Becoming Fluent with the Unit Circle and Trig

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Student Worksheets and Handouts

Purpose:

- Develop an understanding and fluency of unit circle values/relationships using manipulatives.
- Use the unit circle to graph $\sin\theta, \cos\theta$ functions by "unwrapping" the unit circle values.
- Understand why the graphs of $\tan\theta, \cot\theta, \sec\theta, \csc\theta$ appear as they do.
- Understand what $\arcsin\theta, \arccos\theta$ mean and how to evaluate in terms of principal solution.
- Extend graphing of trig functions to polar graphs.
- Extend the trigonometric knowledge to visualize/understand identities with manipulatives.

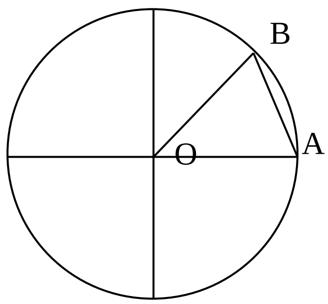
Materials:

- Paper plates
- String
- Scissors
- Sets of card stock triangles cut into 45° - 45° - 90°, 30° - 60° - 90° and 60° - 30° - 90° (yes, I know they are the same but you need to do these in three different)
- Large Unit Circle for classroom display & post-it notes
- Large display paper for graphs, small circular color coding labels
- Card stock of angle measures

Becoming Fluent with the Unit Circle

What is a Radian?

Paper plate activity – each student is given a flattened round paper plate and a piece of string. Find the center by folding the plate twice to form the axes. Label the center O. Use a marker to draw the axes. Label a point on the rim of the circle, A to form radius $\overline{OA}$. Have students cut a piece of string the length of the radius and then tape the string on the rim of the circle, starting at point A and ending at point B. **The measure of the central angle whose arc length is equal to the radius is 1 radian.**



Ask the students to estimate the number of degrees in a radian. *Hopefully*, someone will suggest 60 degrees. Mark $\overline{OB}$. Cut the plate along $\overline{OA}$ so that they can fold the plate over $\overline{OB}$ to estimate the number of radians in a semicircle . . . a little more than 3! Since they will come up with a little more than three radians, this is the time to connect the idea that there are π radians in a semicircle and there are 2π radians in a circle.

Angle Measures on the Unit Circle

Label the angles measures of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ on the paper plate with 0 and 2π at A. Fold the *upper* half of the plate into thirds. Draw a line (use a different color marker, if possible) along these folds all the way across the plate. Label as $\frac{\pi}{3}, \frac{2\pi}{3}$, etc. Fold these in half again and label (different color again) as $\frac{\pi}{6}, \frac{5\pi}{6}$, etc. making the point that $\frac{2\pi}{6}$ is the same as $\frac{\pi}{3}$ etc. Finally, make the folds and label (different color again) for $\frac{\pi}{4}, \frac{3\pi}{4}$, etc.

Identifying Unit Circle Values Using Reference Triangles

You will want to use a new paper plate for this activity.

Students should mark the x and y-axes on this new paper plate. Label the quadrants as well as the signs of the x and y-coordinates found in each quadrant.

Give each student 2 of each color of your pre-cut triangles.
If we determine the plate to have a radius of 1, then the hypotenuse of each of these triangles will have a measure of 1 unit. Using some geometry students can determine the lengths of the remaining sides.

Students should notice that two of the triangles are congruent.

Tape these triangles on the paper plate so that the right angle is on the x-axis for each in such a way that they can flip over the x-axis to demonstrate reference triangles in either QI or QIV and QII or QIII. The first should be placed so that it is at the central angle of $\pi/6$, the second, at the central angle of $\pi/4$ and the third, at the central angle of $\pi/3$. The second set of triangle will be placed at a central angle of $\frac{2\pi}{3}, \frac{3\pi}{4}$ and $\frac{5\pi}{6}$.

Label the lengths of each side of the triangles with the appropriate measure and sign, when positioned in QI and QII. Flip them to lay in QIII and QIV and label accordingly. This gives real meaning to the terms "terminal position" and how it relates to "reference angles".

Activities to Reinforce Unit Circle Fluency

Create post-it notes with left and right coordinates of unit circle values. (I use two different colors, one for cosine and the other for sine.) Give students these post-it notes randomly as they enter the classroom. Students are instructed to place them on the Large Unit Circle display.

Unit Circle Trading Cards – Inside/Out Activity
See attached document to make trading cards.

Using the Unit Circle to graph $\sin\theta, \cos\theta$ functions by "unwrapping" the unit circle values

Create large graphical displays. Label colored stickers with ordered pairs of $(\text{radian measure, function value})$. Use a different colored label for each function. Give students a label to place on the graphical display to create a graph of the trig function.

Graphs of Tangent, Cotangent, Cosecant and Secant Functions

Create large graphical display for $\tan\theta$. Label colored stickers with ordered pairs of $(\text{radian measure, function value})$. Give students a label to place on the graphical display to create a graph of the trig function.

See Graphs of the Tangent, Cotangent, Cosecant and Secant Functions handout.

Inverse Trig Functions and Principal Solutions

You will need to create a large space in the room to do this activity.
Students will select a unit circle angle measure as they enter the room. These measures should include positive and negative values for the same terminal position. Have them create a "human" unit circle. If two or more students have the same terminal position they should stand in front of one another. Ask students to kneel down

Time to play "Simon Says . . ."

Simon says stand if the cosine of your angle is $\frac{1}{2}$. (and similar statements)

When finished playing "Simon Says . . .", discuss inverses and relate to the problem of the periodicity of trig functions. This will give the students a reason to need to learn this material.

The most important thing to discuss is the need to have only one acceptable answer as a principal solution. Most students will accept this but will not understand why they can't answer $\frac{7\pi}{4}$ for $\arctan(-1)$ when it has the same terminal position as $-\frac{\pi}{4}$. Go back to the graph for $\tan\theta$ and point out the part of the graph that we selected. Although both $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ have a tangent of -1, only one of those angles is in the correct position.

See Trigonometric Principal Solutions Packet.

Creating Polar Graphs from Transformed Trigonometric Graphs

Students use rectangular graphs of transformed trigonometric graphs to investigate the graphs of roses, cardioids and limacons.

See Polar Graphing of Roses, Cardioids & Limacons worksheets.

Graphs of the Tangent, Cotangent, Cosecant and Secant Functions

I. Tangent

Recall $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. $\tan\theta$ will be undefined when $\cos\theta = 0$. For what values of θ does this occur?

Complete the table using the unit circle values from the **first quadrant**.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ Rounded to nearest tenth
0				
$\dfrac{\pi}{6}$				
$\dfrac{\pi}{4}$				
$\dfrac{\pi}{3}$				
$\dfrac{\pi}{2}$				

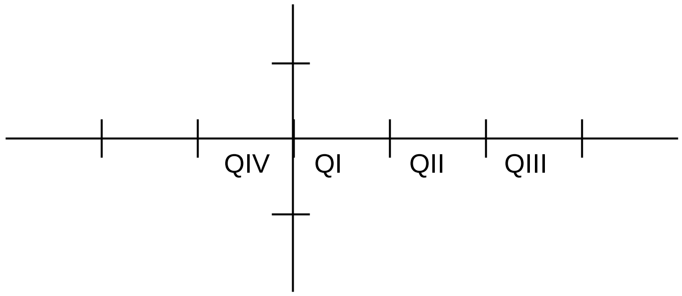
Examine the values of $\tan\theta$ as the measure of the angle goes from 0 to $\dfrac{\pi}{2}$. Do the values for $\tan\theta$ get larger or smaller as you go from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$?

Look at the values of $\tan\theta$ in **Quadrant IV**.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ Rounded to nearest tenth
0				
$-\dfrac{\pi}{6}$				
$-\dfrac{\pi}{4}$				
$-\dfrac{\pi}{3}$				
$-\dfrac{\pi}{2}$				

What happens numerically to $\tan\theta$ as you move **clockwise** from $\theta = 0$ to $\theta = -\dfrac{\pi}{2}$?

Use the information in the previous two tables to sketch a graph of $\tan\theta$ for $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.



Confirm with your graphing calculator. Be sure that your calculator is in ***radian*** mode and set your window to ***ZTRIG***.

What is happening in Quadrant II and Quadrant III?

What does this tell you about the period of $\tan\theta$?

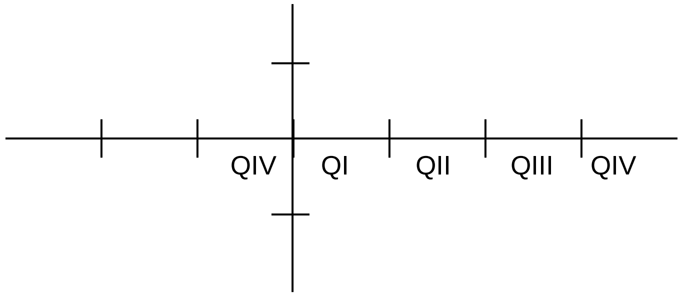
II. Cotangent $\cot\theta = \frac{\cos\theta}{\sin\theta}$

θ	$\cos\theta$	$\sin\theta$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$ Rounded to nearest tenth
0				
$\frac{\pi}{6}$				
$\frac{\pi}{4}$				
$\frac{\pi}{3}$				
$\frac{\pi}{2}$				
$\frac{2\pi}{3}$				
$\frac{3\pi}{4}$				
$\frac{5\pi}{6}$				
π				

The graph of the cotangent function has asymptotes at integral multiples of π. Explain why.

How do these results impact the graph of $\cot\theta$ **in comparison** to the graph of $\tan\theta$?

Sketch the graph of the cotangent function.



III. Cosecant & Secant

Use your TI to graph $\csc\theta = \dfrac{1}{\sin\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$.

Where do the asymptotes occur in the graph of the cosecant function? Hint: $\csc\theta = \dfrac{1}{\sin\theta}$.

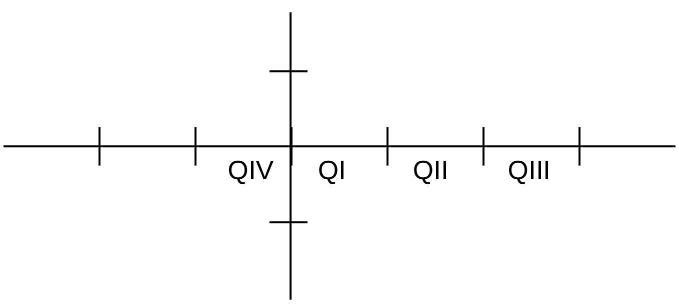
Explain why.

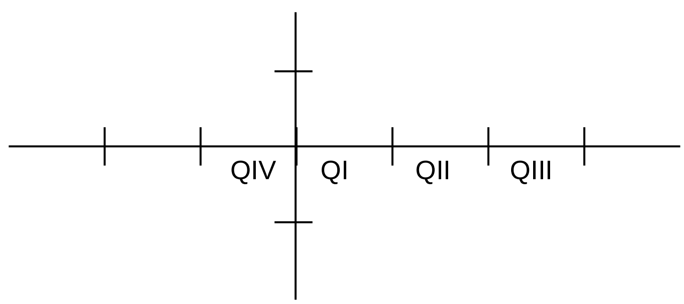
Where do the asymptotes occur in the graph of the secant function? Hint: $\sec\theta = \dfrac{1}{\cos\theta}$.

Explain why.

Explain why the range for these functions excludes values of y between -1 and 1.

Graph the cosecant and secant functions.

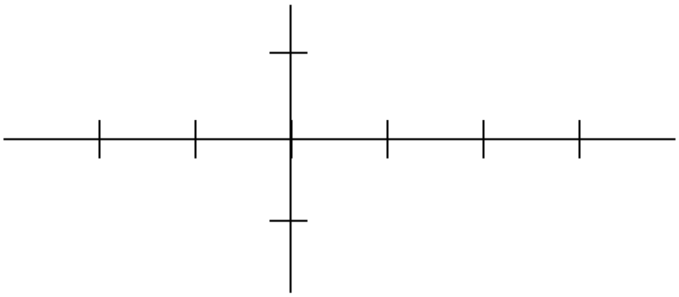


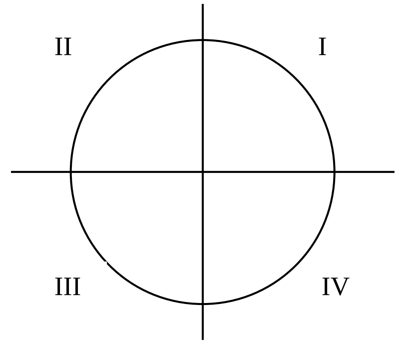


Trigonometric Principal Solutions

I) Graph $y = \sin x$. Determine an interval of *x* such that all values for $\sin x$, that is $-1 \le \sin x \le 1$, are captured.
You also want to have the interval be continuous.
Which of the following intervals will be continuous and capture all values for $\sin x$?

a) $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ b) $0 \le x \le \frac{3\pi}{2}$ c) $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ d) $-\pi \le x \le \pi$

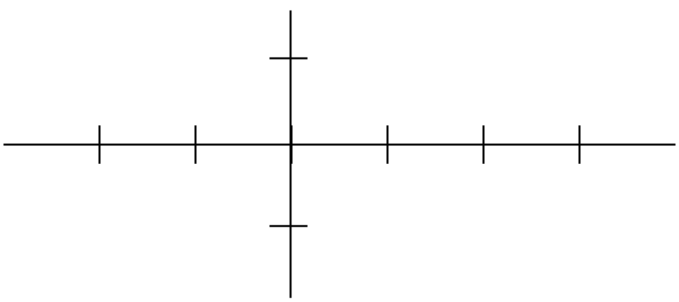


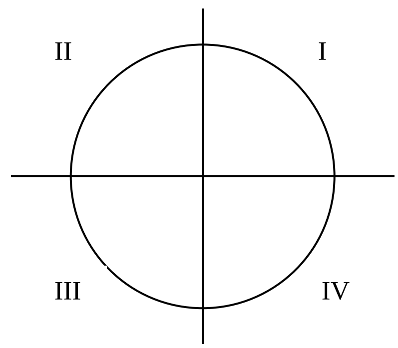


If we would always like to include the first quadrant, which interval is <u>most</u> appropriate?

This is your principal solution interval for $\sin x$.

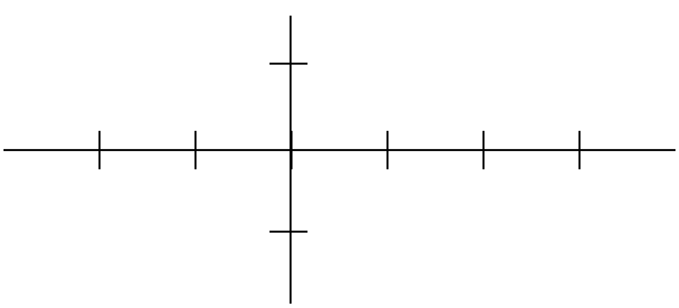
II) Graph $y = \cos x$. Determine an interval of *x* such that all values for $\cos x$, that is $-1 \le \cos x \le 1$, are captured.
You also want to have the interval be continuous.
What quadrants on the unit circle will you find these?

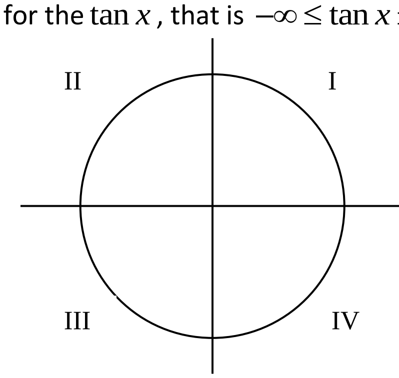




Using the same criteria as above, what is the principal solution interval for $\cos x$?

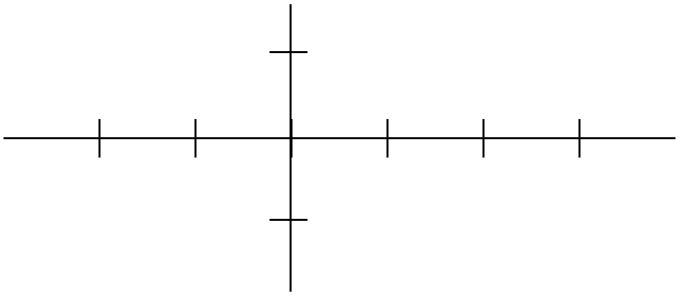
III) Graph $y = \tan x$. Determine an interval of *x* such that all values for the $\tan x$, that is $-\infty \le \tan x \le \infty$, are captured.

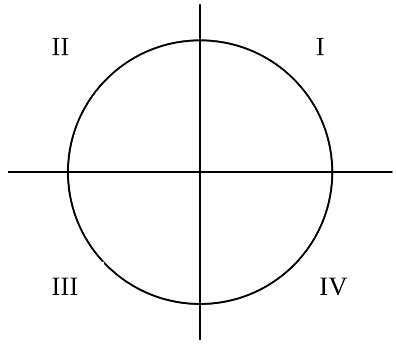




Again, using the same criteria as above, what is the principal solution interval for $\tan x$?

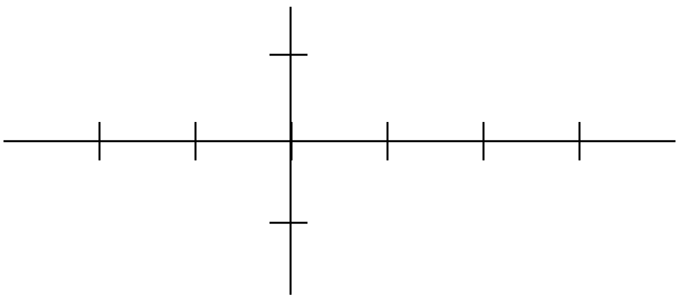
IV) Graph $y = \cot x$. Determine an interval of *x* such that all values for the $\cot x$, that is, $-\infty \le \tan x \le \infty$, are captured.

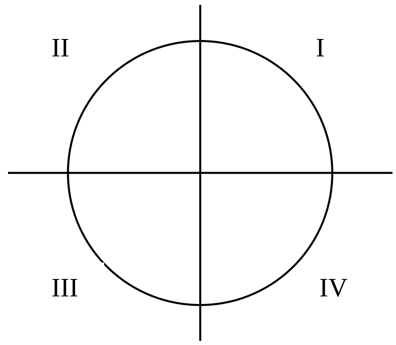




Again, using the same criteria as above, what is the principal solution interval for $\cot x$?

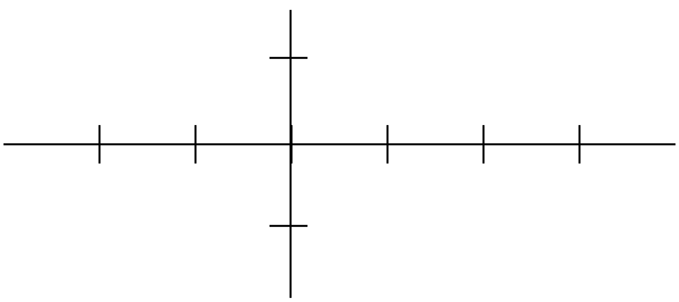
V) Graph $y = \csc x$. Determine an interval of *x* such that all values for the $\csc x$ are captured. What are these values? Obviously the interval cannot be continuous. Find two ***adjacent*** quadrants where all the values of $\csc x$ are captured.

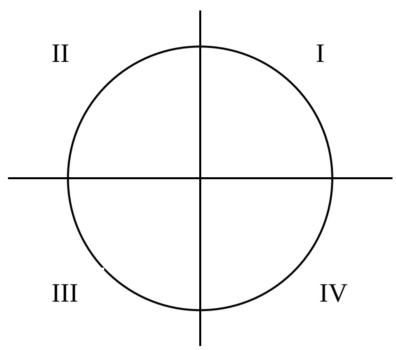




What is the principal solution interval for $\csc x$?

VI) Graph $y = \sec x$. Determine an interval of *x* such that all values for the $\sec x$ are captured. What are these values? Obviously the interval cannot be continuous. Find two adjacent quadrants where all the values of $\sec x$ are captured.





What is the principal solution interval for $\sec x$?

Now use the results from above to evaluate each of the following:

a) $\arccos(1)$

b) $\arctan(-1)$

c) $\arcsin(-1)$

d) $\arccos\left(-\frac{1}{2}\right)$

e) $\arctan\left(\sqrt{3}\right)$

f) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

g) $\arccos\left(\frac{\sqrt{2}}{2}\right)$

h) $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

i) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

j) $\arctan(1)$

Use your TI to evaluate the following. Set the mode to Radians. **Be careful – you must answer with the appropriate value in terms of the principal solutions!**

a) $\arcsin(-0.75)$

b) $\arccos(-0.7)$

c) $\arctan(0.98)$

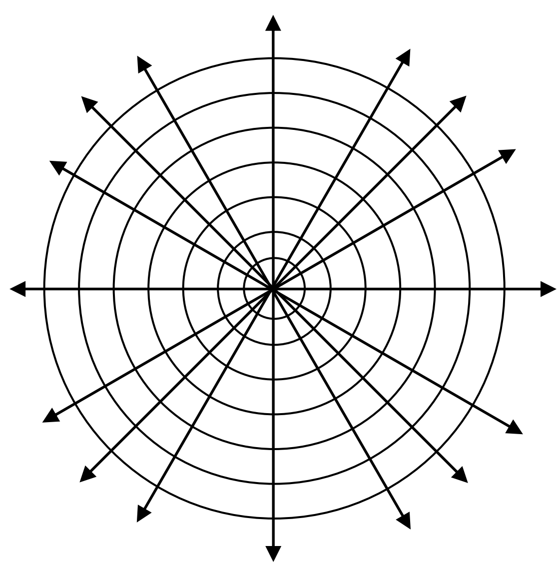
d) $\arctan(4.7)$

Graphing Roses from Transformed Trigonometric Graphs

Graph $y = 2\cos 2\theta$ on $[0, 2\pi]$ on the rectangular plane. Examine the graph for maximum, minimum, midline values and intercepts.

Make a table of values from your graph above where $r = y$. You will use this to create the polar graph.

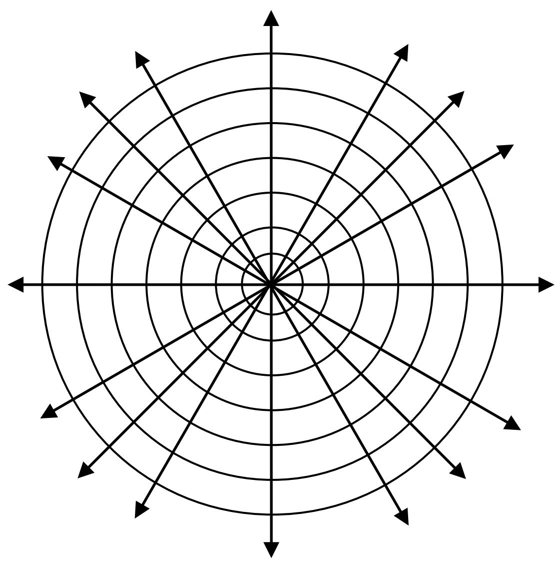
θ	r



Graph $y = 2\cos 3\theta$ on $[0, 2\pi]$ on the rectangular plane. Examine the graph for maximum, minimum, midline values and intercepts.

Make a table of values from your graph above where $r = y$. You will use this to create the polar graph.

θ	r

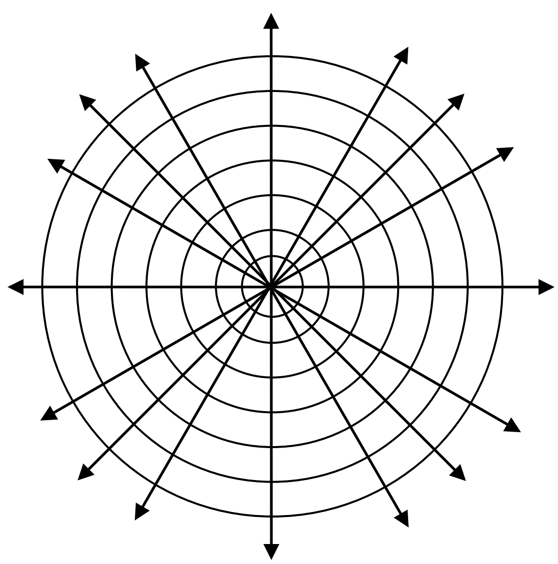


Graphing Cardioids from Transformed Trigonometric Graphs

Graph $y = 2 + 2\sin\theta$ on $[0, 2\pi]$ on the rectangular plane. Examine the graph for maximum, minimum, midline values and intercepts.

Make a table of values from your graph above where $r = y$. You will use this to create the polar graph.

θ	r

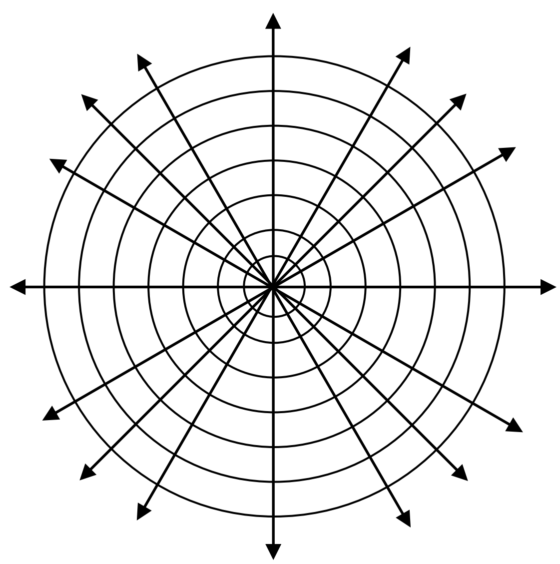


Graphing Limacons from Transformed Trigonometric Graphs

Graph $y = 1 + 2\cos\theta$ on $\left[0, 2\pi\right]$ on the rectangular plane. Examine the graph for maximum, minimum, midline values and intercepts.

Make a table of values from your graph above where $r = y$. You will use this to create the polar graph.

θ	r



Extension of Knowledge to Visualize Basic Identities:

Give students random sized triangles (that will fit their paper plate). They can use them to set up a triangle from the given information. It is nice to assign different quadrants to different students to place their triangle in and then have them compare/confirm their answers with one another and the entire class.

Set #1

Given $\sin\theta = \frac{\Delta}{\square}$ and $\cos\theta = \frac{\Omega}{\square}$, find

a) $\sin(\theta+\pi)$

b) $\cos(\theta+2\pi)$

c) $\tan\left(\theta+\frac{\pi}{2}\right)$

d) $\csc(-\theta)$

e) $\sec^{-1}\frac{-\square}{\Omega}$ (there are two possible answers)

Set #2

If cos θ = 0.3, find cos (θ + π).

If tan θ = 3, find tan (θ + π).

If sin θ = 4/5, find sec (θ + π).

Did you consider what quadrant the angle could originate in? If not, do so now as there may be more than one answer.

Set #3

Given $\sin\theta = \frac{\sigma}{\mathrm{N}}$ and $\tan\theta = \frac{\sigma}{\mathrm{A}}$, find

a) $\cos(\theta)$

b) $\sin(\theta+\pi)$

c) $\tan\left(\theta+\frac{\pi}{2}\right)$

d) $\csc(-\theta)$

e) $\cot^{-1}\frac{-\mathrm{A}}{\sigma}$ (there are two possible answers)