Geometry

Using Manipulatives

& Activities

Christine (Chris) Mikles
Director of Teacher Education
(888) 808-4CPM (4276)
mikles@cpm,org
www.cpm.org

1.1.5 What shapes can you find?

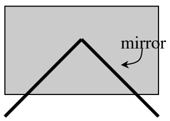
Building a Kaleidoscope

Today you will learn about geometric shapes as you study how a kaleidoscope works.

1-38. BUILDING A KALEIDOSCOPE

How does a kaleidoscope create the complicated, colorful designs you see when you look inside? A hinged mirror and a piece of paper can demonstrate how a simple kaleidoscope creates its beautiful repeating designs.

Your Task: Place a hinged mirror on a piece of paper so that its sides intersect the edge of the paper as shown at right. Explore what shapes you see when you look directly at the mirror, and how those shapes change when you change the angle of the mirror. Discuss the questions below with your team. Be ready to share your responses with the rest of the class.



Discussion Points

What is this problem about? What is it asking you to do?

What happens when you change the angle (opening) formed by the sides of the mirror?

How can you describe the shapes you see in the mirror?

1-39. To complete your exploration, answer these questions together as a team.

a. What happens to the shape you see as the angle formed by the mirror gets bigger (wider)? What happens as the angle gets smaller?

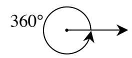
b. Does it matter if the sides of the mirror intersect the edge of the paper the same distance from the point where the mirrors are hinged? What happens to the shapes if they are not equal?

c. What is the smallest number of sides the shape you see in the mirror can have? What is the largest?

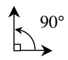
d. With your team, find a way to form a **regular hexagon** (a shape with six equal sides and equal angles).

e. How might you describe to another team how you set the mirrors to form a hexagon? What types of information would be useful to have?

1-40. A good way to describe an angle is by measuring how *wide* or *spread apart* the angle is. For this course, you should think of the measurement of an angle as representing the amount of turn needed when you start looking in one direction and then turn, rotating in one spot, to face another direction. For example, if you turn all the way around to face the direction you started, you turn 360°. This is a called a **circular angle** and can be represented in the diagram at right.

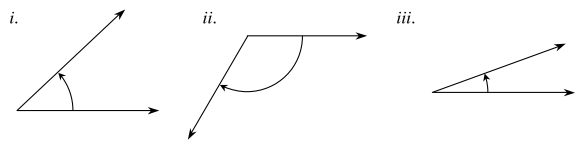


a. Other angles may be familiar to you. For example, an angle that forms a perfect "L" or a quarter turn is a 90° angle, called a **right angle** (shown at right). You can see that four of these angles would form a circular angle.

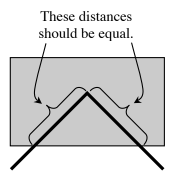


How many degrees would measure a half turn, or turning around to face the opposite direction from where you started? Draw this angle and label its degrees. How is this angle related to a circular angle?

b. Based on the examples above, estimate the measures of these angles shown below. Then confirm your answer using a **protractor**, a tool that measures angles.



1-41. Now use your understanding of angle to create some specific shapes using your hinged mirror. Be sure that both mirrors have the same length on the paper, as shown in the diagram at right.



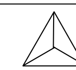
a. Antonio says he can form an **equilateral triangle** (a triangle with three equal sides and three equal angles) using his hinged mirror. How did he do this? Once you can see the triangle in your mirror, place the protractor on top of the mirror. What is the measure of the angle formed by the sides of the mirror?

b. Use your protractor to set your mirror so that the angle formed is 90°. Be sure that the sides of the mirror intersect the edge of the paper at equal lengths. What is this shape called? Draw and label a picture of the shape on your paper.

Problem continues on next page →

1-41. *Problem continued from previous page.*

c. What other shapes with equal sides can you create? How must your mirror be positioned to create these shapes? Sketch each shape you find on your paper, including all lines that you can see in the mirror. Organize the information you gather about the shapes into a table. A possible table is started for you below. For each shape, state how many sides it has, its name (if you know it), and the angle formed by the mirrors.

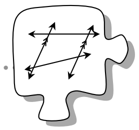
Number of sides	Angle	Name of shape	Sketch
3	120°	Equilateral Triangle	
	90°		
5			
6			

d. What patterns do you see in the table? In the shapes you have sketched?

e. Discuss with your team and predict how many sides a shape would have if the angle that the mirror makes is 40°. Use a picture, words, and the patterns in your table to explain your prediction. Then, check your prediction using the mirror and a protractor. Describe the shape you see with as much detail as possible.

f. Based on the patterns you observed in your table and your drawings, are there any angle measures in your chart that you would change? Explain which angles you would change and why, or demonstrate how you know they are correct.

2.1.1 Which angles are equal?



Complementary, Supplementary, and Vertical Angles

In Chapter 1, you compared shapes by looking at similarities between their parts. For example, two shapes might have sides of the same length or equal angles. In this chapter you will examine relationships between parts within a *single* shape or diagram. Today you will start by looking at angles to identify relationships in a diagram that make angle measures equal. As you examine angle relationships today, keep the following questions in mind to guide your discussion:

How can we name the angle?

What is the relationship?

How do we know?

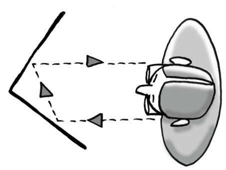
2-1. SOMEBODY'S WATCHING ME

Usually, to see yourself in a small mirror you have to be looking directly into it—if you move off to the side, you can't see your image any more. But Mr. Douglas knows a neat trick. He claims that if he makes a right angle with a hinged mirror, he can see himself in the mirror no matter from which direction he looks into it.

a. By forming a right angle with a hinged mirror, test Mr. Douglas's trick for yourself. Look into the place where the sides of the mirror meet. Can you see yourself? What if you look in the mirror from a different angle?

b. Does the trick work for *any* angle between the sides of the mirror? Change the angle between the sides of the mirror until you can no longer see your reflection where the sides meet.

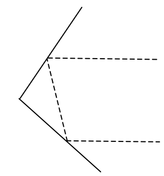
c. At right is a diagram of a student trying out the mirror trick. How many of the angle measures in the diagram can you find? Can you explain why Mr. Douglas's trick works? Talk about this with your team and be ready to share your ideas with the class.



2-47. SOMEBODY'S WATCHING ME, Part Two

You now know enough about angle and line relationships to analyze why a 90° hinged mirror reflects your image back to you from any angle. Since your reflection is actually light that travels from your face to the mirror, you will need to study the path of the light. A 90° hinged mirror bounces that light back towards you, as shown in the diagram below.

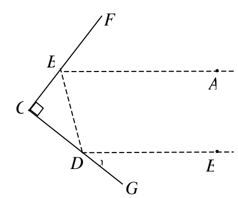
Your task: By finding angle measurements and relationships between segments, explain why the mirror bounces your image back to you from any angle. Be sure to analyze:



- How can you represent on the diagram light coming in from *any* angle?
- What angle relationships can you find in the diagram?
- What do you know about the paths the light takes as it leaves you and as it returns to you? How do you know?
- Why does this trick work? Would it work if the angle between the mirrors were not 90°?

Further Guidance

2-48. Since you are trying to show that the trick works for *any* angle at which the light could hit the mirror, each team will work with a different angle in this problem. Your teacher will tell you what angle x will be in the diagram below for your team.



a. Using angle relationships and what you know about how light bounces off mirrors, find the measure of every other angle in the diagram.

b. What is the relationship between angle ADB and angle EDB? What does this tell you about the relationship between $\overline{BA}$ and $\overline{DE}$?

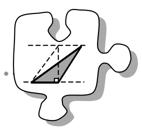
2-49. Explain why the 90° hinged mirror always sends your image back to you, no matter which angle you look into it from.

Further Guidance section ends here.

2-50. EXTENSION

Hold a 90° hinged mirror at arm's length and find your own image. Now close your right eye. Which eye closes in the mirror? Look back at the diagram from problem 2-48. Can you explain why this eye is the one that closes?

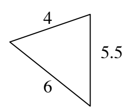
2.2.4 How can I find the height?



Heights and Area

In Lesson 2.2.2, you learned that triangles with the same base and height must have the same area. But what if multiple dimensions of a shape are labeled? How can you determine which dimension is the height?

2-87. Candice missed the lesson about finding the area of the triangle. Not knowing where to start, she drew a triangle and measured its sides, as shown to the right. After drawing her triangle, Candice said, "Well, I've measured all of the sides. I must be ready to find the area!"



If you think she is correct, write a description of how to use the side lengths to find the area. If you think she needs to measure anything else, copy the figure on your paper and show her where to place her ruler.

2-88. HEIGHT LAB

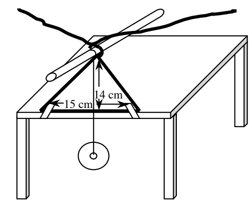
What is the height of a triangle? Is it like walking up any side of the triangle? Or is it like standing at the highest point and looking straight down? Today your team will build triangles with string and consider different ways height can be seen for triangles of various shapes.

a. Use the materials given to you by your teacher to make a triangle like the one in the diagram below.

(1) Tape a 15 cm section of the long string along the edge of a desk or table. Be sure to leave long ends of string hanging off each side.

(2) Tie one end of the short string to the weight and the other to the end of a pencil (or pen). Mark a distance 14 cm from the pencil toward the weight by placing a piece of tape on the string.

(3) Bring the loose ends of string up from the table and cross them as shown in the diagram. Then put the pencil with the weight over the crossing of the string. Cross the strings again on top of the pencil.



b. Now, with your team, build and sketch triangles that meet the three conditions below. To organize your work, assign each team member one of the jobs described at right.

(1) The height of the triangle is inside the triangle.

(2) The height of the triangle is a side of the triangle.

(3) The height of the triangle is outside of the triangle.

Student jobs:
- hold the pen with the weight.
- make sure that the height stays the same and all strings stay taut.
- draw accurate sketches.
- hold the strings that make the sides of the triangles.

c. Now make sure that everyone in your team has sketches of the triangles that you made. Do all three triangles have the same base and height? Do they look the same? Do they all have the same area?

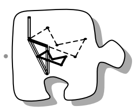
2-89. How can I find the height of a triangle if it is not a right triangle?

a. On the Lesson 2.2.4 Resource Page there are four triangles labeled (a) through (d). Assume you know the length of the side labeled "base." For each triangle, draw in the height that would enable you to find the area of the triangle. **Note:** You do not need to find the area.

b. Find the triangle for part (b) on the same resource page. For this triangle, draw all three possible heights. First choose one side to be the base and draw in the corresponding height. Then repeat the process of drawing in the height for the other two sides, one at a time.

c. You drew in three pairs of bases and heights for the triangle in part (b). Using the same triangle, measure the length of all three sides and all three heights. Find the area three times using all three pairs of bases and heights. Since the triangle remains the same size, your answers should match.

3.1.1 What do these shapes have in common?



Similarity

In Section 1.3, you organized shapes into groups based on their size, angles, sides, and other characteristics. You identified shapes using their characteristics and investigated relationships between different kinds of shapes, so that now you can tell if two shapes are both parallelograms or trapezoids, for example. But what makes two figures look alike? Today you will be introduced to a new transformation that enlarges a figure while maintaining its shape, called a **dilation**. After creating new enlarged shapes, you and your team will explore the interesting relationships that exist between figures that have the same shape.

In your teams, you should keep the following questions in mind as you work together today:

What do the shapes have in common?

How can you demonstrate that the shapes are the same?

What specifically is different about the shapes?

Teacher note for problem 3-1

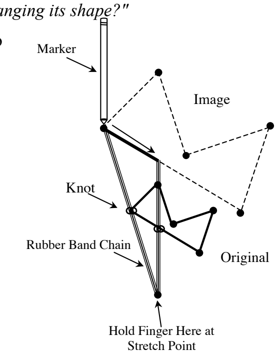
Begin the lesson with a 15-minute rubber band activity as described in problem 3-1. Rubber bands are used as a tool to help students draw an enlarged similar version of a shape. This is a fun activity, but it cannot be emphasized enough how important it is for you to practice this activity ahead of time to make sure you understand it thoroughly.

Have a student volunteer read the lesson introduction. Emphasize that students will be exploring the relationships between figures that preserve shape while changing size. You might ask, *"How can you change the size of something without changing its shape?"*

Distribute two rubber bands to half of the teams and three to the other half. Have each team tie their bands together to make a rubber band chain. A chain made with three rubber bands will have two knots in it.

Instruct each team to draw a simple picture (such as a heart or a stick figure) on a chalkboard or white board. Then instruct each team to mark and label a "stretch point" fairly close to the figure.

While one student anchors one end of the rubber band chain with a finger (or the eraser end of a pencil) at this stretch point, another should thread a piece of chalk (or a dry-erase marker) through the other end of the rubber band chain. For clarity, have students use a different color than they used to draw the original image.



As one student keeps the stretch point fixed, the other student will move the chalk or marker to draw a new figure. The student moving the chalk or marker should watch the knot in the rubber band chain so that it passes exactly over the perimeter of the original figure. As the knot traces the perimeter of the figure, the rubber bands will stretch as you draw the new, enlarged figure. See diagram at right.

If they are using three rubber bands, they should trace the original figure with the knot closest to the stretch point. That is, the first of the two knots counting from the stretch point. A team with a two-rubber-band chain should end up with an image with sides roughly twice the length of the original shape. A team with a three-rubber-band chain should end up with an image with sides roughly three times the length of the original shape. Consider experimenting with the number of rubber bands in the chain and/or the knot used to trace the original figure for different results.

Next have a brief class discussion. Point out that the students have done a new type of transformation, called a "dilation," and that the figure drawn using the rubber band chain is an image of the original. When a figure is dilated, it is stretched proportionally from a point. The result is an enlarged or reduced figure that is similar to the original figure. While the location of the image depends the location of the point of dilation, this course will not dwell on this dependence. It is sufficient for this course for students to understand that a dilation enlarges (or reduces) a figure proportionally, maintaining its shape.

Discuss the relationship between the image and the original. At this point, students should notice that the image and original have the same shape but not the same size. Some students may also notice that the image is approximately two (or three) times as large as the original. They might also notice that the corresponding angles of the original and the image are equal. You shouldn't make any observations that the students don't bring up — you'll have a chance to do this later.

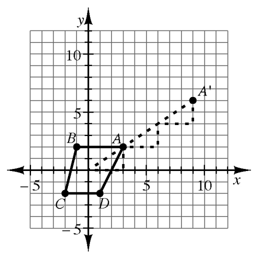
3-1. WARM-UP STRETCH

Before computers and copy machines existed, it sometimes took hours to enlarge documents or to shrink text on items such as jewelry. A pantograph device (shown in use below) was once used to duplicate written documents and artistic drawings. You will now employ the same geometric principles by using rubber bands to draw enlarged copies of a design. Your teacher will show you how to do this.

3-2. In problem 3-1, you created designs that were **similar**, meaning that they have the same shape. But how can you determine if two figures are similar? What do similar shapes have in common? To find out, your team will need to create similar shapes that you can measure and compare.

a. Obtain a Lesson 3.1.1 Resource Page from your teacher and **dilate** (stretch) the quadrilateral from the origin by a factor of 2, 3, 4, or 5 (each team member should pick a different enlargement factor). You may want to imagine that your rubber band chain is attached at the origin and stretched so that the knot traces the perimeter of the figure.

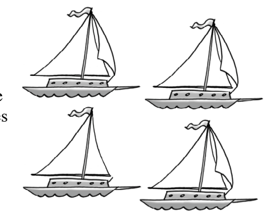
Another way to approach this task is by using slope triangles. If you draw a slope triangle from the origin to a vertex, the hypotenuse of this triangle represents the first rubber band of your chain. The hypotenuse of a second slope triangle continuing on from this vertex represents the second rubber band in your chain. Repeat this process for each vertex to create an enlargement of the original figure. Note that if you are making a figure three times as large as your original, you will need three slope triangles for each vertex. See the diagram at right.



b. Carefully cut out your enlarged shape and compare it to your teammates' shapes. How are the four shapes different? How are they the same? As you investigate, make sure you record what qualities make the shapes different and what qualities make the shapes the same. Be ready to report your conclusions to the class.

3-3. WHICH SHAPE IS THE EXCEPTION?

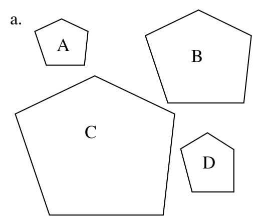
Sometimes shapes look the same and sometimes they look very different. What characteristics make figures alike so that we can say that they are the same shape? How are shapes that look the same but are different sizes related to each other? Understanding these relationships will allow us to know if shapes that appear to have the same shape actually do have the same shape.



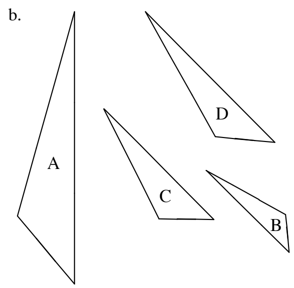
Your task: For each set of shapes below, three shapes are similar, and one is an exception, which means that it is not like the others. Find the exception in each set of shapes. Your teacher may give you tracing paper to help you in your investigation. Answer each of these questions for both sets of shapes below:

- Which shape appears to be the exception? What makes that shape different from the others?
- What do the other three shapes have in common?
- Are there commonalities in the angles? Are there differences?
- Are there commonalities in the sides? Are there differences?

a.



b.



3.2.6 What can I do with similar triangles?

Applying Similarity

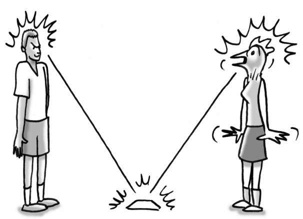
In previous lessons, you have learned methods for finding similar triangles. Once you find triangles are similar, how can that help you? Today you will apply similar triangles to analyze situations and solve new applications. As you work on today's problems, ask the following questions in your team:

What is the relationship?

Are any triangles similar? What similarity conjecture can we use?

3-93. YOU ARE GETTING SLEEPY…

Legend has it that if you stare into a person's eyes in a special way, you can hypnotize them into squawking like a chicken. Here's how it works.



Place a mirror on the floor. Your victim has to stand exactly 200 cm away from the mirror and stare into it. The only tricky part is that you need to figure out where you have to stand so that when you stare into the mirror, you are also staring into your victim's eyes.

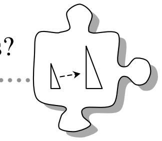
If your calculations are correct and you stand at the *exact* distance, your victim will squawk like a chicken!

a. Choose a member of your team to hypnotize. Before heading to the mirror, determine where you will have to stand in order for the hypnosis to work. Measure the heights of both yourself and your victim (heights to the eyes, of course), then draw a diagram to represent this situation. Label all the lengths you can on the diagram. (Remember: your victim will need to stand 200 cm from the mirror.)

b. How many pairs of equal angles can you find in your diagram? (Hint: Remember what you know about how images reflect off mirrors.) What is the relationship of the two triangles? Explain how you know. Then calculate how far you will need to stand from the mirror to hypnotize your victim.

c. Now for the moment of truth! Have your teammate stand 200 cm away from the mirror, while you stand at your calculated distance from the mirror. Do you make eye contact? If not, check your measurements and calculations and try again.

1.2.4 What shapes can I create with triangles?



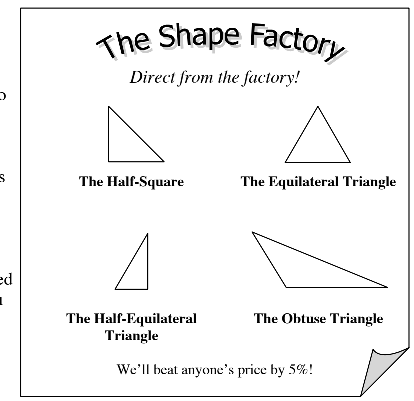
Using Transformations to Create Shapes

In Lesson 1.2.3, you practiced reflecting, rotating and translating figures. Since these are rigid transformations, the image always had the same size and shape as the original. In this lesson, you will combine the image with the original to make new, larger shapes from four basic "building-block" shapes.

As you create new shapes, consider what information the transformation gives you about the resulting new shape. By the end of this lesson, you will have generated most of the shapes that will be the focus of this course.

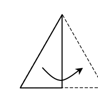
1-79. THE SHAPE FACTORY

The Shape Factory, an innovative new company, has decided to branch out to include new shapes. As Product Developers, your team is responsible for finding exciting new shapes to offer your customers. The current company catalog is shown at right.



Since your boss is concerned about production costs, you want to avoid buying new machines and instead want to reprogram your current machines.

The factory machines not only make all the shapes shown in the catalog, but they also are able to translate, rotate, or reflect a shape. For example, if the half-equilateral triangle is reflected across its longest leg, as shown at right, the result is an equilateral triangle.



Your Task: Your boss has given your team until the end of this lesson to find as many new shapes as you can. Your team's reputation, which has suffered recently from some unfortunate accidents, could really benefit by an impressive new line of shapes formed by a triangle and its transformations. For each shape in the catalog, determine which new shapes you can create using a reflection, rotation, or translation. Be sure to make as many new shapes as possible. Use tracing paper or any other reflection tool to help.

Discussion Points

What are your goals for this task?

What tools would be useful to complete this task?

What counts as a shape?

Will translations create new shapes? Why or why not?

Further Guidance

1-80. Since there are so many possibilities to test, it is useful to start by considering the shapes that can be generated just from one triangle. To get the simplest figures, try reflections and rotations that require a side of the image to lie on top of a side of the same length in the original triangle. For example:

a. Test what happens when the half-square is reflected across each side. For each result (original plus image), draw a diagram and describe the shape that you get. If you know a name for the result, state it.

b. It's easy to match sides when reflecting — just reflect over the side. But how can you match sides with rotation? Try using tracing paper to rotate a half-square so that a side matches with itself. How many degrees do you need to rotate? Where is the center of rotation?

c. The point in the middle of each side is called its middle point, or **midpoint** for short. Try rotating the half-square 180° about the midpoint of each side to make a new shape. For each result, draw a diagram. If you know its name, write it near your new shape.

d. Repeat parts (a) through (c) with each of the other triangles offered by the Shape Factory.

Further Guidance section ends here.

1-81. BUILDING A CATALOG

Your boss now needs you to create a catalog page that includes your shapes. Each entry should include a diagram, a name, and a description of the shape. List any special features of the shape, such as if any sides are the same length or if any angles must be equal. Use color and arrows to highlight these attributes in the diagram.

1-82. EXTENSIONS

What other shapes can be created by reflection and rotation? Explore this as you answer the questions below. You can **investigate** these questions in any order. Remember that the resulting shape includes the original shape and all of its images. Remember to record and name each result.

- What if you reflect an equilateral triangle twice, once across one side and another time across a different side?
- If you reflect an equilateral triangle across a side, you get two points which are endpoints of three segments. Pick one of these points and label it A. Continue reflecting across new sides that have A as one endpoint until new triangles are no longer possible. Describe the resulting shape.
- What if you rotate a trapezoid 180° around the midpoint of one if its non-parallel sides?
- What if you take an arbitrary triangle (no equal sides or angles) and rotate it 180° around the midpoint of one of its sides?