Problems for Breakfast

Shaking Hands

Seven people in a room start shaking hands. Six of them shake exactly two people's hands. How many people might the seventh person shake hands with?

Soccer Schedules

Six soccer teams are competing in a tournament in Waterloo. Every team is to play three games, each against a different team. Judene is in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible?

Crossing Curves

Six points are drawn in the plane. All pairs of points are joined by a curve. What is the fewest number of pairs of curves that intersect?

Connect the Dots

Graph Theory in High School

J.P. Pretti

CENTRE for EDUCATION in MATHEMATICS and COMPUTING
University of Waterloo

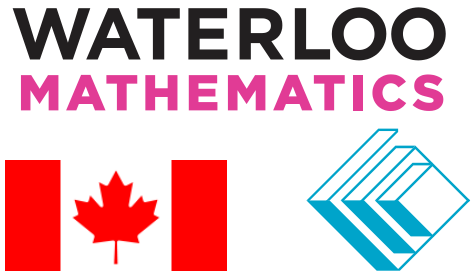
April 28, 2012

Outline

Origins and Objectives

Source of these ideas

- personal interest
- University of Waterloo
- CEMC teacher conferences and student workshops



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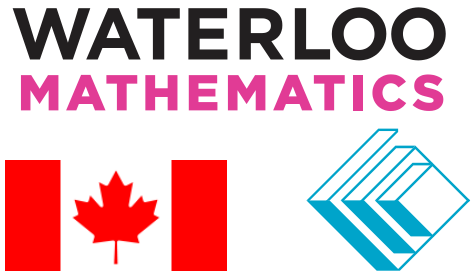

Our hour this morning

- fun and interesting
- challenging
- tangible takeaways

Central Themes

Connections

- number sense, counting, patterns, algebra, geometry
- problem solving, reasoning and proof, communication
- modeling and applications
- edges and vertices

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Problems

- contest problems
- exploratory problems
- open problems
- extra exercises

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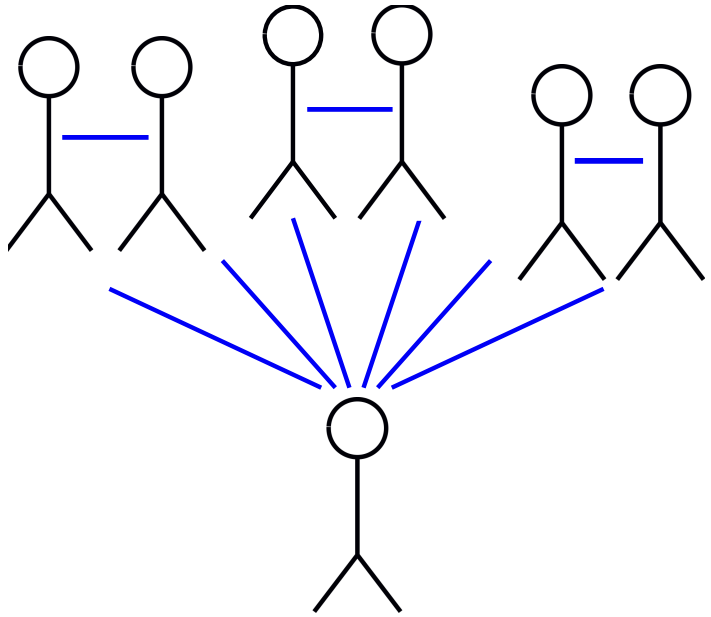

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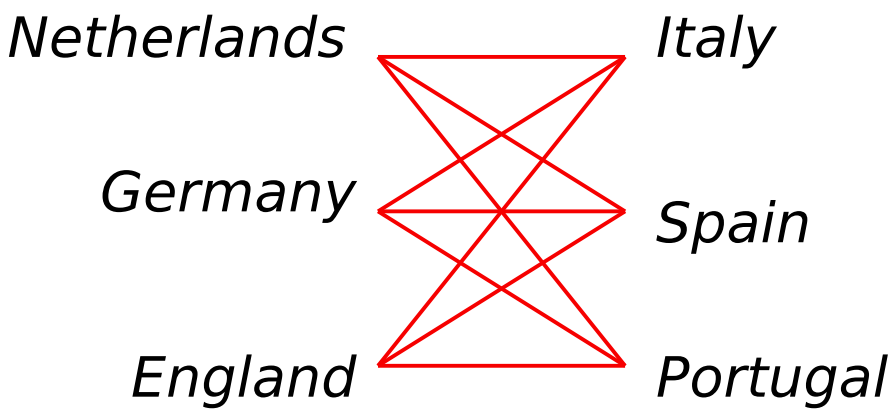

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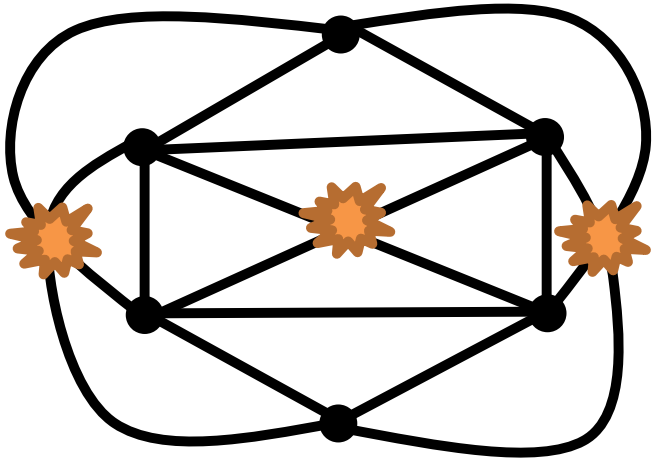


Graph Theory

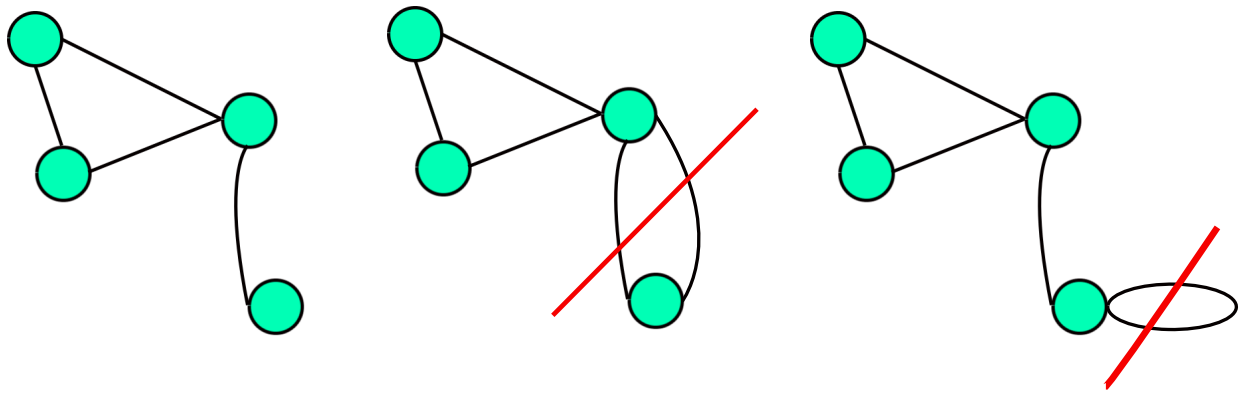
Formal Definition

A *graph* G is

- a set V, and
- a set E of unordered pairs of distinct elements of V.

We call the elements of V *vertices* and elements of E *edges*.

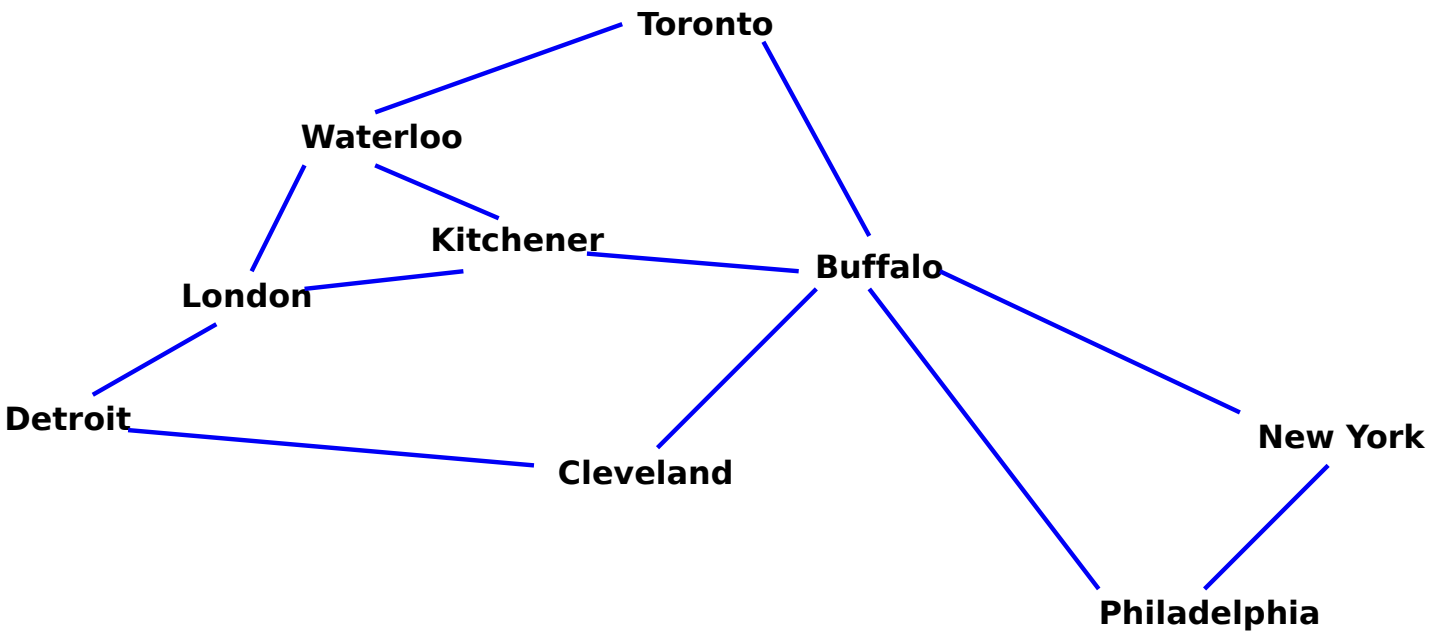
Examples



City Colouring

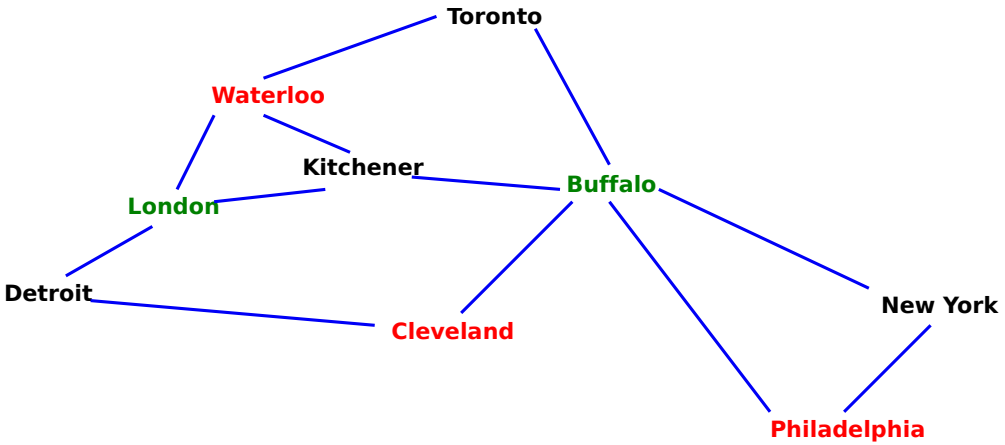
The Problem

What is the fewest number of colours needed to colour the cities in the road map below so that no two cities joined by a road are the same colour?



City Colouring

Answer: 3 colours



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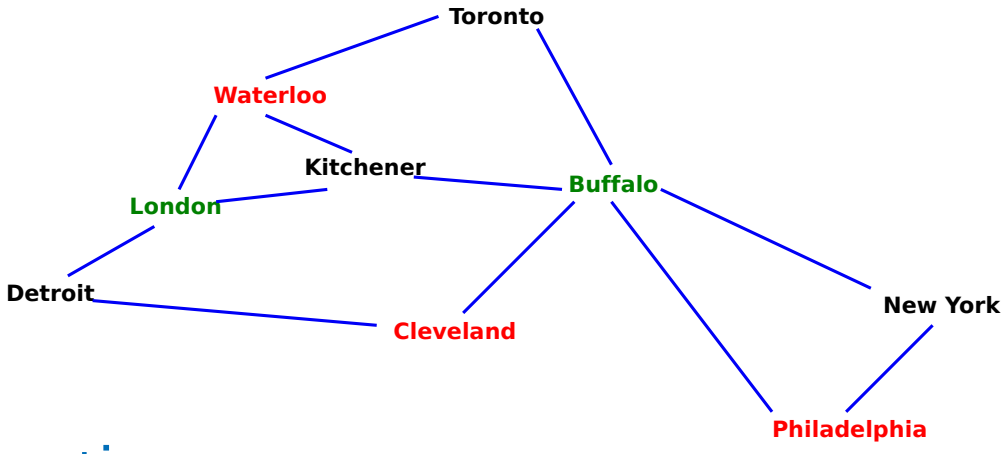

Follow-up Questions

1. Can you prove that it is impossible to use 2 colours?
2. What happens when you start removing roads?
3. Can you draw a map requiring at least 3 colours where no group of 3 cities is fully connected (i.e. no "triangles")?

Facebook Friends

The Problem

Bob, Jamal, Erin, Hina and Ying have Facebook accounts. Maybe nobody is friends with anyone else. Alternately, it could be that everyone is friends with everyone else. A third different possibility is that every two people are friends except for Bob and Erin. How many possibilities are there in total?

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Approaching a Solution

- list the possibilities
- clever counting
- consider fewer people

Facebook Friends - Groups of Size 1, 2 and 3

1 Person

 (1 possibility)

2 People

 (2 possibilities)

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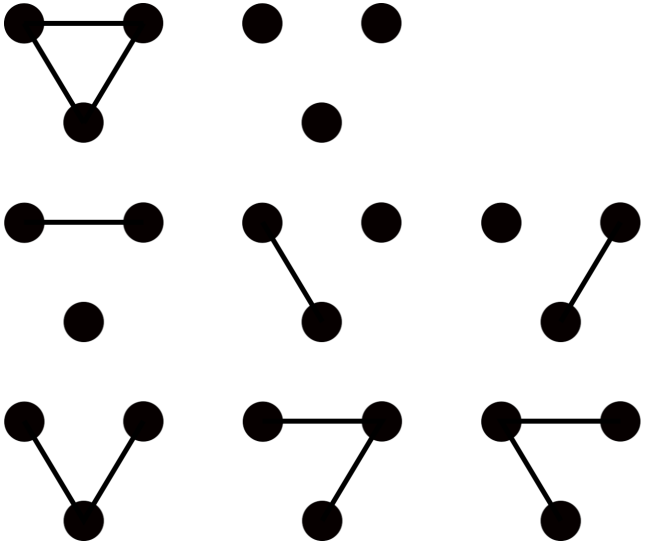
3 People



(8 possibilities)

Facebook Friends - Solution

Maximum Possible Number of Friendships

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Maximum Possible Number of Friendships

- BJ, BE, BH, BY, JE, JH, JY, EH ,EY, HY
- pair 1st person with 4 others, 2nd with 3 others, 3rd with 2 others, and 4th with 1 other
- 5 choices for 1st person, 4 for 2nd person, double counts

The number of potential friendships is 10.

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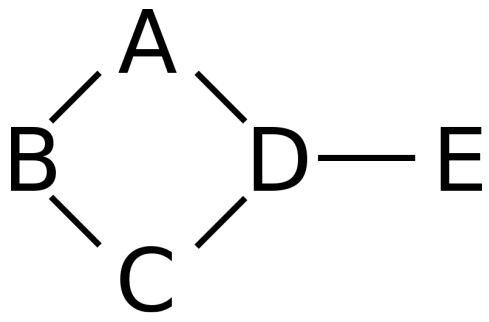
How many possibilities are there for the group of people in this room?

Diameter



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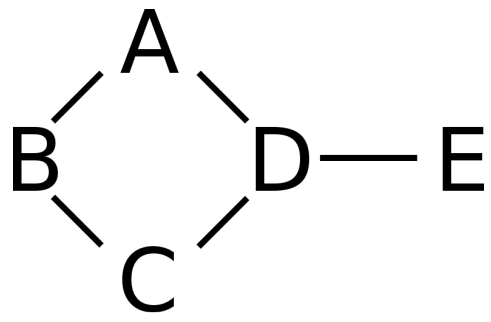

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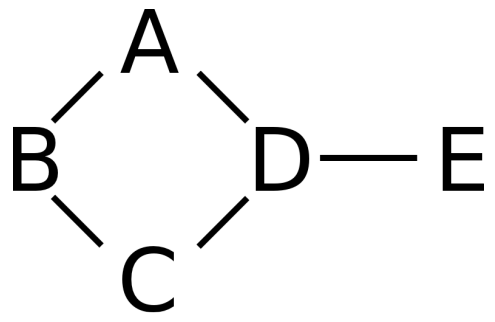

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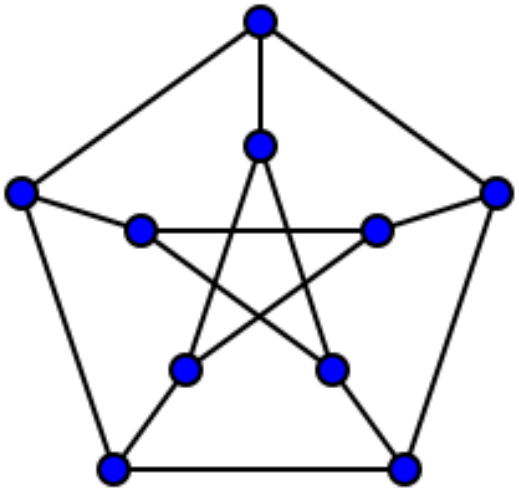
The **diameter** of a graph is the largest distance between any two vertices in the graph. (The diameter of the graph above is 3.)

Diameter of the Petersen Graph

Larger Example

What is the diameter of the following graph?



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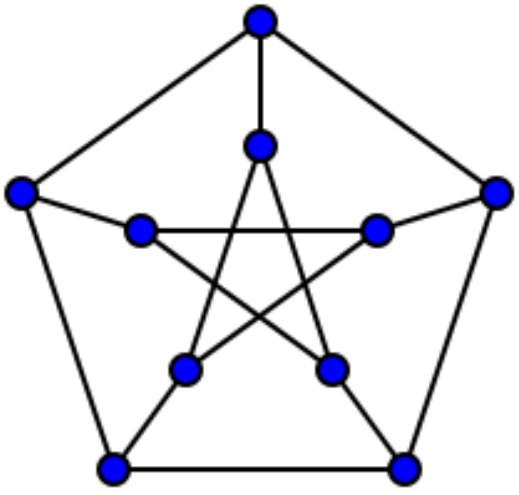


How do you convince someone else that the answer is 2?
What is the largest possible diameter for a graph with 100 vertices?
How do you find the diameter of a very large graph?

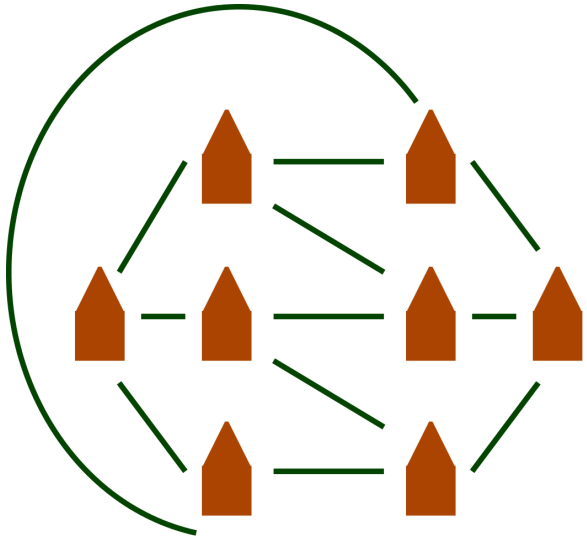
Villages and Canals

The Problem

Villages are joined by canals. Exactly three canals are joined to each village. Canals do not cross each other. There is a way to travel between every pair of villages using canals by visiting no more than two other villages. For some pair of villages, the only way to travel between them by canal is by visiting two other villages along the way. What is the smallest and largest possible number of villages?

Degree, Diameter Problem for Planar Graphs

Example



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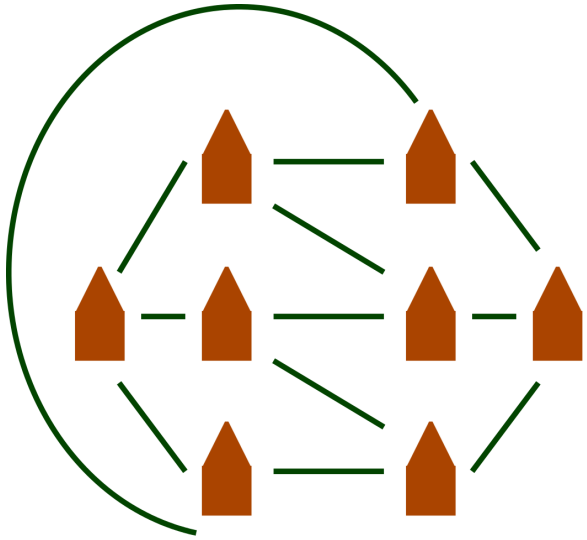

The Same Problem

What are the smallest and largest graphs (number of vertices) with **diameter 3** where every vertex is joined to **exactly 3 other vertices** that can be drawn **without any crossing edges**?

Exploring the Problem(s)

The number of vertices is even.

For each village, count the adjoining canals. This totals three times the number of villages and counts each canal twice. Hence, the total number of canals is $\frac{3}{2}$ the number of villages. This means the number of villages must be an even positive integer.

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Can you prove that no such graph with six vertices exists?

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Is there a graph with more than 8 vertices?

Yes.
Try to find one with 10 vertices. Then try 12 vertices.

Is there a Limit?

An upper bound

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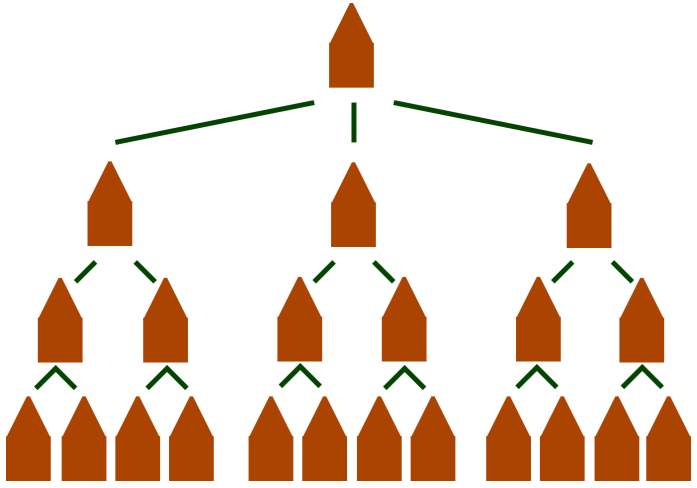
Start at some vertex and consider "levels" of neighbours.



Since the diameter is three, then no other vertices can exist.
Hence, there are at most $1 + 3 + 3 \times 2 + 3 \times 2 \times 2 = 22$ vertices.

This is a nice connection to computer science!

Villages and Canals Final Answer

The right answer is 12

Proving this is difficult.
This was an **open problem** until around 10 years ago.
Many variations are still unsolved.

Solving the Breakfast Problems

Shaking Hands

- uses parity and number sense (similar to villages and canals)

Soccer Schedules

- 2010 Pascal Contest #25

Crossing Curves

- uses $V - E + F = 2$ and inequalities
- 13 points is an open problem!

Parting Thoughts

Graph Theory in Grades 9 to 12

Graphs, visual mathematical structures that are fun to play with, are rich with applications illustrating how modeling relates to many different standards and expectations.

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www.cemc.uwaterloo.ca

- Real-World Problems Being Solved by Mathematicians
- Master of Mathematics for Teachers
- Math contests for Grades 7 through 12
- ...

Thank you!

Any questions?

???

jpretti@uwaterloo.ca

I am happy to answer questions by e-mail.

www.cemc.uwaterloo.ca

Lots to explore!