Avoiding the Consolation Prize: The Mathematics of Game Shows

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Who Are We?

If a tree falls on a philosopher in the woods, and no one is there to hear his screams for help, did he really make a sound?

Let's Make a Deal: To Switch or Not to Switch

It's time to play for **FABULOUS** prizes.

Let's Make a Deal: To Switch or Not to Switch

There are three doors. Behind one of the three is a fabulous prize. Behind the other two doors are Zonks! The host asks you to choose one of the three doors. After you have selected your door, he knowingly reveals a Zonk behind one of the other two doors and gives you the opportunity to switch doors.

Should you switch?

Let’s Make a Deal

Let’s begin by playing lots and lots of times and see what happens. This will allow us to establish an **experimental probability**.

Experimental probability is the probability of an event occurring (ratio of favorable outcomes to total outcomes) in a particular experiment. An experiment of this sort is generally a series of trials of flipping coins, rolling dice, picking doors, etc.

The Monty Hall Experiment

Let's Make a Deal

In our experiment we won approximately 1/3 of the time when we did not switch and around 2/3 of the time when we switched. That seems odd.

But what have we really shown? Do these results constitute a **mathematical proof** demonstrating that one should always switch?

Let's Make a Deal

A **mathematical proof** is a demonstration that a mathematical statement is necessarily true; that the statement couldn't be otherwise without resulting in a contradiction.

Though we ran 1000 trials for each case, our results could be aberrant. It is possible, for instance, though highly unlikely that I could flip a coin 1000 times and each time it lands on heads though the probability of getting heads on a single flip is ½. We could have just experienced one of the most impressive statistical anomalies in recorded history…

Let’s Make a Deal

…but we didn’t.

Let us now provide a logical demonstration, a conceptual proof, one should always switch. It is not quite a formal mathematical proof, but it will explain the experimental results and demonstrate that one should always switch.

We’ll use a branch of mathematics known as probability theory to provide the analysis.

Let’s Make a Deal

What is the probability that the prize is behind the door you chose?

1/3

You choose your door, the host opens a different door, revealing a Zonk, and gives you the opportunity to switch. You keep your door. How often will the prize be behind your door?

1/3

Let's Make a Deal

So, we've established that by keeping the door, your odds of winning are 1/3, which matches our experimental results.

But, what if you switch? We know that the prize is behind the door you chose 1 out of every 3 times. That means the prize is in one of the other two doors 2 out of every 3 times.

Let's Make a Deal

Here's the catch…

We know that 2 out of 3 times, the prize is behind one of the doors you didn't choose.

By revealing a Zonk behind on of those two doors, the host shows you (in those 2 out of 3 cases) exactly which remaining door the prize is behind. (As long as you didn't pick the right door in the first place, you're guaranteed to win by switching)

Therefore **2/3 of the time** the prize is behind the door to which you can switch!

Let’s Make a Deal

Let’s look at it another way…

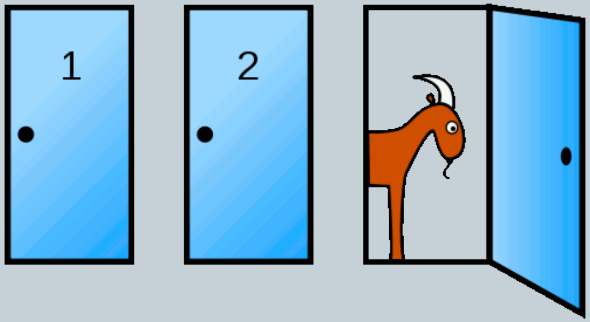
Suppose you choose door #1 and decide to switch. Here are the three **equally likely** outcomes:

1. The prize is behind door #1. Doors #2 and #3 contain Zonks. You switch. Zonk!
2. The prize is behind door #2. The host reveals door #3. You switch. Prize!
3. The prize is behind door #3. The host reveals door #2. You switch. Prize!

Let's Make a Deal

It works exactly the same way if you initially choose door #2 or door #3 and then switch.

In each case, you win 2 out of 3 times when you switch!



Let's Make a Deal

We can give a more formal analysis using Bayes' Theorem.

Bayes' Theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Without loss of generality we name the doors as follows:

a=door initially picked by contestant
b=door opened by host
c=third door

Let’s Make a Deal

Let A, B, and C represent the prize being behind a, b, and c, respectively. Let O represent the host opening b.

Bayes’ Theorem for Monty Hall:

$$P(C|O) = \frac{P(O|C) \cdot P(C)}{P(O)}$$

P(O|C)=1. The host has to open b given that the prize is behind c.

P(C)=1/3. It is equally probable that the prize is located behind any one of the three doors.

Let's Make a Deal

$$P(C|O) = \frac{1 \cdot 1/3}{P(O)}$$

What is P(O)?

The probability that the host opened door b is conditional upon where the prize is located.

More formally:

$$P(O)=(P(O|A)\cdot P(A))+(P(O|B)\cdot P(B))+(P(O|C)\cdot P(C))$$

Let's Make a Deal

$$P(O)=(P(O|A)\cdot P(A))+(P(O|B)\cdot P(B))+(P(O|C)\cdot P(C))$$

$P(A)$, $P(B)$, and $P(C)$ are each 1/3. Remember, it is equally probable that the prize is located behind any one of the three doors.

$P(O|A)=1/2$. The host could have opened either b or c.
$P(O|B)=0$. The host can't open the prize door.
$P(O|C)=1$. The host could only open door b.

$$P(O)=(1/2\cdot 1/3)+(0\cdot 1/3)+(1\cdot 1/3)$$
$$=1/2$$

Let's Make a Deal

Therefore:

$$P(C|O) = \frac{1 \cdot 1/3}{1/2}$$

$$P(C|O) = 2/3$$

The probability that the prize is in the third door, c, given that the host opened b is 2/3.

Deal or No Deal: Besting the Banker

Deal or No Deal: Besting the Banker

In front of you, you have the choice of 26 suitcases, all containing a cash prize. The prizes range from $0.01 to $1,000,000. To play the game, you pick a case. Then, in subsequent rounds you pick cases to reveal the prize amounts within. This is followed by an offer from the "banker."

Do you take the deal or keep going to eventually claim the prize in the case you originally chose?

Deal or No Deal

Can we use mathematics to help us know when to take the banker's deal and when not to?

In order to know whether or not the deal being offered is a good one, we have to be able to come up with a value for our case. To do so, we need to use the concept of **expected value**.

The **expected value (EV)** of a variable is the weighted average of all possible values it could have. It tells us not the actual value, but, on average, what the value of the variable is. For Deal or No Deal, it tells us, on average, how much our case is worth.

Deal or No Deal

An example: I tell you that I will give you a dollar for each pip (little dot) facing up when you roll a regular six-sided die. What is the EV of your roll?

To find the EV, we multiply the value of each possible outcome ($1, $2, $3...$6) by its probability (1/6 in each case) and then find the sum.

In this case:

EV(roll)= $1(1/6) + $2(1/6) + $3(1/6) + $4(1/6) + $5(1/6) + $6(1/6)=$3.50

Notice, because all outcomes are equally probable, EV=sum of outcomes ($21)/number of outcomes (6)

Deal or No Deal

But what does that mean for playing Deal or No Deal?

To find the EV of your suitcase, simply sum up all of the remaining suitcase values and divide by the number of remaining suitcases (because all suitcase values are equally likely to be in your suitcase).

In theory, if the banker offers something less than the expected value of your suitcase, NO DEAL. If the banker offers something more, DEAL.

So, playing rationally (in a mathematical sense) means you may have to turn your nose up at $200,000!

Plinko:
Let the Chips Fall Where They're Most Likely To

Plinko: Let the Chips Fall Where They're Most Likely To

In the game of Plinko on the Price is Right, contestants have the chance to win five Plinko chips by playing a pricing game. They then send these chips plinking down the Plinko board to win cash prizes. The amount of the prize is determined by the slot in which the chip lands. Simple, but strangely compelling…

With all of the random bouncing, does it really matter from whence the Plinko chip plinks?

Plinko

Good question. How do we find the answer?

We could start by just dropping chips over and over and tracking the results. That would be mesmerizing fun, but would only get us as close to an answer as **experimental probability** can. And we want to be certain. We want a proof.

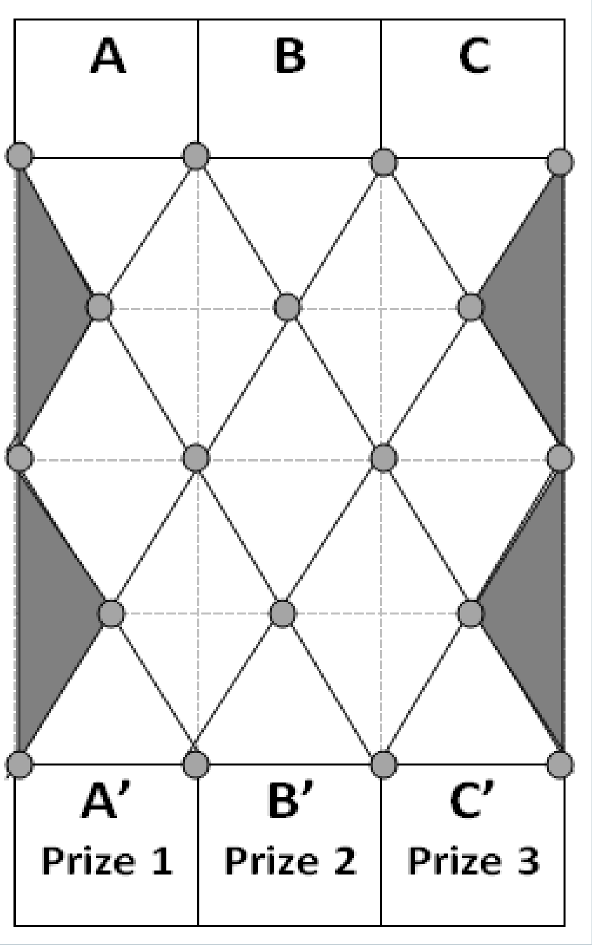
To see if the outcome depends on where the chip is released, we need to be able to demonstrate the probabilities with which a chip will land in the various slots based on the release points. Let's start with a smaller board…

Plinko

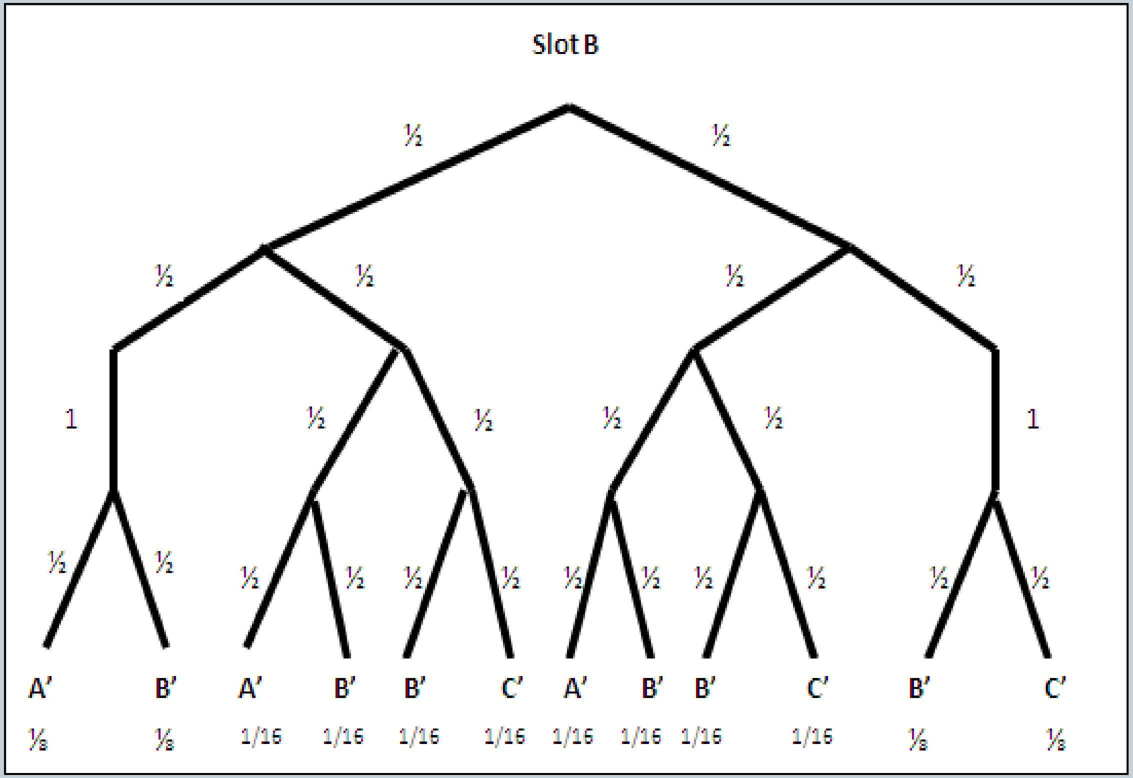
What happens when we start a chip in B?

At the first peg it has a ½ probability of going left and a ½ probability of going right. If it falls left at peg 1, it hits another peg and again has a ½ probability of going left and a ½ probability of going right. If it falls right at peg 1…

We can represent the possible paths as follows.



Plinko



Slot B
½
½
½
½
½
½
1
½
½
½
½
1
½
½
½
½
½
½
½
½
½
½
½
½
½
½
A′
B′
A′
B′
B′
C′
A′
B′
B′
C′
B′
C′
⅛
⅛
1/16
1/16
1/16
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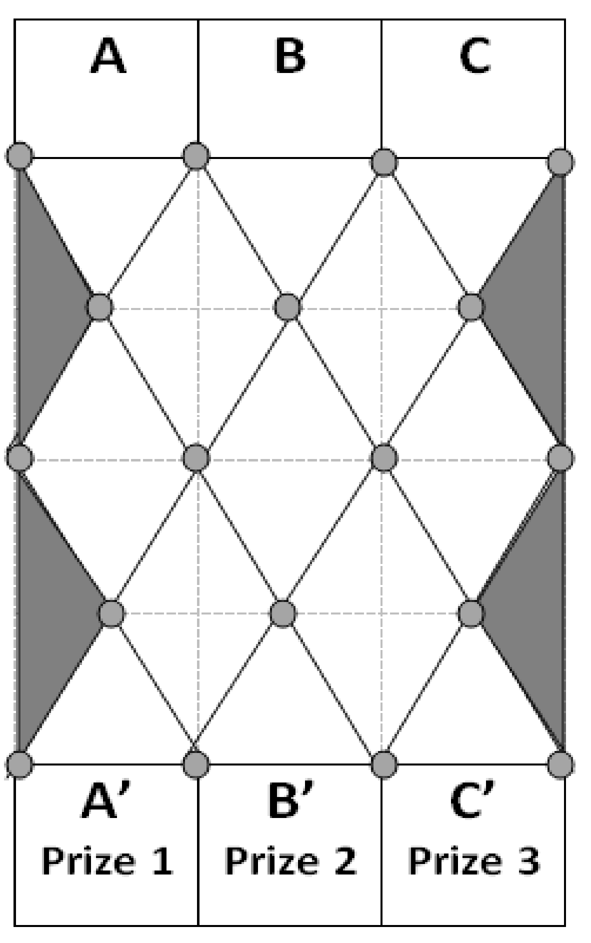
⅛



A
B
C
A′
Prize 1
B′
Prize 2
C′
Prize 3

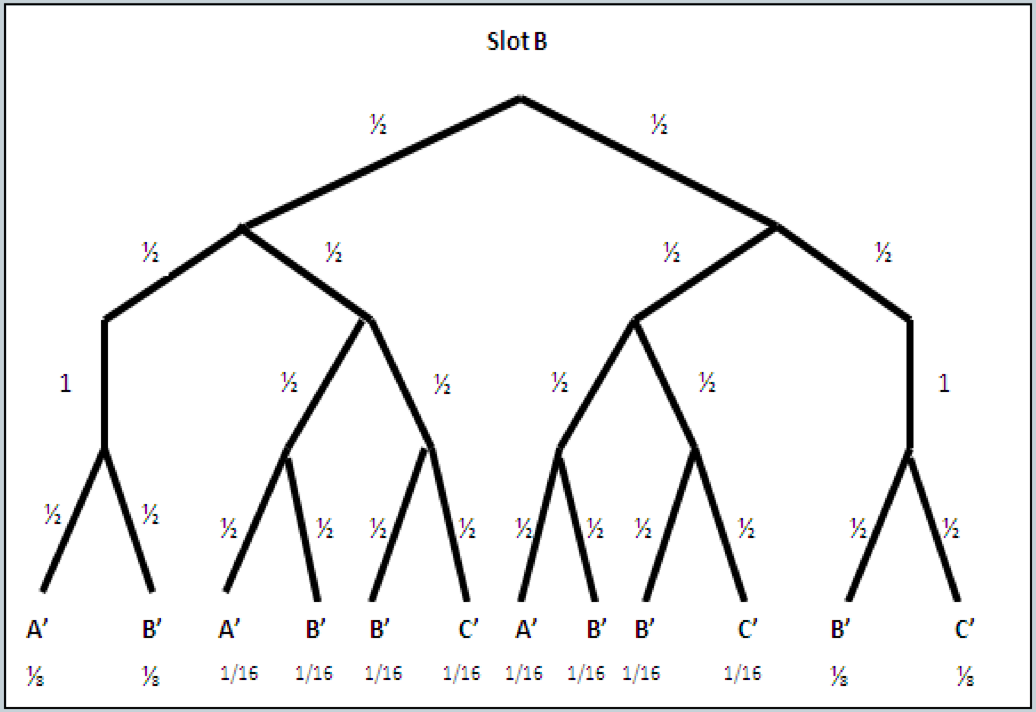
Plinko

Multiplying down each branch of the tree and then adding the probabilities for each slot, we get:

P(A’ | B) = 1/4

P(B’ | B) = 1/2

P(C’ | B) = 1/4

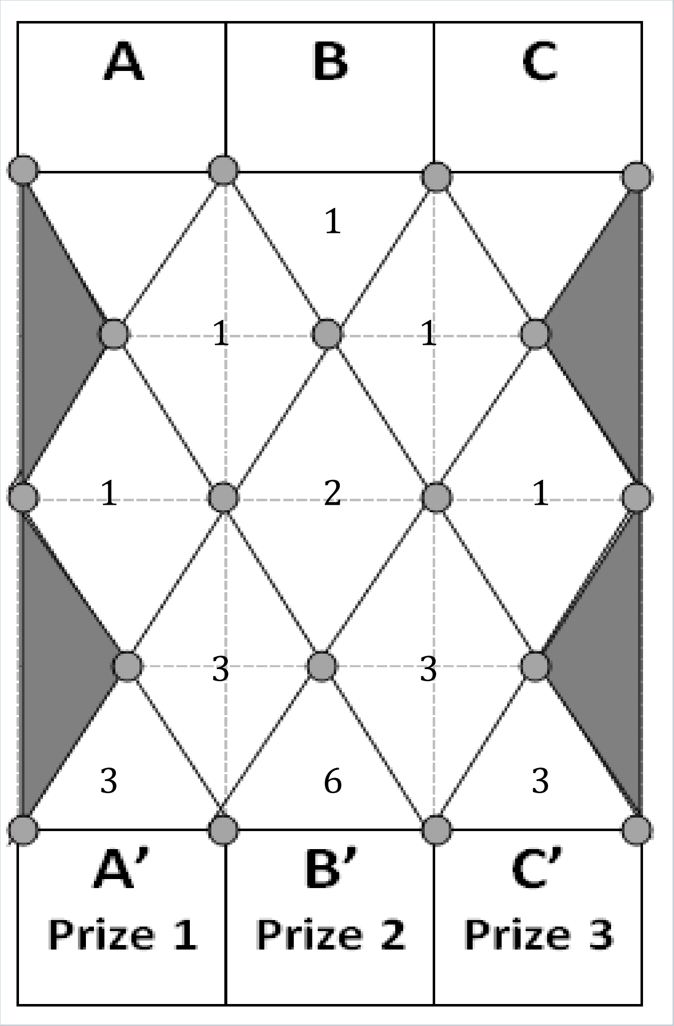


Plinko

Let's look at it another way…

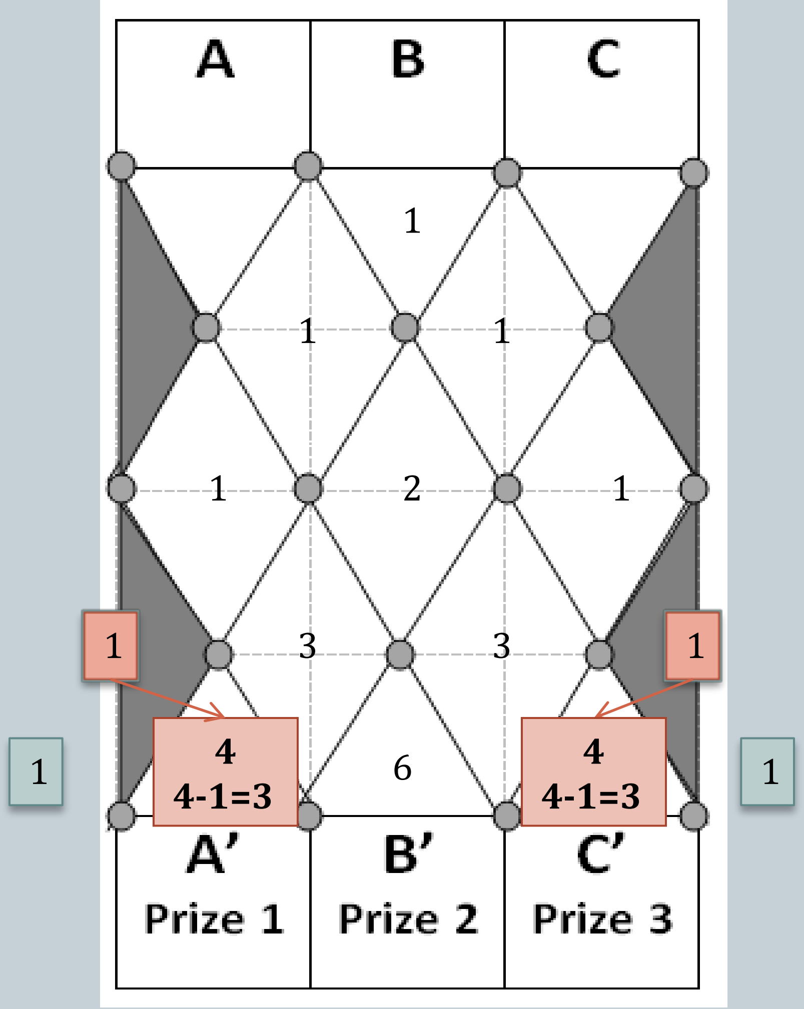
The numbers on the mini-Plinko board represent the number of possible paths to the slot. Is the pattern familiar?

You can transpose Pascal's Triangle onto the Plinko board to count the paths and determine the probabilities.



Plinko

To use Pascal's Triangle to count the number of paths from a given slot, we put the top of the triangle at the slot from which we are counting and then subtract appropriately when a chip hits a wall. We do this because the wall takes away one of the paths.

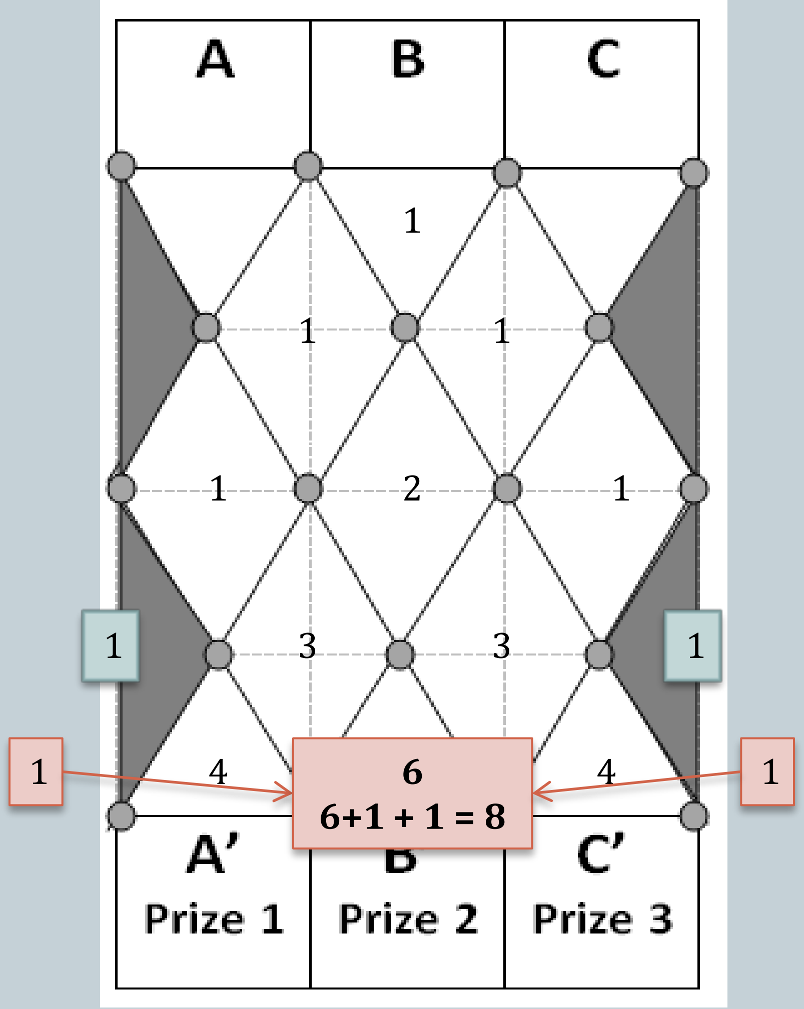


Plinko

We can also use the triangle to tell us the probabilities. This time we have to reflect the triangle numbers back onto the board when a chip hits the wall (because, as in the tree, the subsequent paths aren't halved so they get double counted).

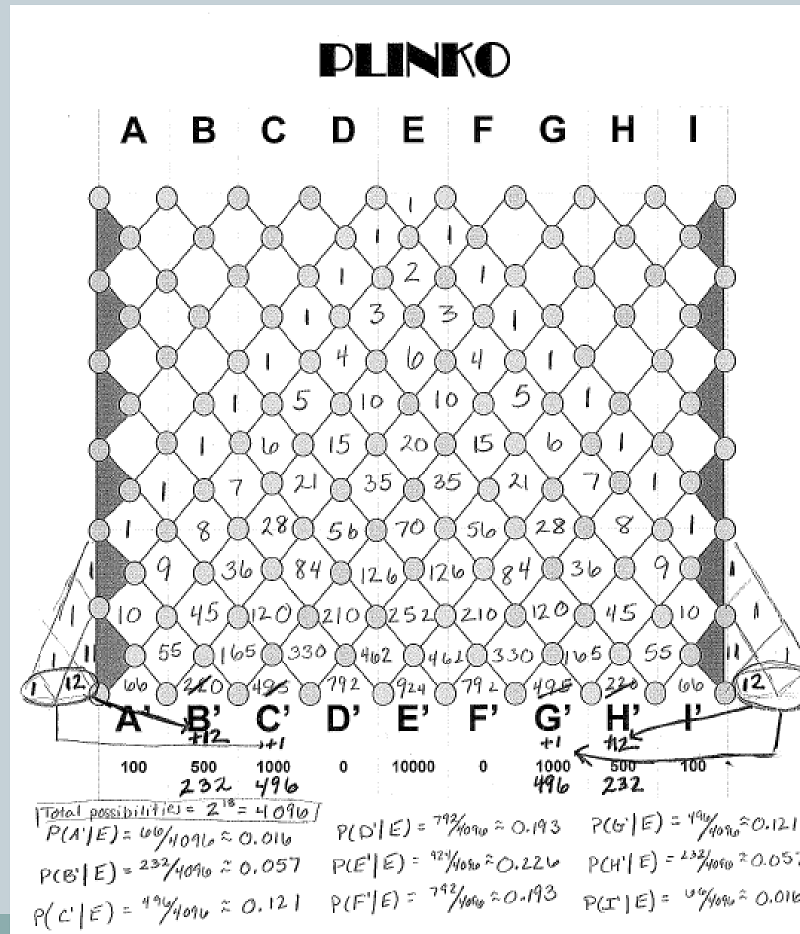
To get the probability for a slot, put the value of that slot in the numerator and the sum of all of the slot values in the denominator.

E.g., P(A' | B) = 4/16



Plinko

When you do it for the big board, this is what you get…



Plinko

And the complete table of probabilities with the EV calculated for each slot…

The moral of the story: **just drop it in the middle.**

	A	B	C	D	E	F	G	H	I
A' 100	0.226	0.193	0.121	0.054	0.016	0.003	0.000	0	0
B' 500	0.387	0.346	0.247	0.137	0.057	0.016	0.003	0.000	0
C' 1000	0.242	0.247	0.241	0.196	0.121	0.054	0.016	0.003	0.000
D' 0	0.107	0.137	0.196	0.226	0.193	0.121	0.054	0.016	0.006
E' 10,000	0.032	0.057	0.121	0.193	0.226	0.193	0.121	0.057	0.032
F' 0	0.006	0.016	0.054	0.121	0.193	0.225	0.196	0.137	0.107
G' 1000	0.000	0.003	0.016	0.054	0.121	0.196	0.241	0.247	0.242
H' 500	0	0.000	0.003	0.016	0.057	0.137	0.247	0.346	0.387
I' 100	0	0	0.000	0.003	0.016	0.054	0.121	0.193	0.226
Expected Value	**$778.10**	**$1012.80**	**$1605.10**	**$2262.20**	**$2561.70**	**$2262.20**	**$1605.10**	**$1012.80**	**$778.10**

Golden Balls:
To Steal or Not to Steal

Golden Balls

Golden Balls

If you were playing Golden Balls, what should you do?

We'll use a branch of mathematics known as **game theory** to provide an answer.

Game theory is the mathematical exploration of games and strategic behavior.

Golden Balls

A “game” is defined as an interaction containing the following elements:

1. There are two or more players.
2. At least one player has a choice of actions.
3. The game has a set of outcomes for each player.
4. The outcomes depend on the choices of actions by the players.

A “strategy” is the choice of actions.

Golden Balls

Before analyzing Golden Balls, let's look at a related game: The Prisoner's Dilemma.

Two suspects are arrested by the police. The police have separated both prisoners, and visit each of them to offer the following deal. If one testifies for the prosecution against the other and the other remains silent, the betrayer goes free and the silent accomplice receives a ten-year sentence. If both remain silent, both prisoners are sentenced to only one year in jail. If each accuses the other, each receives a five-year sentence. Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know about the betrayal before the end of the investigation.

Which strategy should the prisoners choose?

Golden Balls

We can determine the best strategy for each prisoner by analyzing the possible outcomes using a **payoff matrix**.

		Prisoner B	
		Stay Silent	**Betray**
Prisoner A	**Stay Silent**	1,1	10,0
	Betray	0,10	5,5

Golden Balls

Prisoner B

Prisoner A

	Stay Silent	Betray
Stay Silent	1,1	10,0
Betray	0,10	5,5

In all cases, Prisoner A does less time if he chooses betray.

If Prisoner B chooses to stay silent, Prisoner A serves no time instead of one year.

If Prisoner B chooses to betray, Prisoner A serves five years instead of ten years.

And likewise for Prisoner B.

Golden Balls

Prisoner B

Prisoner A

	Stay Silent	Betray
Stay Silent	1,1	10,0
Betray	0,10	5,5

Conclusion: For both prisoners betray is the best strategy.

In fact, betray is a **strictly dominant** strategy. That is, it performs better than any other strategy in every single case.

Golden Balls

Prisoner B

Prisoner A

	Stay Silent	Betray
Stay Silent	1,1	10,0
Betray	0,10	5,5

If the two prisoners are rational, they will both pick betray and end up in the 5,5 square.

Both prisoners choosing the betray strategy is the Nash Equilibrium for this game.

A **Nash Equilibrium** is a set of strategies for each player such that no player can improve his outcome by unilaterally changing his strategy.

Golden Balls

Prisoner B

Prisoner A

	Stay Silent	Betray
Stay Silent	1,1	10,0
Betray	0,10	5,5

Interestingly, both would do better if they each chose to stay silent, but they can't rationally get there.

This square is known as the **Pareto optimum**. A Pareto improvement is a change that would make at least one person better off without making anyone else worse off. If no Pareto improvement can be made, we say that the situation is Pareto efficient or is a Pareto optimum.

Golden Balls

Back to Golden Balls… What does the payoff matrix look like for this game?

		Contestant B	
		Split	**Steal**
Contestant A	**Split**	£50k, £50k	0, £100k
	Steal	£100k,0	0,0

Golden Balls

Contestant B

		Split	Steal
Contestant A	**Split**	£50k, £50k	0, £100k
	Steal	£100k,0	0,0

If contestant A steals and contestant B splits, contestant A wins 100k instead of 50k.

If contestant A steals and contestant B steals, contestant A gets the same as if she would have split. (If contestant B steals, contestant A has no way of getting any money.)

Golden Balls

Contestant B

Contestant A

	Split	Steal
Split	£50k, £50k	0, £100k
Steal	£100k,0	0,0

Stealing is the best strategy. In all cases it results in an outcome which is better than or equal to any other strategy.

A strategy that does this is called a **weakly dominant strategy**.

Golden Balls

Contestant B

		Split	Steal
Contestant A	**Split**	£50k, £50k	0, £100k
	Steal	£100k,0	0,0

What is the Nash Equilibrium for this game?

The Nash Equilibrium is the set of strategies that results in the 0, 0 square.

What a brilliant game show for the producers—no one should ever win any money!

Golden Balls

Do all games have a Nash Equilibrium, that is, a set of optimal strategies?

Answer: All multiplayer games with finite payout matrices have at least one Nash Equilibrium.

Some have more than one!

Golden Balls

Consider, for example, the simple game of chicken.

Two bumper cars are speeding towards each other. Each driver can swerve away to avoid the collision or stay the course. If one driver swerves and the other does not, the one who swerved loses pride. If they both swerve, they both lose some pride, but not as much. If neither swerves, they collide and have sore necks for the rest of the day.

Which strategy should the drivers choose?

Golden Balls

What does the payoff matrix look like for chicken?

For payoff amounts, we'll just use a ranking of 1 for worst through 4 for best.

		Driver B	
		Swerve	**Stay**
Driver A	**Swerve**	3,3	2,4
	Stay	4,2	1,1

Golden Balls

		Driver B	
		Swerve	**Stay**
Driver A	**Swerve**	3,3	2,4
	Stay	4,2	1,1

Whether swerving or staying is better for Driver A depends on what Driver B does, and the same is true for Driver B.

The 4,2 and 2,4 squares are both Nash Equilibria.

Golden Balls

Is it always best to pick one action and do it every time?

Let's look at a game that involves repeated plays.

Consider the game of Matching Pennies:

> Each of two players simultaneously shows either a head (H) or a tail (T). If the pennies match, one player wins both coins. If they do not match, then the other player wins both.

Which strategy should the players choose?

Golden Balls

What does the payoff matrix look like for matching pennies?

Player B

		H	T
Player A	H	1, -1	-1, 1
	T	-1, 1	1, -1

Golden Balls

		Player B	
		H	T
Player A	H	1, -1	-1, 1
	T	-1, 1	1, -1

The best outcome for Player A depends on what Player B does.

If Player A always plays H, then player B will start to always play T, and vice versa.

Golden Balls

Player B

		H	T
Player A	H	1, -1	-1, 1
	T	-1, 1	1, -1

Clearly it's best for player A to "mix it up" and do each some percentage of the time.

A strategy like that is called a **mixed strategy** (as opposed to a **pure strategy**).

Golden Balls

		Player B	
		H	**T**
Player A	**H**	1, -1	-1, 1
	T	-1, 1	1, -1

Let's say Player A chooses heads with probability p. Since the total probability has to add up to one, that means Player A chooses T with probability 1 – p.

Golden Balls

		Player B	
		H	**T**
Player A	**H**	1, -1	-1, 1
	T	-1, 1	1, -1

If p is greater than ½ , then Player B should always play T. In that case, Player A should expect to win

value = (-1)(p) + (1)(1 – p) = – p +1 – p = 1 – 2p.

But since p is greater than ½, that means that 1 – 2p is negative!

Golden Balls

		Player B	
		H	**T**
Player A	**H**	1, -1	-1, 1
	T	-1, 1	1, -1

If p is less than ½ , then Player B should always play H. In that case, Player A should expect to win

value = (-1)(1 – p) + (1)(p) = – 1 + p + p = 2p – 1.

But since p is less than ½, that means that 2p – 1 is negative!

Golden Balls

		Player B	
		H	**T**
Player A	**H**	1, -1	-1, 1
	T	-1, 1	1, -1

A strategy is typically optimal if the opponent can't change their strategy to gain an advantage. In other words, Player B should do no better if she plays H or T against Player A. We can find that by setting the cases equal and solving for p.

$$1 - 2p = 2p - 1$$

$$4p = 2$$

$$p = 1/2$$

Golden Balls

		Player B	
		H	**T**
Player A	**H**	1, -1	-1, 1
	T	-1, 1	1, -1

If p is exactly ½, then Player B can do anything and the value is the same, 0.

That's the best Player A can do, adopt a mixed strategy of ½ H and ½ T, and the same is true for Player B.

Golden Balls

Let’s consider the Prisoner’s Dilemma and Golden Balls again. Is there any way for the players to get the better result from cooperating, even though the Nash Equilibrium says they should not?

Answer: Not if they only play one time.

Golden Balls

But if they play repeatedly, then players can induce cooperation by setting a pattern of rewards for cooperation and penalties for noncooperation.

This is known as a **tit-for-tat** strategy.

Golden Balls

For a tit-for-tat strategy to work, it must meet several conditions:

Nice: The player should not defect before an opponent does.

Retaliating: The player should sometimes retaliate when the opponent defects.

Forgiving: After retaliating, the player must fall back into cooperating if the opponent stops defecting.

Non-envious: The player is not trying to score more than the opponent.

Golden Balls

For a tit-for-tat strategy to work, the game must also not last a known, finite number of turns.

Otherwise it's best for a player to defect on the last turn, since the opponent won't get the chance to punish the player. Therefore, if both players are rational, they'll both defect on the last turn. Thus the player might as well defect on the second-to-last last turn, since the opponent will defect on the last turn regardless. And so on.

Conclusion

What have we learned:

- Always switch doors on Let's Make a Deal. Probabilities change with new information!
- The Deal or No Deal banker is tricky and cheap.
- Drop the Plinko chip in the middle.
- Always steal from your opponent on Golden Balls if you're playing once. It's mean, but gives the best results no matter what your opponent does.
- Play nice in a game like Golden Balls if you play over and over again. Cooperating can benefit you both.

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