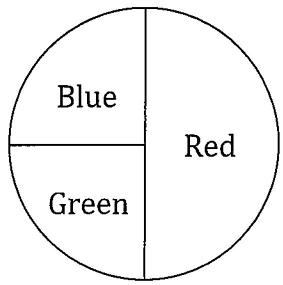
a. What would be the probability of spinning a red on your first spin? Write this as a fraction. What would be the probability of spinning a red on your second spin (also as a fraction)? Are the outcomes of these two spins dependent or independent? **Explain**.

b. Work with your team to make a probability table to represent this situation. What is the probability for spinning each color on the first spin? The second spin? How can you see these fractions in your table?

c. What is the probability for spinning a red on the first spin and then spinning a red on the second spin? Work with your team to find **at least two ways** to find this result.

d. How are the probabilities of two individual events and the probability of the combined events related? For example, how could you find the probability of spinning a blue and then a yellow without using a table?

10-41. Consider the spinner at right.



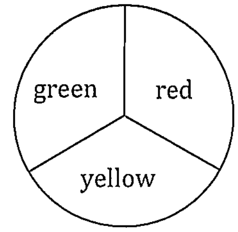
a. How is the probability for spinning each color on this spinner different from the spinner in problem 10-40? Find the probability for spinning each color for this spinner.

b. If this spinner is spun two times, will the results of the two spins be independent of each other? **Explain**.

c. Make a probability table to represent the possible outcomes of spinning this spinner twice. How is your probability table for this situation similar to and how is it different from the one in problem 10-40?

d. For this spinner, what is the probability of spinning a red and then another red? Work with your team to find this probability in **at least two ways**. That is, show how this probability can be found in the probability table and how it can be found without the use of the table.

10-42. At the school fair, students play a game called Flip and Spin, in which a player first flips a coin and then spins the spinner shown at right. If the coin comes up heads and the spinner lands on red, the player wins a stuffed animal. If the coin comes up tails and the spinner lands on yellow, the player gets another turn to play. If the spinner lands on green, the player's turn is over (whether the coin comes up heads or tails).



a. Calculate the probability that you will win a stuffed animal on the first play. Also find the probability of getting another turn and of losing your turn. Use **more than one method** to show your thinking. Be prepared to share your strategies with the class.

b. How would the outcomes change if someone were to spin first and then flip the coin? How would the outcomes change if someone was to spin and another person was to flip a coin at the same time? **Explain.**

10-43. Find the probability of each of the outcomes of compound events described below.

a. Flipping a coin and getting "heads" and picking a Jack from a standard deck of cards.

b. Rolling a five on a number cube and then rolling a six.

c. Flipping a coin and getting "tails" and then choosing the one blue marble in a bag of 12 marbles.

d. Picking the Ace of Spades from a standard deck of cards, putting it back, mixing them up and picking the Ace of spades again.