A Spotlight on Problem Solving for Math Contests

Mike Eden & J.P. Pretti
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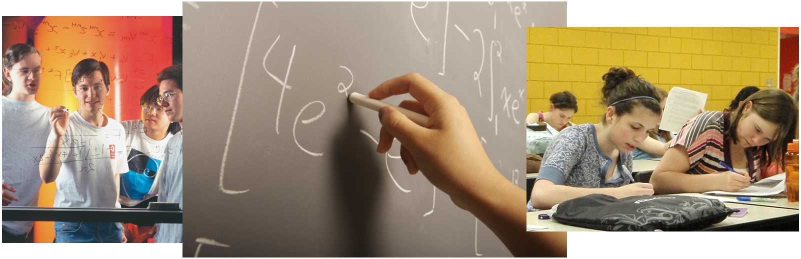
The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
UNIVERSITY OF WATERLOO

Outline

People and Places

- Who are we?
- Where are we from?
- What is the CENTRE for EDUCATION in MATHEMATICS and COMPUTING? (CEMC)

Incorporating Problem Solving into the Curriculum



- There is no time! How can this be done?
- Why would we do this?

Example #1

If $a - 3b - 9c + 27d = 19$ and each of a, b, c, and d equals 1 or 2 or 3, then find the value of $a + b + c + d$.

Example #1 - Solution

Since $a - 3b - 9c + 27d = 19$, then $a - 19 = 3(b + 3c - 9d)$.
The right side is a multiple of 3 and so $a - 19$ is also.
Therefore, $a = 1$.
After substitution and division by 3, $-6 = b + 3c - 9d$ and so $-6 - b = 3(c - 3d)$.
The right side is a again a multiple of 3 and so $-6 - b$ is also.
Therefore, $b = 3$.
After substitution and division by 3, $-3 = c - 3d$ and so $-3 - c = -3d$.
The right side is a again a multiple of 3 and so $-3 - c$ is also.
Therefore, $c = 3$.
After substitution and division by 3, we have $d = 2$.
Therefore $a + b + c + d = 1 + 3 + 3 + 2 = 9$.

Example #2

Maria has a red package, a green package, and a blue package.
The sum of the masses of the three packages is 60 kg.
The sum of the masses of the red and green packages is 25 kg.
The sum of the masses of the green and blue packages is 50 kg.
What is the mass of the green package, in kg?

Example #2 - Solution

Let r, g and b be the masses of the red, green and blue packages, respectively.
We are told that $r+g+b=60$, $r+g=25$, and $g+b=50$.
Subtracting the second equation from the first, we obtain $b=60-25=35$.
Substituting into the third equation, we obtain $g=50-b=50-35=15$.
Therefore, the mass of the green package is 15 kg.

Example #3

If $(x + 1)(x - 1) = 8$, then determine the numerical value of $(x^2 + x)(x^2 - x)$.

Example #3 - Solution

Solution 1
Since $(x+1)(x-1)=8$, then $x^2-1=8$ or $x^2=9$.
Thus, $(x^2+x)(x^2-x)=x(x+1)x(x-1)=$
$x^2(x+1)(x-1)=9(8)=72$.

Solution 2
Since $(x+1)(x-1)=8$, then $x^2-1=8$ or $x^2=9$, so $x=\pm 3$.
If $x=3$, then
$(x^2+x)(x^2-x)=(3^2+3)(3^2-3)=12(6)=72$.
If $x=-3$, then
$(x^2+x)(x^2-x)=((-3)^2+(-3))((-3)^2-(-3))=6(12)=72$.
In either case, $(x^2+x)(x^2-x)=72$.

Example #4

What is the largest two-digit number that becomes 75% greater when its digits are reversed?

Example #4 - Solution

Let n be the original number and N be the number when the digits are reversed. Since we are looking for the largest value of n, we assume that $n > 0$.

Since we want N to be 75% larger than n, then N should be 175% of n, or $N = \frac{7}{4}n$.

Suppose that the tens digit of n is t and the units digit of n is u.

Then $n = 10t + u$.

Also, the tens digit of N is u and the units digit of N is t, so $N = 10u + t$.

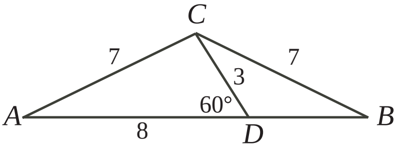
We want $10u + t = \frac{7}{4}(10t + u)$ or $4(10u + t) = 7(10t + u)$ or $40u + 4t = 70t + 7u$ or $33u = 66t$, and so $u = 2t$.

Example #4 - Solution (cont'd.)

This tells us that that any two-digit number $n = 10t + u$ with $u = 2t$ has the required property.
Since both t and u are digits then $u < 10$ and so $t < 5$, which means that the possible values of n are 12, 24, 36, and 48.
The largest of these numbers is 48.

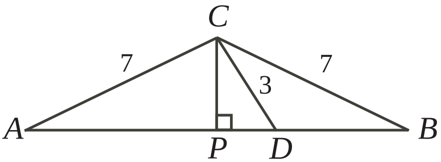
Example #5

In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60^\circ$, $AD = 8$, and $CD = 3$. Determine the length of BD.



Example #5 - Solution 1

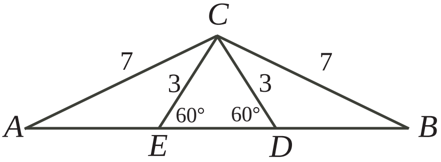
Drop a perpendicular from C to P on AD.



Since $\triangle ACB$ is isosceles, then $AP = PB$.
Since $\triangle CDP$ is a 30°-60°-90° triangle, then $PD = \frac{1}{2}(CD) = \frac{3}{2}$.
Thus, $AP = AD - PD = 8 - \frac{3}{2} = \frac{13}{2}$.
This tells us that $DB = PB - PD = AP - PD = \frac{13}{2} - \frac{3}{2} = 5$.

Example #5 - Solution 2

Since $\triangle ACB$ is symmetric about the vertical line through C, we can reflect CD in this vertical line, finding point E on AD with $CE = 3$ and $\angle CED = 60^\circ$.



Then $\triangle CDE$ has two 60° angles, so must have a third, and so is equilateral.
Therefore, $ED = CD = CE = 3$ and so
$DB = AE = AD - ED = 8 - 3 = 5$

Example #5 - Solution 3

Since $\angle CDB = 180^\circ - \angle CDA = 180^\circ - 60^\circ = 120^\circ$, then using the cosine law in $\triangle CDB$, we obtain

$$\begin{aligned}
CB^2 &= CD^2 + DB^2 - 2(CD)(DB)\cos(\angle CDB) \\
7^2 &= 3^2 + DB^2 - 2(3)(DB)\cos(120^\circ) \\
49 &= 9 + DB^2 - 6(DB)\left(-\frac{1}{2}\right) \\
0 &= DB^2 + 3DB - 40 \\
0 &= (DB-5)(DB+8)
\end{aligned}$$

Since $DB > 0$, then $DB = 5$.

Example #6

The numbers a, b, c, in that order, form a three term arithmetic sequence and $a + b + c = 60$.
The numbers $a - 2$, b, $c + 3$, in that order, form a three term geometric sequence.
Determine all possible values of a, b and c.

Example #6 - Solution

Since a, b, c form an arithmetic sequence, then we can write $a = b - d$ and $c = b + d$ for some real number d.
Since $a + b + c = 60$, then $(b - d) + b + (b + d) = 60$ or $3b = 60$ or $b = 20$.
Therefore, we can write a, b, c as $20 - d, 20, 20 + d$.
(We could have written a, b, c instead as a, $a + d$, $a + 2d$ and arrived at the same result.)
Thus, $a - 2 = 20 - d - 2 = 18 - d$ and $c + 3 = 20 + d + 3 = 23 + d$, so we can write $a - 2, b, c + 3$ as $18 - d, 20, 23 + d$.

Example #6 - Solution (cont'd.)

Since these three numbers, $18-d, 20, 23+d$, form a geometric sequence, then

$$\begin{aligned} \frac{20}{18-d} &= \frac{23+d}{20} \\ 20^2 &= (23+d)(18-d) \\ 400 &= -d^2-5d+414 \\ d^2+5d-14 &= 0 \\ (d+7)(d-2) &= 0 \end{aligned}$$

Therefore, $d=-7$ or $d=2$.
If $d=-7$, then $a=27$, $b=20$ and $c=13$.
If $d=2$, then $a=18$, $b=20$ and $c=22$.
(We can check that, in each case, $a-2, b, c+3$ is a geometric

Example #7

Square $OPQR$ has vertices $O(0, 0)$, $P(0, 8)$, $Q(8, 8)$, and $R(8, 0)$. The parabola with equation $y = a(x - 2)(x - 6)$ intersects the sides of the square $OPQR$ at points K, L, M, and N. Determine all the values of a for which the area of the trapezoid $KLMN$ is 36.

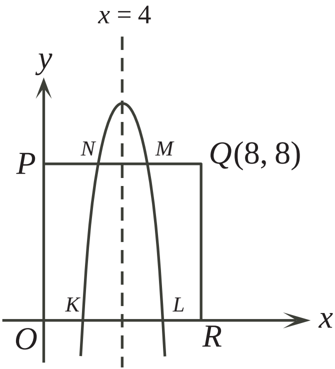
Example #7 - Solution

First, we note that $a \neq 0$. Second, regardless of the value of $a \neq 0$, the parabola has x-intercepts 2 and 6, and so intersects the x-axis at $(2, 0)$ and $(6, 0)$, which we call $K(2, 0)$ and $L(6, 0)$. This gives $KL = 4$. Third, since the x-intercepts of the parabola are 2 and 6, then the axis of symmetry of the parabola has equation $x = \frac{1}{2}(2 + 6) = 4$. Since the axis of symmetry of the parabola is a vertical line of symmetry, then if the parabola intersects the two vertical sides of the square, it will intersect these at the same height, and if the parabola intersects the top side of the square, it will intersect it at two points that are symmetrical about the vertical line $x = 4$. Fourth, recall that a trapezoid with parallel sides of lengths a and b and height h has area $\frac{1}{2}h(a + b)$. We now examine three cases.

Example #7 - Case 1: $a < 0$

Here, the parabola opens downwards. Since the parabola intersects the square at four points, it must intersect PQ at points M and N. (Since the parabola gets “narrower" towards the vertex.) Since the parabola opens downwards, then $MN < KL = 4$.

Since the height of the trapezoid equals the height of the square (or 8), then the area of the trapezoid is $\frac{1}{2}h(KL + MN)$ which is less than $\frac{1}{2}(8)(4 + 4) = 32$. But the area of the trapezoid must be 36, so this case is not possible.



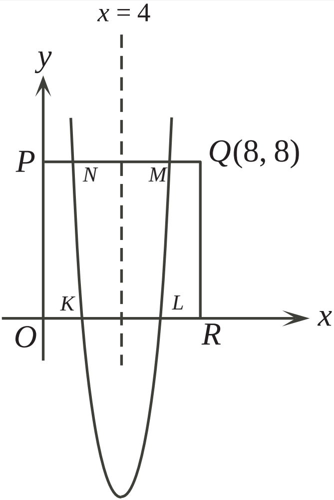
Example #7 - Case 2: $a > 0$; M and N on PQ

Here, the height of the trapezoid is 8, $KL = 4$, and M and N are symmetric about $x = 4$.

Since the area of the trapezoid is 36, then $\frac{1}{2}h(KL+MN) = \frac{1}{2}(8)(4+MN) = 36$ or $4 + MN = 9$ or $MN = 5$.

Thus, M and N are each $\frac{5}{2}$ units from $x = 4$, and so N has coordinates $(\frac{3}{2}, 8)$.

Since this point lies on the parabola with equation $y = a(x-2)(x-6)$, then $8 = a(\frac{3}{2} - 2)(\frac{3}{2} - 6)$ or $8 = a(-\frac{1}{2})(-\frac{9}{2})$ or $8 = \frac{9}{4}a$ or $a = \frac{32}{9}$.



Example #7 - Case 3: $a > 0$; M and N on QR and PO

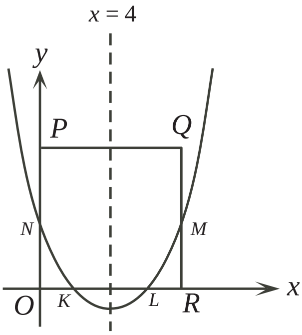
Here, $KL = 4$, $MN = 8$, and M and N have the same y-coordinate.

Since the area of the trapezoid is 36, then $\frac{1}{2}h(KL + MN) = \frac{1}{2}h(4 + 8) = 36$ or $6h = 36$ or $h = 6$.

Thus, N has coordinates $(0, 6)$.

Since this point lies on the parabola with equation $y = a(x-2)(x-6)$, then $6 = a(0-2)(0-6)$ or $6 = 12a$ or $a = \frac{1}{2}$.

Therefore, the possible values of a are $\frac{32}{9}$ and $\frac{1}{2}$.



Exposing Students to More Problems

- Why consider problems that are *not* curriculum dependent?
- Is there time?

Example #8

In the diagram, any ♠ may be moved to any unoccupied space. What is the smallest number of ♠'s that must be moved so that each row and each column contains three ♠'s?

♠	♠	♠	♠	
♠	♠	♠		♠
♠	♠			
♠	♠	♠	♠	
		♠		

Example #8 - Solution

Since there are 4 ♠'s in each of the first three columns, then at least 1 ♠ must be moved out of each of these columns to make sure that each column contains exactly three ♠'s.
Therefore, we need to move at least 3 ♠'s in total.
If we move the ♠ from the top left corner to the bottom right corner

	♠	♠	♠	
♠	♠	♠		♠
♠	♠			
♠	♠	♠	♠	
		♠		♠

and the ♠ from the fourth row, third column to the fifth row, fourth column

	♠	♠	♠	
♠	♠	♠		♠
♠	♠			
♠	♠		♠	
		♠	♠	♠

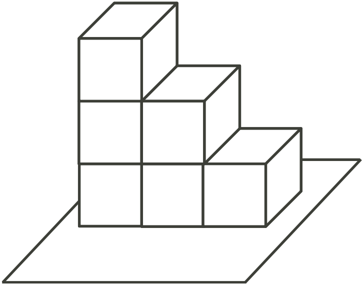
and the ♠ from the second row, second column to the third row, fifth column

	♠	♠	♠	
♠		♠		♠
♠	♠			♠
♠	♠		♠	
		♠	♠	♠

then we have exactly three ♠'s in each row and each column. Therefore, since we must move at least 3 ♠'s and we can achieve the configuration that we want by moving 3 ♠'s, then 3 is the smallest number.
(There are also other combinations of moves that will give the required result.)

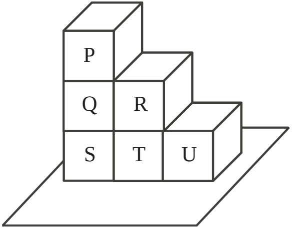
Example #9

Six dice are stacked on the floor as shown. On each die, the 1 is opposite the 6, the 2 is opposite the 5, and the 3 is opposite the 4. What is the maximum possible sum of numbers on the 21 visible faces?

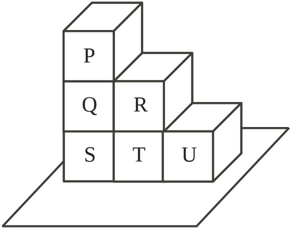


Example #9 - Solution

Label the six dice as shown:

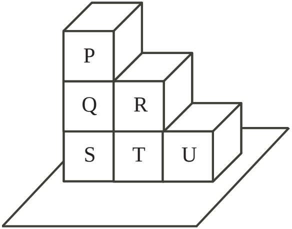


The maximum overall exposed sum occurs when the sum of the exposed faces on each die is maximized.
Die P has 5 exposed faces. The sum of these faces is a maximum when the 1 is hidden, so the maximum exposed sum on die P is $2 + 3 + 4 + 5 + 6 = 20$.



Dice Q and S each have 3 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 as the unpaired exposed face. (This is on the left face of the stack.) Each of these dice has a maximum exposed sum of $6 + 7 = 13$.

Dice R and U each have 4 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 and the 5 as the unpaired exposed faces (on the top and right of the stack).



Each of these dice have a maximum exposed sum of $5 + 6 + 7 = 18$.

Die T has 2 exposed faces, which are opposite each other, so have a sum of 7.

Therefore, the maximum possible sum of the exposed faces is $20 + 13 + 13 + 18 + 18 + 7 = 89$.

Example #10

A 3×3 square frame is placed on a grid of numbers, as shown. In the example, the sum of the numbers inside the square frame is 108, and the middle number is 12. When the square frame is moved to a new position, the sum of its numbers becomes 279. In the frame's new position, what is the middle number?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

Example #10 - Solution

Suppose that the middle number inside the square frame is x. Then the other numbers in the middle row inside the frame are $x-1$ and $x+1$. Also, the other numbers in the left column are $(x-1)-7=x-8$ and $(x-1)+7=x+6$, since there are 7 numbers in each row of the large grid. This tells us that the other numbers in the first and third rows are $x-7$, $x-6$, $x+7$, and $x+8$. Therefore, the sum of the numbers in the frame is $x+x-1+x+1+x-8+x-7+x-6+x+6+x+7+x+8=9x$
Thus, the sum of numbers in the frame is 9 times the middle number.

For the sum of the numbers inside the frame to be 279, the middle number in the frame must be $\frac{1}{9}(279)=31$.

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- Problem of the Week
- Publications
- Masters in Mathematics for Teachers

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Thank you for coming!!

- Please contact us any time with any questions
- Mike Eden: mceden@uwaterloo.ca
- J.P. Pretti: jpretti@uwaterloo.ca

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