Exciting Math for your 4th Year Course

Social Decision Making:

Voting

Fair Division

Game Theory

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Topics	Unifying Themes
• Voting • Fair Division • Game Theory	Fairness Social Decision Making

Math of Voting

- Ranked-Choice Voting
- Weighted Voting

Math of Voting

Ranked-Choice Voting

- Vote by <u>ranking</u> the candidates according to your preferences, rather than just voting for your single favorite candidate.
- Yields richer data about the "will of the people."
- Several methods for analyzing the results, such as:
  - Instant Runoff Voting (IRV)
  - Borda method (assigning points for preferences)
  - Condorcet method (each pair run off head to head)
  - Related: Approval voting

Solve voting problems like:

U.S. mid-term elections, November 2010
(from FairVote.org)

- Democrats, in the final days of the campaign, spent upwards of $15,000 on a mailer to targeted households identifying Mark Vogel [3rd party candidate] as the true conservative candidate in the race.
- Republicans fielded fake Green Party candidates to siphon votes from Democrats in Arizona.

– **Why use this phony (unethical?) strategy?**
To exploit the weakness of non-ranked-choice voting when there are 3 or more candidates.

– **Can it be avoided?**
Yes, use ranked-choice voting.

Solve voting problems like:

The presidential election in France this week (April 22, 2012):

- Close contest between the incumbent (Sarkozy) and left-wing challenger (Hollande), with strong showing for far-right (Le Pen). No candidate received a majority of votes in the first round of voting, so by French law a run off is required.
  - **Run off elections are expensive and have low turnout. Can they be avoided?**
    Yes, use ranked-choice voting.

Ranked-choice voting in the US

For example, Newsweek article …

- **An Electoral Experiment in North Carolina**
- By McKay Coppins, *Newsweek*, October 24, 2010
- The logic of general elections is simple: winner takes all. This, of course, can encourage nasty campaigning—and at the end of a race with more than two candidates, the victor often wins with only a plurality (not a majority) of support. Searching for a solution, a handful of cities have experimented with an alternative approach, in which voters rank candidates in order of preference. If no one receives more than half of the first-place votes, a winner is picked by tallying second and third choices. (You may not get your first pick, but chances are the winner won't be your worst nightmare.)
- On Nov. 2, this approach will be used for the first time in a statewide election. It's a small race—North Carolina court of appeals judge—but proponents hope it will encourage the more than 20 states that have mulled the system since 2000.

Example and Software

Take a field trip, stop for lunch. Everyone must eat at the same restaurant. Where should we eat? How to decide?

… Vote, but use ranked-choice voting …

- Why should Subway win?
- Why should Subway <u>not</u> win?
- Why should KFC win?
- Why should McDonald's win?

	Rankings					
KFC	1	1	2	2	3	3
McDonald's	2	3	1	3	1	2
SUBWAY	3	2	3	1	2	1
Number of Voters	6 voters	4 voters	6 voters	7 voters	5 voters	5 voters

ANSWERS

- Why should Subway win? [most 1st preference votes]
- Why should Subway not win? [most 3rd preference votes]
- Why should KFC win? [least 3rd preference votes]
- Why should McDonald's win? [runoff top two, McD and Subway, McD wins]

	Rankings					
KFC	1	1	2	2	3	3
McDonald's	2	3	1	3	1	2
SUBWAY	3	2	3	1	2	1
Number of Voters	6 voters	4 voters	6 voters	7 voters	5 voters	5 voters

Software can help …

Core Math Tools from NCTM:

www.nctm.org/coremathtools/

Go to Advanced Apps – Ranked-Choice Voting

(More to come.)

Voting Issues

- The plurality winner can be (and often is) the candidate who is least preferred by the most voters! That's just not fair!
- Different methods can produce different winners.
- Is there a perfect method?
  No – Arrow's Impossibility Theorem
- What to do? Plurality is the worst (when $\geq$ 3 candidates). Experts prefer … Borda, Approval

Math of Voting

Weighted Voting

- Each voter has more than one vote (weight).
- Ex: shareholders voting
- Ex: states voting
- <u>Key idea</u>: Weight vs. Power
- Ex: Shareholder may have *twice the # of votes* (weight), but the *same power* (in terms of forming winning coalitions).
- Measure power – Banzhaf, Shapley-Shubik, …

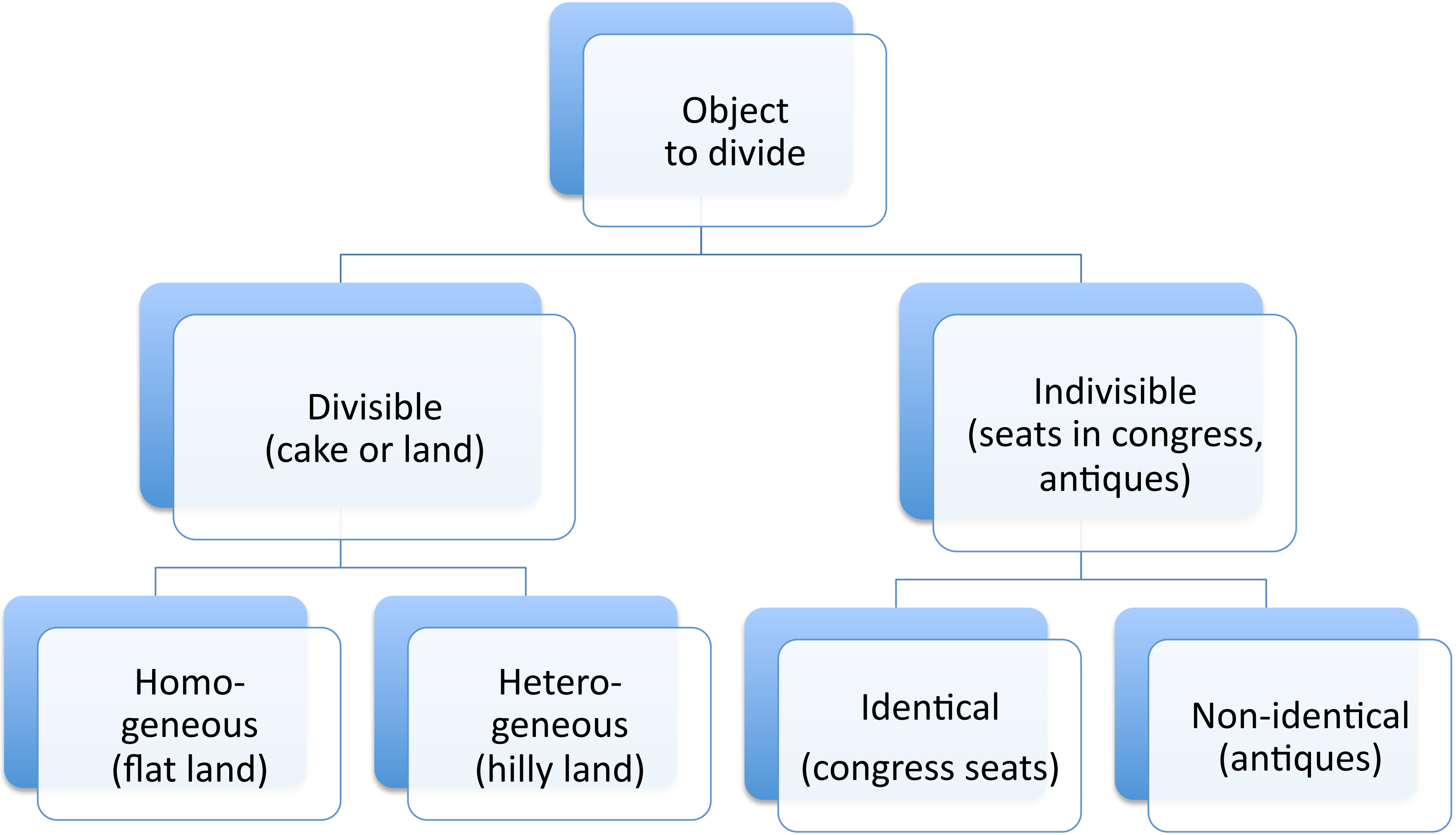
Math of Fair Division

Solve problems like these:

- Article I, Section 2 of the Constitution of the United States – “Representation and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers.”
- Divide a cake fairly.
- Divide an estate fairly.

Categories of Fair Division Problems

1. Divisible or Indivisible objects?
2. Sameness of objects?


Object
to divide
Divisible
(cake or land)
Indivisible
(seats in congress,
antiques)
Homo-
geneous
(flat land)
Hetero-
geneous
(hilly land)
Identical
(congress seats)
Non-identical
(antiques)

Apportioning Seats in Congress

- Rounding fractions
  - up, down, to the closest, based on the geometric mean
- Historical
  - ✓ Hamilton – round down, and adjust
  - ✓ Jefferson – round down, and adjust differently
  - ✓ Adams – round up, and adjust
  - ✓ Webster – round to the closest (arithmetic mean)
  - ✓ Huntington-Hill – round based on geometric mean

Canonical Fair Division Problems with classical and new methods

Cake Cutting

(divisible heterogeneous objects)

- Classical solution: Cut and Choose
- New development: Surplus Procedure

Estate Division

(non-identical, mostly indivisible objects)

- Classical solution: Knaster's procedure
- New development: Adjusted Winner

Surplus Procedure

A fair division procedure for dividing a cake between two people …

Surplus Procedure

- **A fair division procedure for dividing a cake between two people**
- Assumptions: Only parallel, vertical cuts, perpendicular to the horizontal axis, are made. A referee gathers valuation information independently from each person and uses that information to carry out the procedure. The steps of the procedure are then as follows.
- **Step 1.** A mark is made, labeled point *a*, indicating person A's opinion of a 50-50 cut of the cake. Another mark is made, labeled point *b*, indicating person B's opinion of a 50-50 cut.
- **Step 2.** If *a* and *b* coincide, then the cake is cut at *a* = *b*, and each person gets one of two pieces.
- **Step 3.** If *a* and *b* are not the same point, then assume that *a* is to the left of *b*. Person A receives the piece to the left of *a*, and person B receives the piece to the right of *b*. So far, each person feels they have received a 50% share of the cake, according to their own respective valuations.
- **Step 4.** The portion between *a* and *b* is the portion of the cake that remains, called the *surplus*. Let *c* be the cut point between *a* and *b* that divides this surplus so that **each person receives the same proportion of the value of the surplus, relative to their respective valuations of the cake in the surplus**. Then person A receives the surplus portion to the left of *c*, and person B receives the surplus portion to the right of *c*. Thus, person A receives in total everything to the left of *c*, and person B receives everything to the right of *c*.

Fairness Criteria

Voting

- Arrow's fairness criteria

Fair Division

- Proportionality, Envy-freeness, Equitability, etc.

- *Proportionality* is satisfied if each of n people thinks he or she is getting at least 1/n of the total value. This seems fair since each person gets an equal proportion (or better), according to each person's valuation.
- *Envy-Freeness* is satisfied if no person envies another person's share. That is, every person receives a share he or she considers at least tied for the most desirable. This seems fair since everyone is happy with their portion, in that nobody wants to give up their portion in exchange for someone else's portion.
- *Equitability* is satisfied if each person's subjective valuation of his or her own portion is the same as everyone else's valuation of their portion. This seems fair since each person is getting the same value, according to each person's own valuation.

- Cut-and-choose satisfies Proportionality and Envy-Freeness, but not Equitability.
- The surplus procedure (SP) satisfies Proportionality and Envy- Freeness, but not Equitability. However, SP does satisfy proportional equitability.
- Knaster's procedure satisfies Proportionality but not Envy- Freeness (unless just two people) and not Equitability.
- The Adjusted Winner (AW) method satisfies Proportionality, Envy-Freeness, and Equitability.

Game Theory

Would you split or steal?

Popular game show in the UK….

Video:

www.youtube/watch?v=p3Uos2fzlJ0

(Or search for: Golden Balls –100,000 Split or Steal? 14/03/08 – YouTube)

Split or Steal Game Matrix

What would you do?

		Ms. Collum	
		STEAL	**SPLIT**
Mr. Rowe	**STEAL**	0, 0	100,000, 0
	SPLIT	0, 100,000	50,000, 50,000

Nash's Theorem

Every finite non-cooperative game with two or more players has a Nash equilibrium.

Nash Equilibrium:

- Set of strategies, one for each player, such that no player can improve his/her payoff by unilaterally changing strategy. That is:
- Each player has no incentive to change strategy, if the other player does not change. That is:
- If row stays same, is column switch better? (no)
  If column stays the same, is row switch better? (no)

Is STEAL-STEAL a Nash Equilibrium?

SPLIT-SPLIT?

Is SPLIT-SPLIT better than STEAL-STEAL?

No matter what Rowe does, what should Collum do?

No matter what Collum does, what should Rowe do?

		Ms. Collum	
		STEAL	**SPLIT**
Mr. Rowe	**STEAL**	0, 0	100,000, 0
	SPLIT	0, 100,000	50,000, 50,000

ANSWERS

Is STEAL-STEAL a Nash Equilibrium? [yes]

SPLIT-SPLIT? [no]

Is SPLIT-SPLIT better than STEAL-STEAL? [yes]

No matter what Rowe does, what should Collum do? [STEAL]

No matter what Collum does, what should Rowe do? [STEAL]

		Ms. Collum	
		STEAL	**SPLIT**
Mr. Rowe	**STEAL**	0, 0	100,000, 0
	SPLIT	0, 100,000	50,000, 50,000

How Optimal is Optimal?

- A Nash equilibrium (STEAL-STEAL) may not provide the best cumulative payoff.
- A *dominant strategy* (STEAL) may not provide the best cumulative payoff.
- Another Optimality criterion:
  Maximize your minimum gain (*maximin*).
  Nash equilibrium may not be a maximin strategy.

✓ However, everything is straightforward with zero-sum games.

Zero-Sum Games

- One player’s gain is exactly the other player’s loss.
- The sum of the payoffs in any outcome is zero.
- Split or Steal is not a zero-sum game.

Consider a zero-sum game …

- Rowe and Collum are political candidates running against each other.
- Votes gained for Rowe are lost for Collum – zero sum game.
- Votes gained or lost depend on position on issue.
- Table shows votes gained or lost for Rowe.

		Ms. Collum's Position		
		FOR	**AGAINST**	**NEUTRAL**
Mr. Rowe's Position	**FOR**	11,000	8,000	–9,000
	AGAINST	4,000	3,000	2,000
	NEUTRAL	12,000	–11,000	–7,000

Minimax Theorem

Every finite, two-person, zero-sum game has optimal strategies.

- Maximin strategy for the maximizing player (row)
- Minimax strategy for the minimizing player (column)
- maximin = minimax = *value* of the game
- Value of game may or may not be a payoff entry.
- If it is, then it is called a *saddle point*, and it corresponds to *pure* optimal strategies for each player.

Complete the table – row mins and column maxs.

Circle the maximin and minimax.

		Ms. Collum's Position			
		FOR	AGAINST	NEUTRAL	row min
Mr. Rowe's Position	FOR	11,000	8,000	–9,000	
	AGAINST	4,000	3,000	2,000	
	NEUTRAL	12,000	–11,000	–7,000	
column max					

- 2000 is the maximum of Rowe's minimum values.
- 2000 is the minimum of Collum's maximum values.
- Optimal: Rowe–Against, Collum–Neutral

		Ms. Collum's Position			
		FOR	AGAINST	NEUTRAL	
Mr. Rowe's Position	FOR	11,000	8,000	–9,000	–9,000
	AGAINST	4,000	3,000	2,000	**2,000**
	NEUTRAL	12,000	–11,000	–7,000	–11,000
		12,000	8,000	**2,000**	

Minimax & Nash

AGAINST-NEUTRAL is the minimax/maximin solution. It is also a Nash equilibrium.

- Rowe has no incentive to change unilaterally from AGAINST.
- Collum has no incentive to change unilaterally from NEUTRAL.

		Ms. Collum's Position			
		FOR	AGAINST	NEUTRAL	
Mr. Rowe's Position	FOR	11,000	8,000	–9,000	–9,000
	AGAINST	4,000	3,000	2,000	2,000
	NEUTRAL	12,000	–11,000	–7,000	–11,000
		12,000	8,000	2,000	

What if value (= minimax = maximin) is not a payoff entry in the table?

- Use *mixed* strategies, instead of *pure* strategies.
- Mix the strategies using probability.
- Still get minimax and maximin, and they are still equal, now in terms of expected value.

Game Theory

Non-zero-sum games

- Split or Steal (like Prisoner's Dilemma)

Zero-sum games

- Political candidates' position on an issue.

Key Results

- Nash's theorem
- Minimax theorem
- Pure and mixed strategies

Nice Math for 4^{th} Year Course

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