

Connect The Dots: Graph Theory in High School – Exercises

Terminology

A graph G is

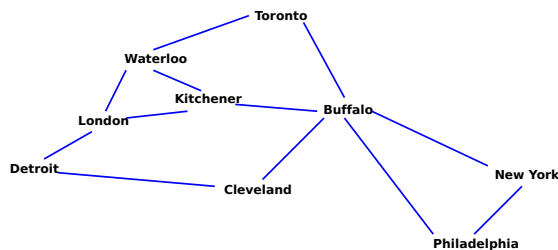
- a set V , and
- a set E of unordered pairs of distinct elements of V .

We call the elements of V *vertices* and elements of E *edges*. If $\{v, w\}$ is an edge in a graph then we say it is *incident* to its *end vertices* v and w , and we say that v and w are *adjacent*. The *degree* of a vertex is the number of edges incident to it. A graph is *n-regular* if the degree of every vertex is n . A *path* of *length* k between vertices v and w is a sequence of edges e_1, e_2, \dots, e_k where e_1 is incident to v , e_k is incident to w , and e_i and e_{i+1} share a common end vertex for all $i = 1, 2, \dots, k - 1$. The *distance* between two vertices is the length of the shortest path between them. The *diameter* of a graph is the largest distance between any two vertices in the graph.

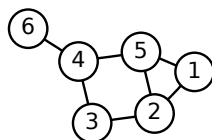
Exercises

The relative difficulty of the problems is indicated by asterisks (*easy, **moderate, ***challenging). The last two sections of this document list hints and final answers for all the problems.

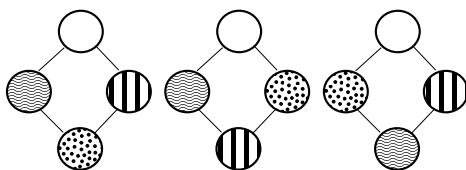
- (a) What is the fewest number of colours needed to colour the cities in the road map below so that no two cities joined by a road are the same colour? *



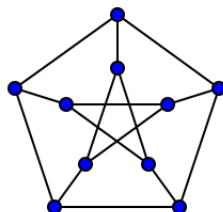
- (b) Can you prove that it is impossible to use 2 colours? **
(c) What happens when you start removing roads? *
(d) Can you draw a map requiring at least 3 colours where no group of 3 cities is fully connected (i.e. no triangles)? ** What about 4 colours with no group of 4 cities fully connected? ***
- Ten points are drawn on a circle. Each pair of points is joined by a straight line. How many straight lines are there? *
- Bob, Jamal, Erin, Hina and Ying have Facebook accounts. Maybe nobody is friends with anyone else. Alternately, it could be that everyone is friends with everyone else. A third different possibility is that every two people are friends except for Bob and Erin. How many possibilities are there in total? **
- Seven points are drawn in the plane. Straight lines are joined between some pairs of points so that no triangles are formed using the points as vertices. What is the maximum number of possible lines? **
- What is the diameter of the graph below? *



6. What is the largest possible diameter for a graph with 100 vertices? *
7. Draw every 2-regular graph with 10 vertices. Draw a 3-regular graph with 10 vertices. Explain why there isn't a 3-regular graph with 9 vertices. **
8. An n -cycle is a graph with n vertices and n edges where the edges form a cyclic path through every vertex. How many different ways can the vertices of an n -cycle be coloured with n different colours? Two colourings are considered different if any two colours are neighbours in one colouring but not the other. This means, for example, that there are 3 ways to colour a 4-cycle as illustrated below. **



9. Nine hockey teams are competing in a tournament. Every team is to play 2 games, each against a different team. (Note that not every pair of teams plays a game together.) Lynn is in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible? ***
10. Seven people in a room start shaking hands. Six of them shake exactly two people's hands. How many people might the seventh person shake hands with? **
11. A schedule is set for an n -team league in which each team plays exactly 3 games against different opponents. What are the possible values of n ? **
12. The cities of the province of Ontario are joined by roads so that no more than 5 roads leave any one city. For any two cities, there is always a route between them that passes through at most two intermediate cities. Prove that there are less than 107 cities in Ontario. ***
13. Villages are joined by canals. Exactly three canals are joined to each village. Canals do not cross each other. There is a way to travel between every pair of villages using canals by visiting no more than two other villages. For some pair of villages, the only way to travel between them by canal is by visiting two other villages along the way. What is the smallest and largest possible number of villages? ***
14. Villages are joined by canals so that no more than 4 canals leave any one village. For any two villages, there is always a canal route between them that passes through at most one intermediate village. Prove that the fewest possible number of villages is 15.
 - (a) Show that it is possible that there are 15 villages. **
 - (b) Show that 15 is the fewest possible number of villages. ***
15. Read about Euler's formula $V - E + F = 2$ for graphs that can be drawn without pairs of edges crossing (*planar* graphs). Use this fact to prove that a graph with 5 vertices and 15 edges cannot be drawn without at least one pair of edges crossing. ***
16. How can the Petersen graph (shown below) be drawn so that fewer edges cross each other? *



- (a) Draw the Petersen graph so that only two edges cross. **
- (b) Prove that the Petersen graph cannot be drawn with at most 1 pair of edges crossing. ***
17. Six points are drawn in the plane. All pairs of points are joined by a curve. What is the fewest number of pairs of curves that intersect? * Justify that your answer is correct. ***
18. An n -cube is a graph where V consists of all binary strings of length n . Two vertices are connected if and only if the corresponding strings differ in exactly one position. How many edges are there in an n -cube? **
19. Six soccer teams are competing in a tournament in Waterloo. Every team is to play three games, each against a different team. Judene is in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible? ***
20. An increasing list of two-digit positive integers is formed so that
- each integer in the list uses only digits from 1, 2, 3, 4, 5, 6,
 - each of the digits 1, 2, 3, 4, 5, 6 appears in exactly three of the integers in the list, and
 - each integer in the list has the property that its units digit is greater than its tens digit.
- How many different lists are possible? ***

Hints

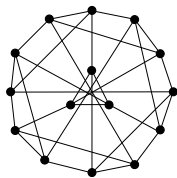
1. (a) Look for triangles. (c) Think cycles and groups nearly fully connected.
2. How many ways are there to select a first and second vertex?
3. Consider the maximum possible number of lines. Each line either exists or does not exist.
4. Consider cases for the vertex of highest degree
5. Focus on the vertex labelled 6.
6. Spread out the vertices.
7. Think cycles. Don't forget small cycles.
8. First fix the colour of one vertex.
9. Combine ideas from the previous two questions.
10. The number of hands involved in handshakes must be an even integer.
11. The sum of the degrees in a graph is an even integer.
12. Build such a graph moving successfully further from a start vertex.
13. Consider the sum of the degrees and draw a graph as hinted for the previous question.
14. Draw 12 vertices in a cycle and place a triangle in the middle.
15. $2|E| \geq 3|F|$
16. a) Slide vertices small distances. b) $2|E| \geq 5|F|$
17. $2|E| \geq 3|F|$



18. There are 2^n binary strings of length n .
19. There are two possible graphs.
20. Is this question very different from the previous question?

Final Answers and Proof Sketches

1. (a) Waterloo, Kitchener and London must all be coloured differently. (c) An odd-lengthed cycle. Next, draw a graph with 4 vertices missing exactly one edge. Draw 3 copies of this graph. Attach each copy to a distinct vertex of the original graph.
2. 45
3. 1024
4. 12
5. 3
6. 99
7. The five 2-regular 10-vertex graphs are a 10-cycle, 7 cycle and 3 cycle, 6-cycle and 4-cycle, pair of 5-cycles, 4-cycle and pair of 3-cycles. The Petersen graph (see #16) is a 3-regular graph with 10 vertices. There isn't a 3-regular graph with 9 vertices because the sum of the degrees (which must be even) would be odd.
8. $\frac{(n-1)!}{2}$
9. 30016
10. 0, 2, 4 or 6 (An instance of each needs to be constructed for full justification.)
11. Even $n \geq 4$
12. A breadth-first search tree from a start vertex will span the entire village and contain at most $1 + 5 + 5 * 4 + 5 * 4 * 4 = 106$
13. The smallest possible number of villages is 8 and the largest is 12.



14. For the proof, again consider a breadth-first search tree and consider cases at the “bottom” or “leaves” of the tree.
15. Assume Euler’s formula holds, substitute the hint into the equation, and reach a contradiction.
16. Move the top horizontal edge up over the vertex just above it and then slide the two bottom vertices of the inner vertex up. Assume Euler’s formula holds, substitute the hint into the equation, and reach a contradiction.
17. Assume Euler’s formula holds, substitute the hint into the equation, and reach a contradiction.
18. $n2^{n-1}$
19. 70
20. 70

