21

Algebra Wasn't Always about Symbol Manipulation

(6–12) Session

Symbols have only been in common use for 300 years. For 5,000 years Babylonians, Arabs, and others have thought algebraically and represented their thinking numerically, geometrically, and verbally. Come look at this history and find clues about how humans learned to think algebraically, which can reflect how our students learn the same thing.

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Here's a famous Algebra problem found on a Babylonian Clay Tablet:

The area of a square,
added to the side of the square,
comes to .75.

What is the side of the square?

I have added the area and the side of my square. (.75) $x^2+x=.75$	$a=1$ $b=1$ $c=-.75$
You write down 1, the coefficient. 1	$b=1$ (The coefficient of x; the unknown side)
You break half of 1. (0.5) $\frac{1}{2}=.5$	$\frac{b}{2}=.5$
You multiply 0.5 and 0.5. (0.25) $\left(\frac{1}{2}\right)^2=\frac{1}{4}=.25$	$(b/2)=.25$
You add 0.25 and 0.75. (1.0) ← I am looking for 1 "x;" the length of one side $\left(\frac{1}{2}\right)^2+.75=1$	$\left(\frac{b}{2}\right)^2+(-c)=1$
This is the square of 1. $\left(\frac{1}{2}\right)+.75=(1)^2$	$x^2=(b/2)^2-c$ $x=1=\sqrt{(b/2)^2-c}$
Subtract 0.5 which you multiplied. $\sqrt{1}-\frac{1}{2}=\frac{1}{2}$	$\sqrt{\left(\frac{b}{2}\right)^2-c}-\frac{b}{2}=x$
0.5 is the side of the square. $x=.5$	$x=\frac{-b}{2(a)}+\frac{\sqrt{b^2-4c}}{2(a)}$

Problem 48 from the Papyrus:

An Octagon is inscribed in a square of side 9. The problem isn't overtly stated, but it's thought that the author was trying to find the area of a circle or radius 4.5 by approximating that circle with the octagon. (In Problem 50, the formula Is developed to find the area of a circle, $A_{\odot} = \left(\frac{8}{9}d\right)^2$ A pretty decent approximation of pi, arrived at geometrically.)

$$A_{\odot} = \left(\frac{8}{9}d\right)^2$$

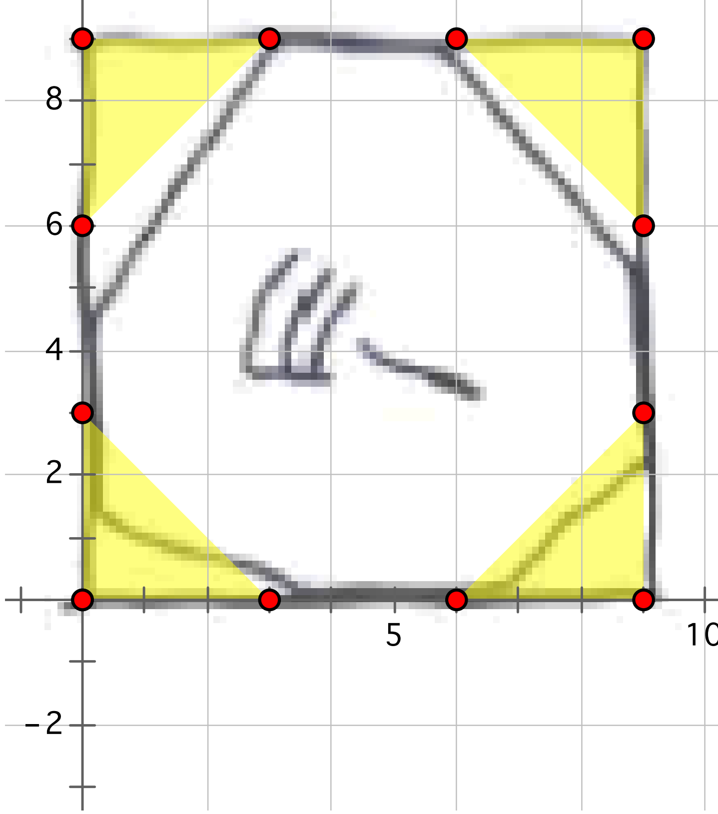
$$A_{\odot} = \frac{64}{81}(2r)^2$$

$$A_{\odot} = \frac{64}{81} * 4(r)^2$$

$$\rightarrow \pi \approx 3.160493$$

In problem 48, there is no problem statement, But the scribe has made the notations 8x8=64 and 9x9=81.

The author is trying to get to the constant, $\frac{64}{81} * 4$ so that it can be applied to a formula to be used in future calculations. There are different ideas about how this value was found.



$$\frac{1}{2}bh = A_{\triangle}$$

$$\frac{1}{2}(3)(3) = 4.5$$

$$81 - 4(4.5) = 63$$

$$\frac{\text{octagon}}{\text{square}} = \frac{63}{81} \approx \frac{64}{81}$$

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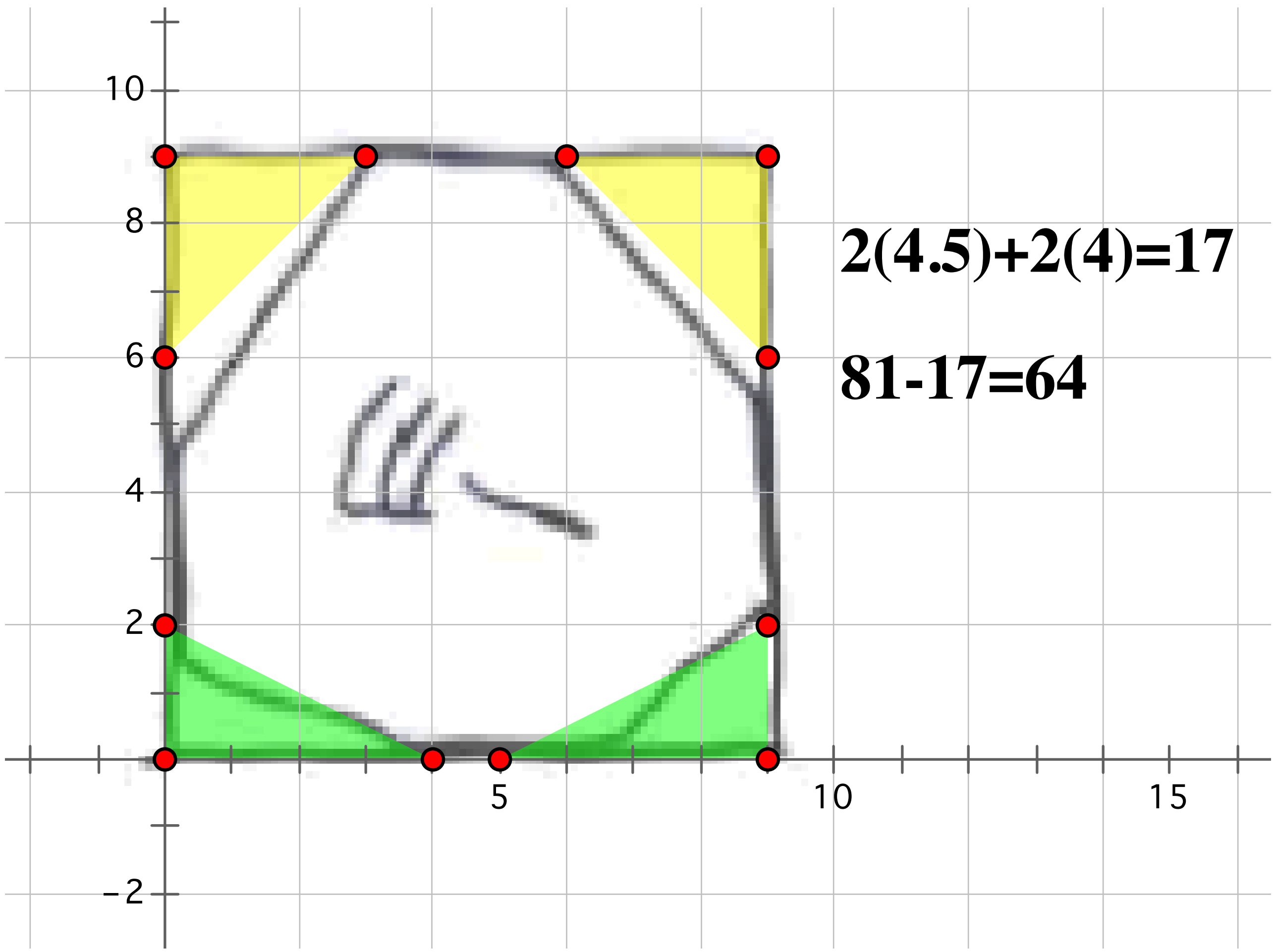
$$A_{\odot} = \frac{64}{81} * 4(r)^2$$

$$\rightarrow \pi \approx 3.160493$$

Or perhaps the author used 2 sets of triangles, two (3 x 3)'s and two (2 x 4)'s

(given the notations, 8x8=64 and 9x9=8 this may make more sense...)



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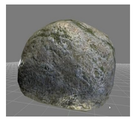
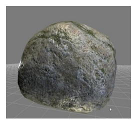
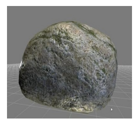
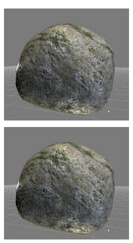
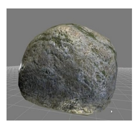
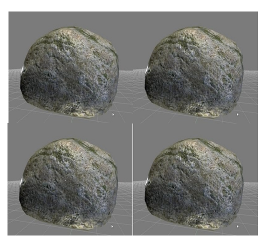
Pythagoras! What a guy!

- Felt that "everything is number" and that number meant a discrete, dividable amount.
- Numbers could best be represented as piles of stones
- If this is "one" 
- Then this is "two"  

- If this is "one," then this is one- half

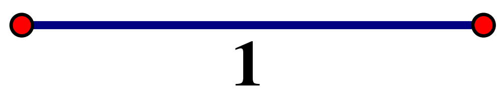
- But here is a problem- in the 2x2 area below, how to represent the diagonal?

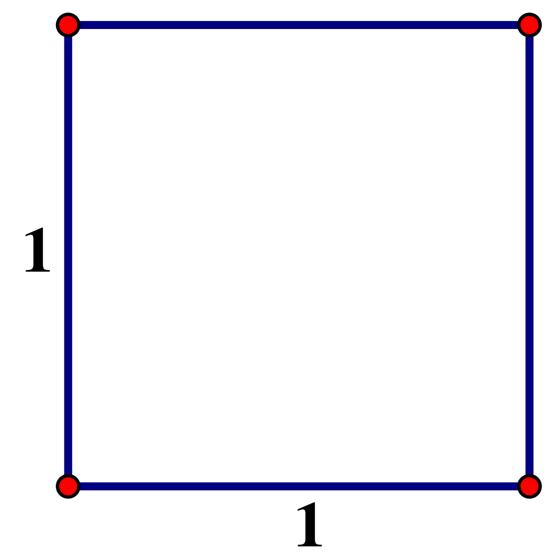


Here is where a great mathematical tradition has a quasi formal beginning.

If the numbers you have can't solve the problem, invent some new ones….

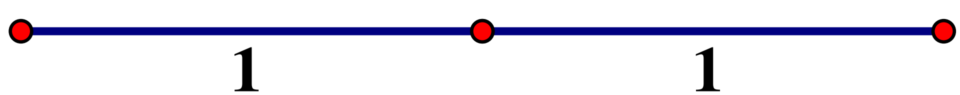
Consider the square of side 1.



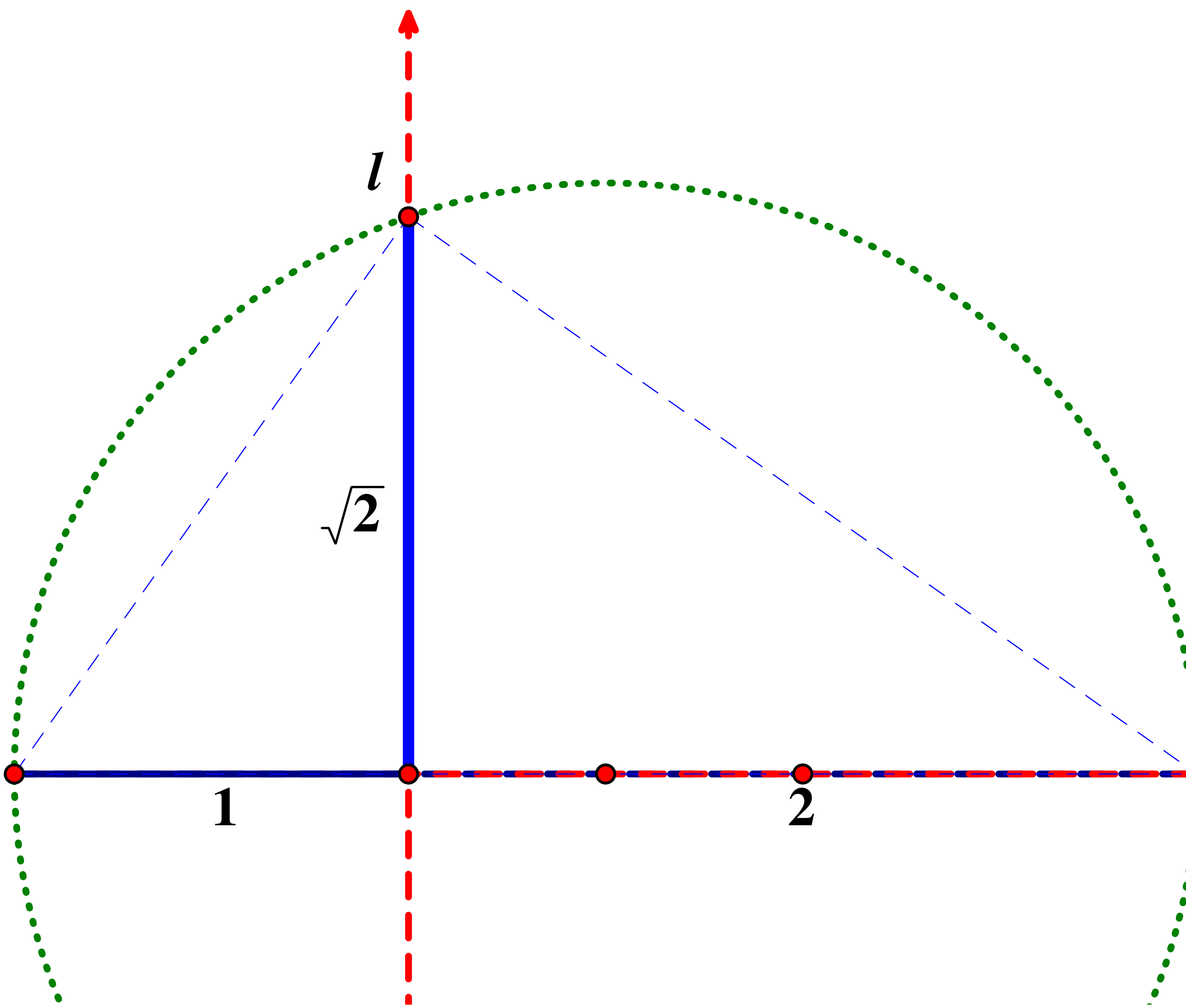


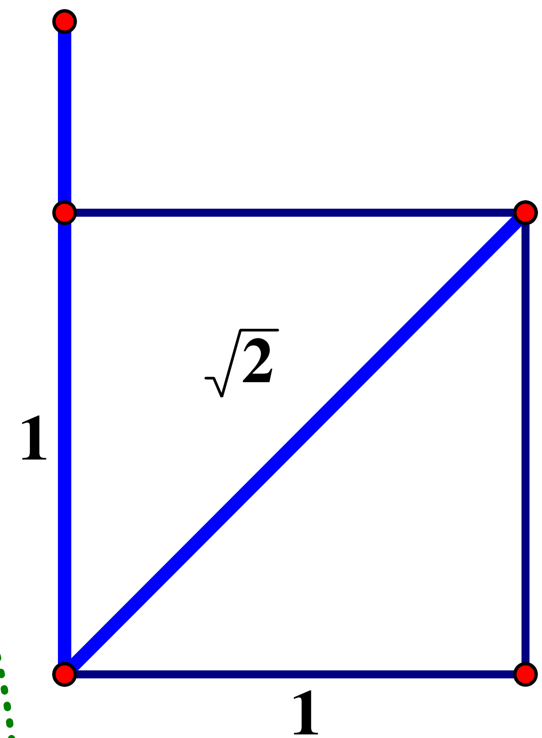
Pythagoras knows that the diagonal of this square should be the root of 2, but there is No way to represent this as a quantity of stones.

I can put 2 "one's" together to make a "two."
But no rational number of "ones" will work together to make root (2).



Once I have segments of length 1 and length 2, I can construct the Geometric mean between 1 & 2, which has a length of root 2. That length does cut a diagonal across the square of side 1; but we have had to see numbers as lengths of rope rather than piles of stones in order to solve this problem.



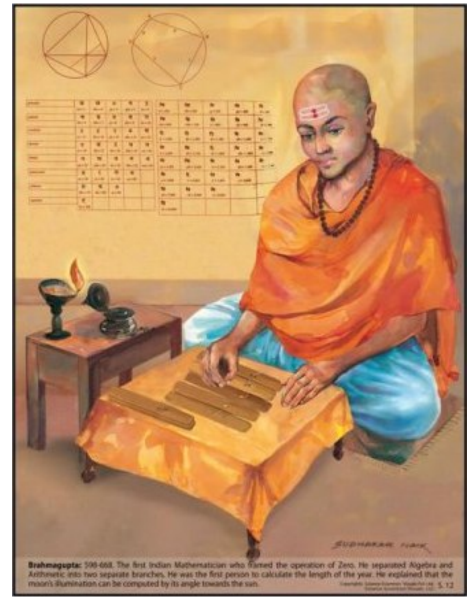


- Pre-Greeks, problem-solving was centered on solving particular problems, by using an established formula, or a variation of "Guess, Check & Revise," including the Egyptian approach to solving simultaneous linear equations, called "the Method of False Position." Representations were numeric and/or Geometric, and one representation could verify the other.
- The Greeks used Roman Numerals, so numerical representations were excruciating! They tended to see everything Geometrically, and wanted absolute certainty. Numbers were lengths of segments, and *Elements* is largely concerned with number theory, presented entirely from a Geometric viewpoint.
- To the Greeks, problem solving was interesting, but Problem Proving/ Problem Categorizing, and setting up a system of numbers that would include all possible solutions was more important.
- Once the number system (Natural, Rational and Irrational Numbers – still no zero!) had been settled, interest in types of problems increased.

Enter Diophantus & *Arithmetica*

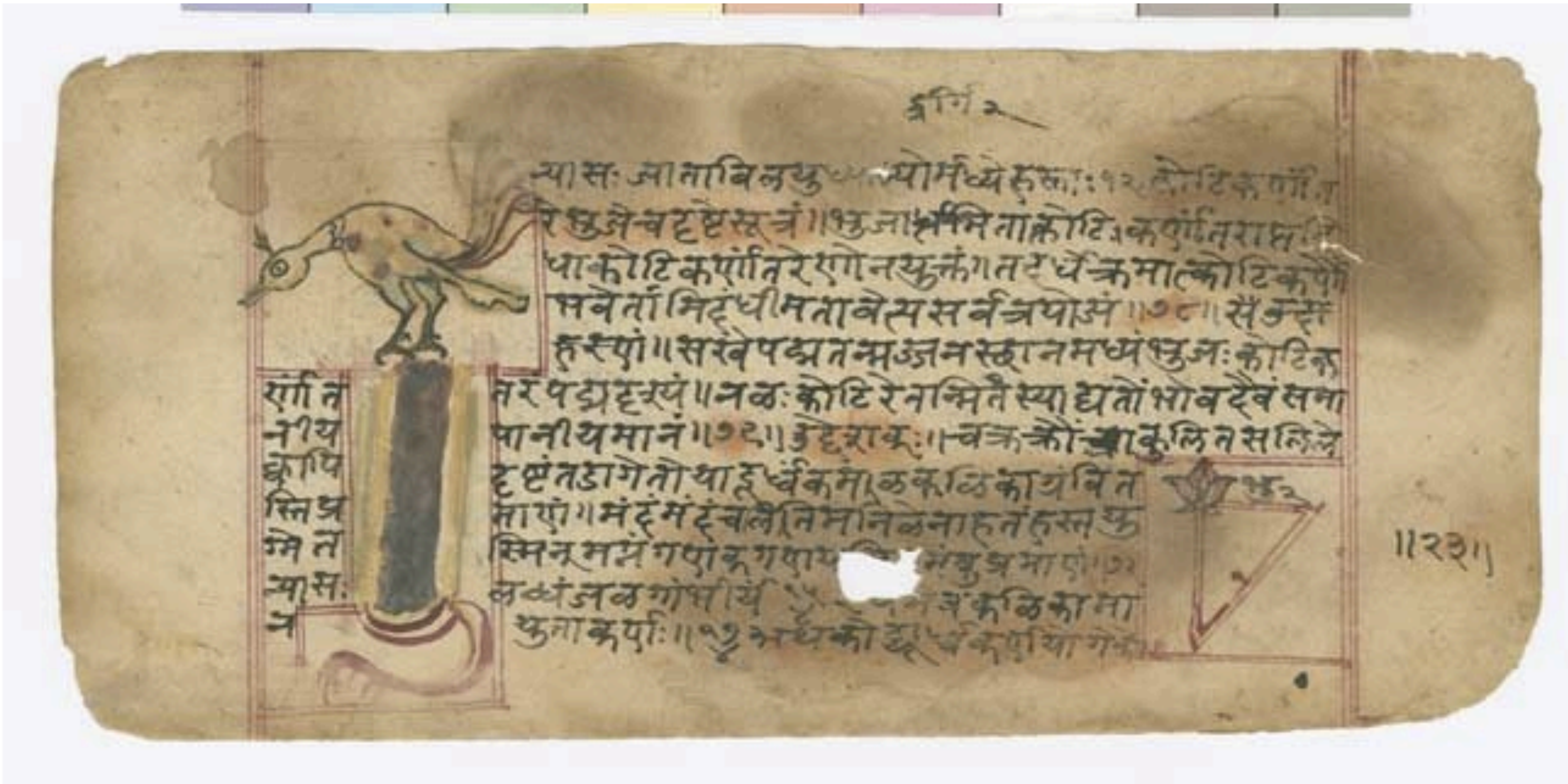
- Diophantus lived around 500 years after Euclid. There are 10 Books of *Arithmetica*, and they were very popular with other Greeks (Hypatia worked on some Diophantine problems) and with the Arab and Persian mathematicians at the House of Wisdom.
- Diophantus used some symbols in his work, but they are more like abbreviations than what we use currently.
- He dealt with powers higher than the third- this was new.
- The solutions to Diophantine Equations are Natural or Rational Numbers.
- He offers one solution, not generalized, and does not prove his results.

In India, the Hindus were also thinking about Algebraic Problems


Bramagupta

- They made progress in algebra as well as arithmetic. They developed some symbolism which, though not extensive, was enough to classify Hindu algebra as almost symbolic and certainly more so than the syncopated algebra of Diophantus. Only the steps in the solutions of problems were stated; no reasons or proofs accompanied them.
- The Hindus recognized that quadratic equations have two roots, and included negative as well as irrational roots. They could not, however, solve all quadratics since they did not recognize square roots of negative numbers as numbers. In indeterminate equations the Hindus advanced beyond Diophantus. Aryabhata (b. 476) obtained whole number solutions to $ax \pm by = c$ by a method equivalent to the modern method. They also considered indeterminate quadratic equations.

http://www.ucs.louisiana.edu/~sxw8045/history.htm



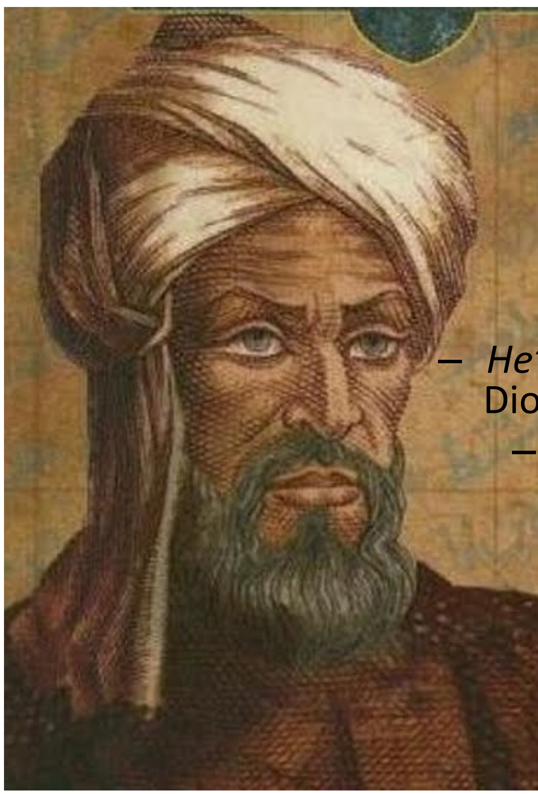
This is a page from a manuscript of the *Lilavati* of Bhaskara II (1114-1185). This manuscript dates from 1650. The rule for the problem illustrated here is in verse 151, while the problem itself is in verse 152:

> 151: The square of the pillar is divided by the distance between the snake and its hole; the result is subtracted from the distance between the snake and its hole. The place of meeting of the snake and the peacock is separated from the hole by a number of *hastas* equal to half that difference.
>
> 152: There is a hole at the foot of a pillar nine *hastas* high, and a pet peacock standing on top of it. Seeing a snake returning to the hole at a distance from the pillar equal to three times its height, the peacock descends upon it slantwise. Say quickly, at how many *hastas* from the hole does the meeting of their two paths occur? (It is assumed here that the speed of the peacock and the snake are equal.)

These verses and much else from the *Lilavati* may be found in Kim Plofker, "Mathematics in India", in Victor Katz, ed., The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook (Princeton: Princeton University Press, 2007), pp. 385-514.

Among the faculty: **Mohammed ibn-Musa al-Khwarizimi** (d. before 850)

- *Concerning the Hindu Art of Reckoning*- based on Bhramagupta, he uses the Indian system of numeration and is therefore responsible for the mistaken impression that our current system of numerals is Arabic in origin. These numerals were known by the mispronunciation of al-Khwarizimi's name- al-gorisme, which is further perverted into algorithm, which now means a rule for reckoning.
  - *Al-iabr wa' muqābalah*- the origin of Algebra, and this is the book that eventually teaches Europeans about representations of relationships between variables.
    - » Quadratics organized- but zero is not recognized as a root
    - » the approach is Geometric
    - » much synthesis between Greek, Indian and Arabian understanding
    - » Operations on binomial expressions
    - » Rules for operating on signed numbers
    - » This book did for Algebra what *The Elements* did for Geometry
  - *He's the REAL "Father of Algebra!"* though his work is more elementary than Diophantus's, and less symbolic- even the numbers are written out in words.
  - he gives praise to Mohammed for encouraging him to study-most of the Islamic scholars do this, and therefore some parts of the later Latin translations of their work are missing.



Al-Khwarizmi sorted problems into three types, and wrote rules for how to solve each.
He starts out with one of my favorite Math Text Lines of all time:

"When I considered what people generally want in calculating, I found that it always is a number."

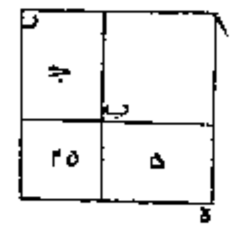
Roots and Squares are equal to Numbers	$bx + ax^2 = c$
Squares and Numbers are equal to Roots	$ax^2 + c = bx$
Roots and Numbers are equal to Squares	$bx + c = ax^2$

- No symbols were used, though the addition of workable numerals was a huge improvement.
- All of the work was communicated verbally.
- Because there had been no need for them, and because solutions were seen as geometric entities, there were no "negative" solutions.

As he discusses the solutions of these problems, Al-Khwarizmi offers advice:

- “When you meet with an instance which refers you to this case, try its solution by addition, and if that do not serve, then subtraction certainly will.”
- “And know that, when in a question belonging to this case you have halved the number of the roots and multiplied the moiety by itself, if the product be less than the number of the dirhems connected with the square, then the instance is impossible; but if the product be to the dirhems by themselves, then the root of the square is equal to the moiety of the roots alone, without either addition of subtraction.”
- Al’Khwarizmi actually follows his narrative with a geometric solution, to show that his new fangled method equals the tried and trusted methods of the Greeks

علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته



واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع ه ن والسطح ه ب فصار طول السطحين جميعا ضلع ج ه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع ه ج عشرة اعداد لان
ضلع ج د جذر المال فقسمنا ضلع ج ه بنصفين على نقطة

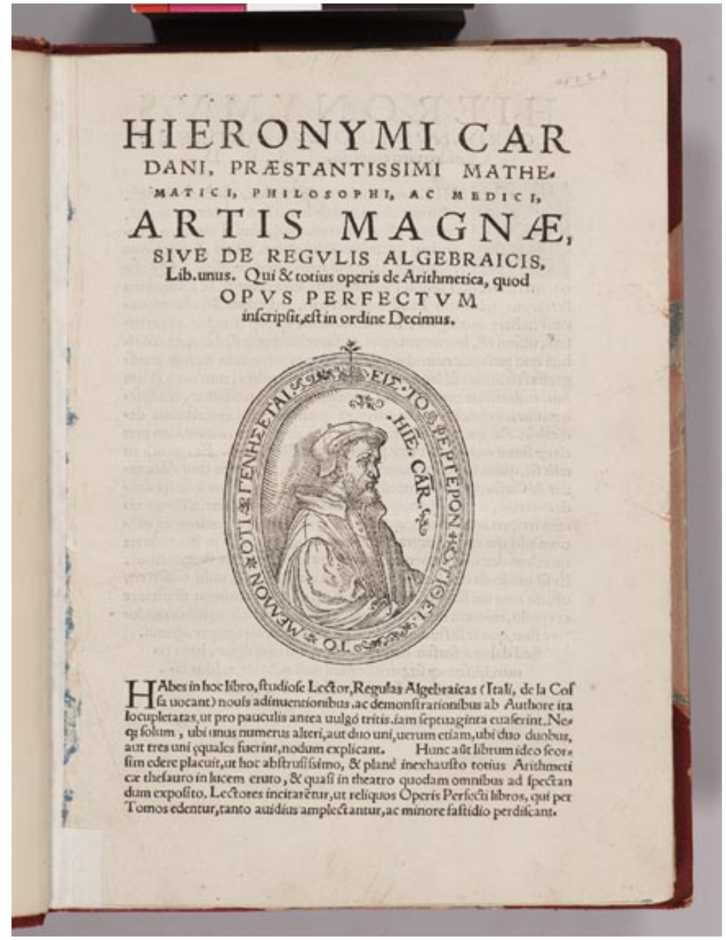
I can't help but notice…

- That Al-Khwarizmi has worked out, by trial and error, a series of steps that he recommends his reader memorize and follow in order to solve problems, that he has carefully divided into "types." This seems to be exactly how my students approach Algebra.
- He's backed up his verbal reasoning (which is new to him) with the tried and true method that he knows (the Geometry) to satisfy himself of the validity of his methods. In fact, the newer, more abstract method is built on the physical models of the past.
- Al-Khwarizmi manages to use Algebraic Thinking with physical and numerical models, and absolutely no facility with symbol manipulation. It's only much later in the history of mathematics that the use of symbols evolves.
  - 1494 Pacioli re-introduces symbols, though they are not as efficient as those of Diophantus 1200 years ago
  - Vieté (1504-1603) introduced the idea of using letters to represent constants.
  - As late as 1630 there is no convention about the use of symbols.

Just my opinion but…

- We tend to look at *Elements* as well as modern Geometry texts as though the progression from undefined term to definition, from axiom or postulate to theorem was an obvious and natural sequence.
- We forget that Euclid had the benefit of hindsight- he was not developing the ideas in *Elements* organically, but rather, he was putting long established and rather chaotically developed ideas together in a logical sequence.
- Likewise, we start teaching Algebra as though the symbol manipulation made its own inherent sense. It took the greatest mathematical minds ever to walk the planet 5000 years to get to the point that a symbolically expressed relationship made any sense. And the people who did figure that out were grown adults. We start with the symbols and with younger students every year.
- I'm all for introducing Algebraic thinking as early as possible- but in concrete, developmentally appropriate ways. Geometric and numerical models can do that.

Of course, there were other missing links- negative and imaginary numbers needed to be developed in order to solve other classes of problems


HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aût librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

- 1501-1576 Girolamo Cardano- quite a character. Cardano was an astrologer, gambler (he did a lot of work with probability and combinatorics as well), physician, and a rogue. He wrote a book called *Arts Magna* or *The Great Art*- also known as the *First Book of the Rules of Algebra* in which he set down the techniques for solving cubics and quadratics- including the notion that all cubics had three solutions and that some could be negative, irrational or **even imaginary.** Unfortunately, there is a lot of evidence that he plagiarized the methods.

From Chapter 37 of *Ars Magna* where he wants to find 2 numbers whose product is 40:

"Putting aside the mental tortures involved, multiply (5+√(-15)) by(5-√(-15)), making 25 – (-15) which [latter] is +15. Hence this product is 40. … This is truly sophisticated…"

But Algebra has had almost nothing to do with symbolic manipulation for most if its history.

- **400 years of symbols ÷ 4500 years of area problems ≈ 9% of Algebra's History devoted to symbol manipulation.**
- **Of course, we have to do it- it's the basis of most of the curriculum (right or wrong.) But by understanding how Algebra developed, and what kinds of problems it was invoked to solve, maybe we can remember to focus a little more on the problems that gave rise to the symbols we cherish so much.**

- http://www.mathsisfun.com/algebra/add-subtract-balance.html
- http://illuminations.nctm.org/ActivityDetail.aspx?ID=216
- http://illuminations.nctm.org/ActivityDetail.aspx?ID=127
- http://illuminations.nctm.org/ActivityDetail.aspx?ID=127
- http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=2591&bodyId=2588