Not "Just" a Fixed-Point Theorem

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by

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		Colin	
		C	D
Rose	C	(0, 0)	(-2, 1)
	D	(1, -2)	(-1, -1)

What is a game? (in Game Theory)

- Mathematical Model of a (competitive) situation
- Involves:
  - Some number of players
  - Each player has some number of strategies that will determine the outcome
  - Payoffs to each player (possibly different) based on the outcome

A Familiar Example: Rock-Paper-Scissors

Let's play! Everyone find a partner, play 3 rounds.

How do we model this as a "game" now?

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

So now we have players, strategies, and payoffs. How do we play rationally?

How did you choose your first move?

How did you choose second and third moves?

The difficulty of Game Theory:

- Payoffs? How do you know what makes sense?
- What strategies are available? More than just R/P/S.
- In the real world, how do you know who (all) the players are?
- How do things change when you play more than once?

In the abstract, there are several types of games:

- 2-person zero-sum - players in direct opposition; your loss is my gain.
- 2-person non-zero-sum - my gain could be your loss, but not always
- N-person - much more complicated

A more abstract example:

- Zero-sum
- Two players, four strategies each
- Payoffs, as written, are for Rose

	A	B	C	D
A	2	-1	1	0
B	5	1	7	-20
C	3	2	4	3
D	-16	0	0	2

How do we "solve" this? What does that mean?

- Best strategy?
- Best for whom?
- Why?

This outcome (Rose C, Colin B) is a Nash Equilibrium!

If both players play these strategies, neither could gain by switching.

Mixed-Strategy Solutions (zero-sum)

- When pure-strategy solutions don't exist
- Some probability of selecting a given pure strategy
- Apply expected value to see how well we will do
- Example: Rock-Paper-Scissors, let's solve it!

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

If, say, Colin played a mixed strategy of 1/2 R, 1/2 P then Rose could expect:

- Rose-R: Pr(Colin-R) * (R/R payoff) + Pr(Colin-P) * (R/P payoff) = (1/2) * 0 + (1/2) * (-1) = -1/2
- Rose-P: (1/2) * 1 + (1/2) * 0 = 1/2
- Rose-S: (1/2) * (-1) + (1/2) * 1 = 0

So P will (on average) yield the highest expected value: 1/2

What is optimal for Colin?

A mixed strategy that Rose can't exploit, i.e.
Rose-R Rose-P and Rose-S all give same expected value!

This means probabilities x, y, (1-x-y). So...

$$\text{Rose-R} : x(0) + y(-1) + (1-x-y)(1) = 1 - x - 2y$$
$$\text{Rose-P} : x(1) + y(0) + (1-x-y)(-1) = -1 + 2x + y$$
$$\text{Rose-S} : x(-1) + y(1) + (1-x-y)(0) = y - x$$

Which implies...

$$1 - x - 2y = -1 + 2x + y = y - x$$
$$2 - 3x - 3y = 0 \qquad 3x = 1$$
$$2 - 1 - 3y = 0 \qquad x = 1/3$$
$$y = 1/3$$

Note:

- The game is symmetric!
- The same calculation works for Rose and her optimal strategy
- Having both players optimal strategies and the value of the game is a solution to the game - our mixed-strategy NE.

Extends to non-zero-sum

	A	B
A	(2,3)	(3,2)
B	(1,0)	(0,1)

Rose A, Colin A is the equilibrium here:

- First number max in column
- Second number max in row

EQUILIBRIUM POINTS IN N-PERSON GAMES

By John F. Nash, Jr.*

Princeton University

Communicated by S. Lefschetz, November 16, 1949

A Nash Equilibrium always exists in finite games

Even if it requires mixed strategies

Existence of Equilibrium Points

A proof of this existence theorem based on Kakutani's generalized fixed point theorem was published in Proc. Nat. Acad. Sci. U. S. A., 36, pp. 48–49. The proof given here is a considerable improvement over that earlier version and is based directly on the Brouwer theorem. We proceed by constructing a continuous transformation T of the space of n-tuples such that the fixed points of T are the equilibrium points of the game.

THEOREM 1. *Every finite game has an equilibrium point.*

PROOF. Let $\mathfrak{s}$ be an n-tuple of mixed strategies, $p_i(\mathfrak{s})$ the corresponding pay-off to player i, and $p_{i\alpha}(\mathfrak{s})$ the pay-off to player i if he changes to his α^{th} pure strategy $\pi_{i\alpha}$ and the others continue to use their respective mixed strategies from $\mathfrak{s}$. We now define a set of continuous functions of $\mathfrak{s}$ by

$$\varphi_{i\alpha}(\mathfrak{s}) = \max\,(0,\, p_{i\alpha}(\mathfrak{s}) - p_i(\mathfrak{s}))$$

and for each component s_i of $\mathfrak{s}$ we define a modification s_i' by

$$s_i' = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})},$$

calling $\mathfrak{s}'$ the n-tuple $(s_1', s_2', s_3' \cdots s_n')$.

We must now show that the fixed points of the mapping $T: \mathfrak{s} \to \mathfrak{s}'$ are the equilibrium points.

First consider any n-tuple $\mathfrak{s}$. In $\mathfrak{s}$ the i^{th} player's mixed strategy s_i will use certain of his pure strategies. Some one of these strategies, say $\pi_{i\alpha}$, must be "least profitable" so that $p_{i\alpha}(\mathfrak{s}) \leqq p_i(\mathfrak{s})$. This will make $\varphi_{i\alpha}(\mathfrak{s}) = 0$.

Now if this n-tuple $\mathfrak{s}$ happens to be fixed under T the proportion of $\pi_{i\alpha}$ used in s_i must not be decreased by T. Hence, for all β's, $\varphi_{i\beta}(\mathfrak{s})$ must be zero to prevent the denominator of the expression defining s_i' from exceeding 1.

Thus, if $\mathfrak{s}$ is fixed under T, for any i and β $\varphi_{i\beta}(\mathfrak{s}) = 0$. This means no player can improve his pay-off by moving to a pure strategy $\pi_{i\beta}$. But this is just a criterion for an eq. pt. [see (2)].

Conversely, if $\mathfrak{s}$ is an eq. pt. it is immediate that all φ's vanish, making $\mathfrak{s}$ a fixed point under T.

Since the space of n-tuples is a cell the Brouwer fixed point theorem requires that T must have at least one fixed point $\mathfrak{s}$, which must be an equilibrium point.

Recall: Brouwer's Fixed-Point Theorem:

- Any continuous function from a closed ball to itself has a fixed-point.

Some Selected Applications

- Penalty kicks in soccer
- Student/parent/school effort and learning
- National Resident Matching Program
- Federal appellate court clerk hiring

Soccer Penalty Kicks

Do people naturally select NE strategies outside a laboratory setting?

- Zero-sum game between keeper and kicker
- Strategies of kick (or jump) to the right, left, or center
- Unique (mixed-strategy) NE
- Similar in structure to classic "matching pennies" game

		Keeper	
	L	C	R
Kicker L	P_L	Π_L	Π_L
C	μ	0	μ
R	Π_R	Π_R	P_R

$P_{L \text{ or } R}$ = Pr (goal | same side)

$\prod_{L \text{ or } R}$ = Pr (goal | different sides)

μ = Pr (goal | kicker selects C, keeper doesn't)

The Role of Effort in Educational Attainment

Game with three players:

- Students
- Parents
- School

Based on data from British National Child Development Study

Showed there's a NE for the three players, and linked effort from each to achievement

National Resident Matching Program

- Matching system (an NGO actually) for medical students and residency programs
- Students and programs both submit preferences Strategy: are your stated preferences your true preferences?
- Successful NRMP matching algorithm implements a NE upon the stated preferences - no program or student that would prefer to be matched are not

Where can NE go wrong?

- NE can result in the worst case
- Prisoner's Dilemma
- Iteration can lead to consistent series of "defecting"
- Example: Federal appellate court clerk hiring
  - Consistent date under-cutting of job offers
  - Led to total revamp of system

Math Review

What math do we see in elementary game theory?

Basic Probability Concepts

- Definitions, simple computations
- Rules such as multiplication rule
- Conditional probability, Bayes' Theorem
- Expected value (especially for mixed-strategy solutions)
- Probability trees

Basic Algebra Concepts

- Solving single equations
- Solving systems of equations (up to linear programming techniques for complicated games)

Also: Modeling!

Selected Resources, Readings

Chess Set (Shatranj in Iranian), glazed fritware, 12th century. New York Metropolitan Museum of Art. November 2006, by Zereshk.

Game Theory and Strategy. Philip D. Straffin, 1993.

Must Try Harder: Evaluating the Role of Effort in Educational Attainment. Fraja, Oliveira, and Zanchi. The Review of Economics and Statistics, 92(3): 577-597.

Non-Cooperative Games. J.F. Nash. The Annals of Mathematics 54(2): 286-295.

Equilibrium Points in N-Person Games. J.F. Nash. PNAS 36(1): 48-49.

The Nash equilibrium: A perspective. Holt and Roth. PNAS 101(12): 3999-4002.

The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory. A.E. Roth. Journal of Political Economy, 92(6): 991-1016.

Testing Mixed-Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer. Chiappori, Levitt, and Groseclose. The American Economic Review 92(4): 1138-1151.

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		C	D
Rose	C	(0, 0)	(-2, 1)
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Questions?
Want to teach game theory in the summer?
http://cty.jhu.edu/