Mathematical Reasoning in an Unreasonable Environment

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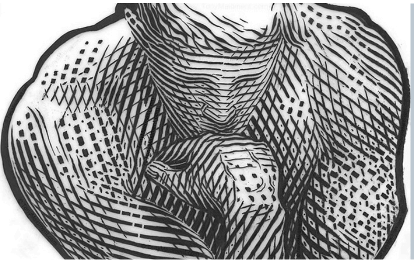
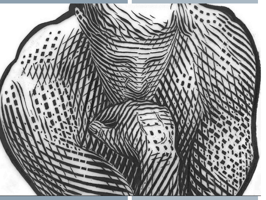
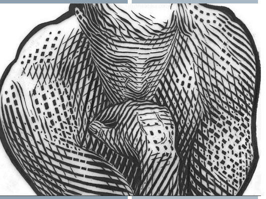




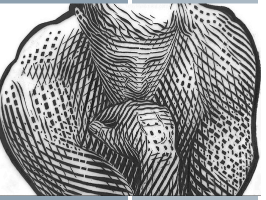
Reasoning: A cornerstone of mathematics

- *Principles and Standards of School Mathematics* (2000)
  - "Instructional programs from prekindergarten through grade 12 should enable all students to
    - Recognize reasoning and proof as fundamental aspects of mathematics
    - Make and investigate mathematical conjecture
    - Select and use various types of reasoning and methods of proof." (p. 402)



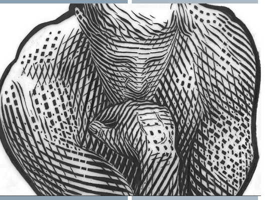
Reasoning: A cornerstone of mathematics

- *Common Core State Standards for Mathematics*.
  - *"Construct viable arguments and critique the reasoning of others.*
    - *Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.*
    - *They make conjectures and build a logical progression of statements to explore the truth of their conjectures.*
    - *They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.*
    - *They justify their conclusions, communicate them to others, and respond to the arguments of others.*
    - *They reason inductively about data, making plausible arguments that take into account the context from which the data arose.*



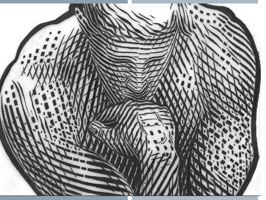
Reasoning: A cornerstone of mathematics

- *Common Core State Standards for Mathematics*.
  - *"Construct viable arguments and critique the reasoning of others.*
    - *Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning form that which is flawed, and—if there is a flaw in an argument—explain what it is.*
    - *Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies.*
    - *Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments."* (pp. 6, 7)



Student demographics too often predict the opportunities that students have for reasoning and sense making.

- *Over last 2 decades, enrollment in advanced and college-preparatory math courses have increased.*
- *High schools continue to use tracking or ability grouping.*
- *With tracking, high schools run the risk of promoting inequitable opportunities for math learning for those students placed in remedial classes.*
- *Schools with multiple levels of the same class have a responsibility to engage all students in every class in reasoning and sense making.*
    - *NCTM (2009)*

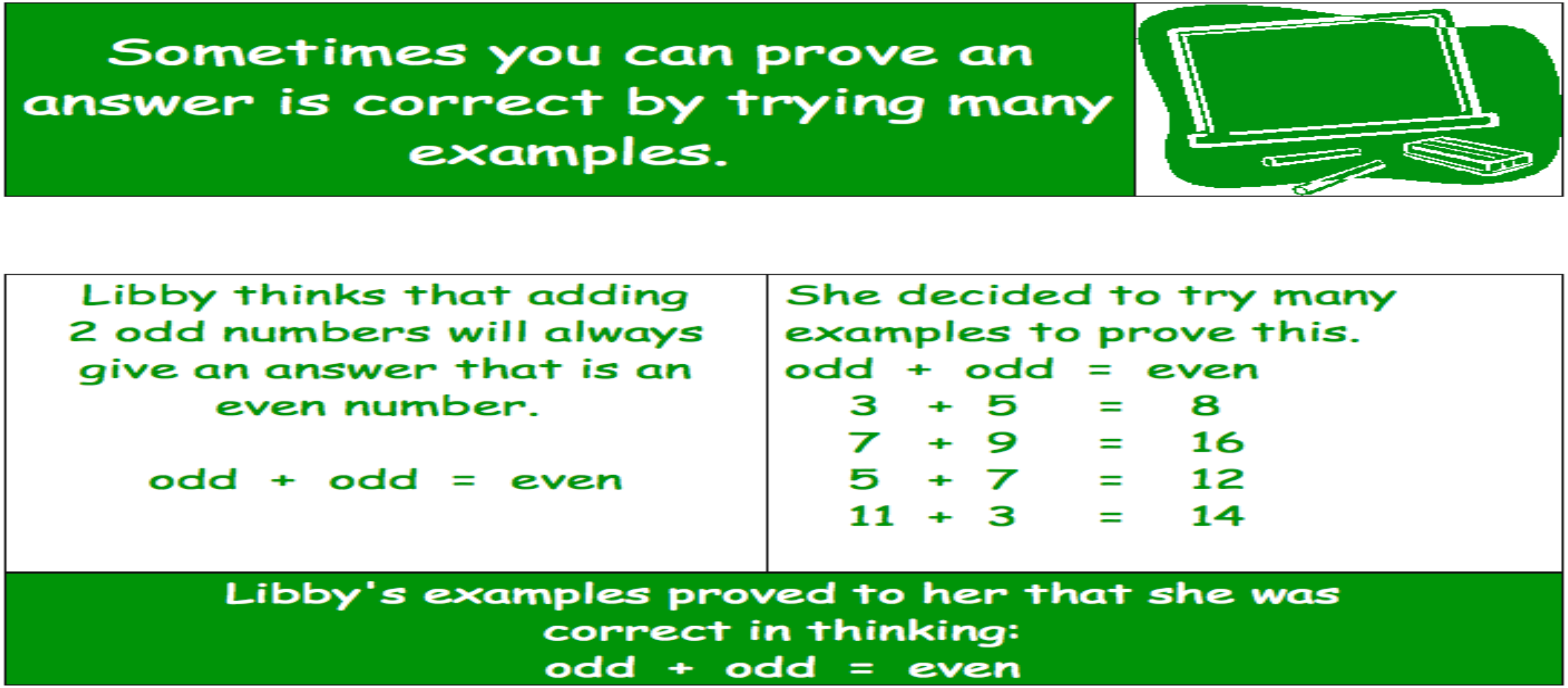


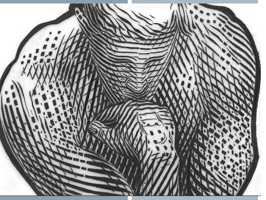
Reasoning: A cornerstone of mathematics

- *State tests have spawned may test prep cottage industries.*
- *Following is a standard from one state under Reasoning and Proof:*
  - **MST3.E.RP.03.08** - Support an argument by trying many cases

Reasoning: A cornerstone of mathematics

- *The photo illustrates an item from a website purporting to test that standard for Grade 3.*



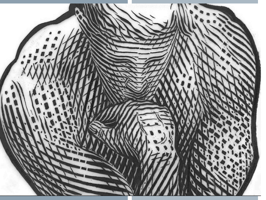


Reasoning and Proof: What constitutes a proof?

Is “proof” using many cases a true proof?

How do you convince students?

Is the test item a good example of “proof”?



Reasoning and Proof: A cornerstone of mathematics

- *From a New York State Exam, 2010*

45 Estimate 805 • 11 ÷ 22.

Estimation ______________________

Calculate the value of 805 • 11 ÷ 22.

Show your work.

Answer ______________________

On the lines below, explain why your estimation is reasonable.

Reasoning: A cornerstone of mathematics

› *From a New York State Exam, 2010*

45 Estimate 805 • 11 ÷ 22.

Estimation ______________________

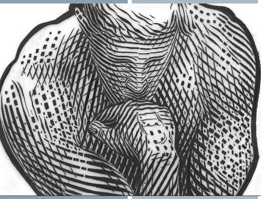
Calculate the value of 805 • 11 ÷ 22.

Show your work.

Answer ______________________

On the lines below, explain why your estimation is reasonable.

"No, I think it was just Divine intervention."



What is a reasonable estimation? What reasoning are you using? Is 402.5 unreasonable?

› *From a New York State Exam, 2010*

45 Estimate 805 • 11 ÷ 22.

Estimation ____________________

Calculate the value of 805 • 11 ÷ 22.

Show your work.

Answer ____________________

On the lines below, explain why your estimation is reasonable.

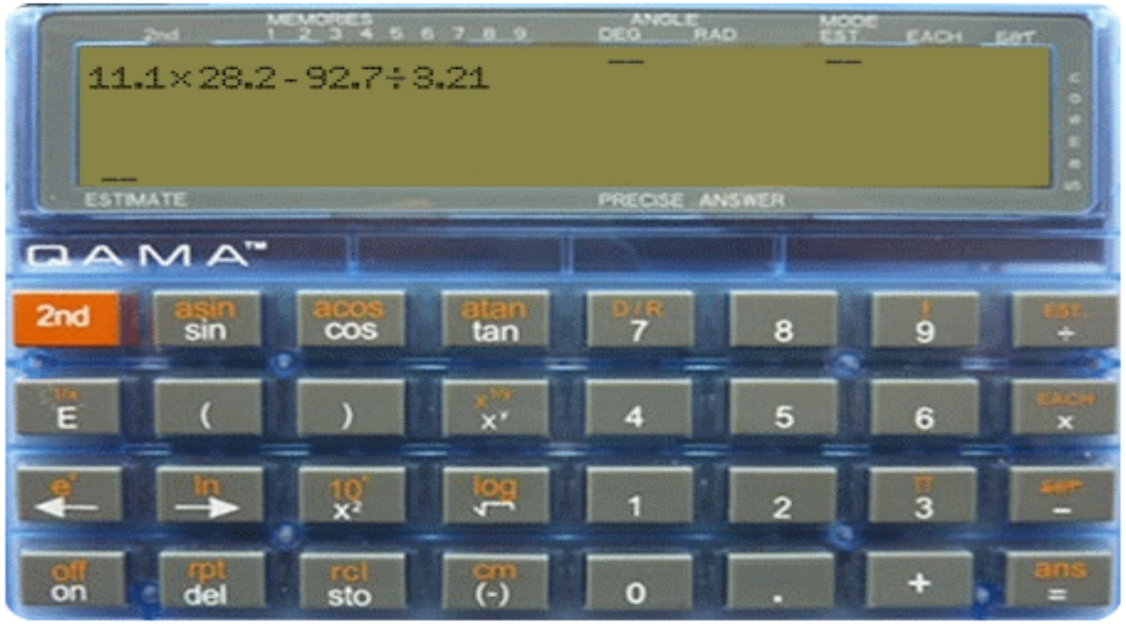
Could you reason (and prove) with a calculator? Consider a Qama!

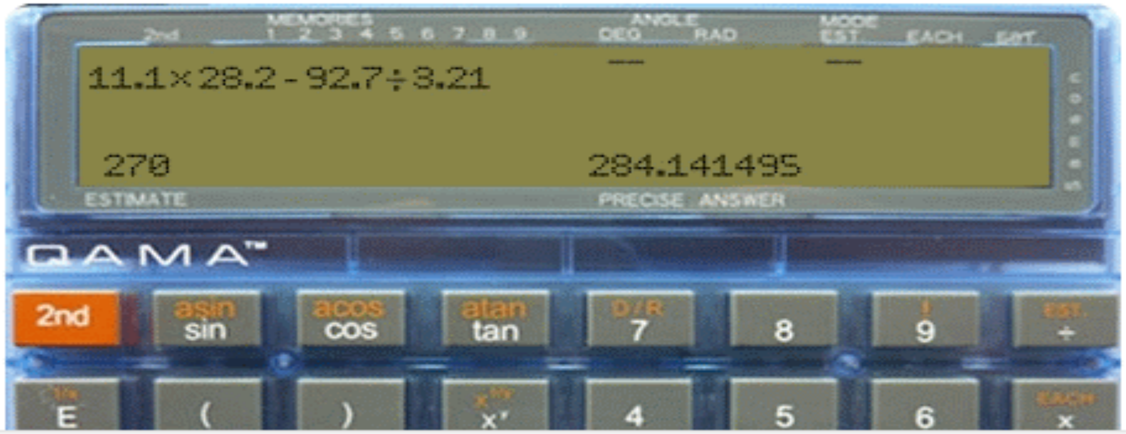
Here is how **QAMA** works:

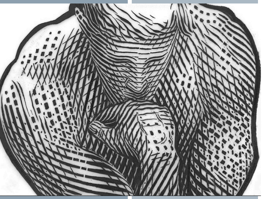
Calculations are entered as usual.
But when the '=' is pressed... nothing happens.

You first need to enter a mental estimate:
If the algorithms programed into **QAMA** consider your estimate as case-specifically reasonable, then and only then it will show the precise answer.

Here, for example, approximate the task as **10 x 30 - 90÷3**





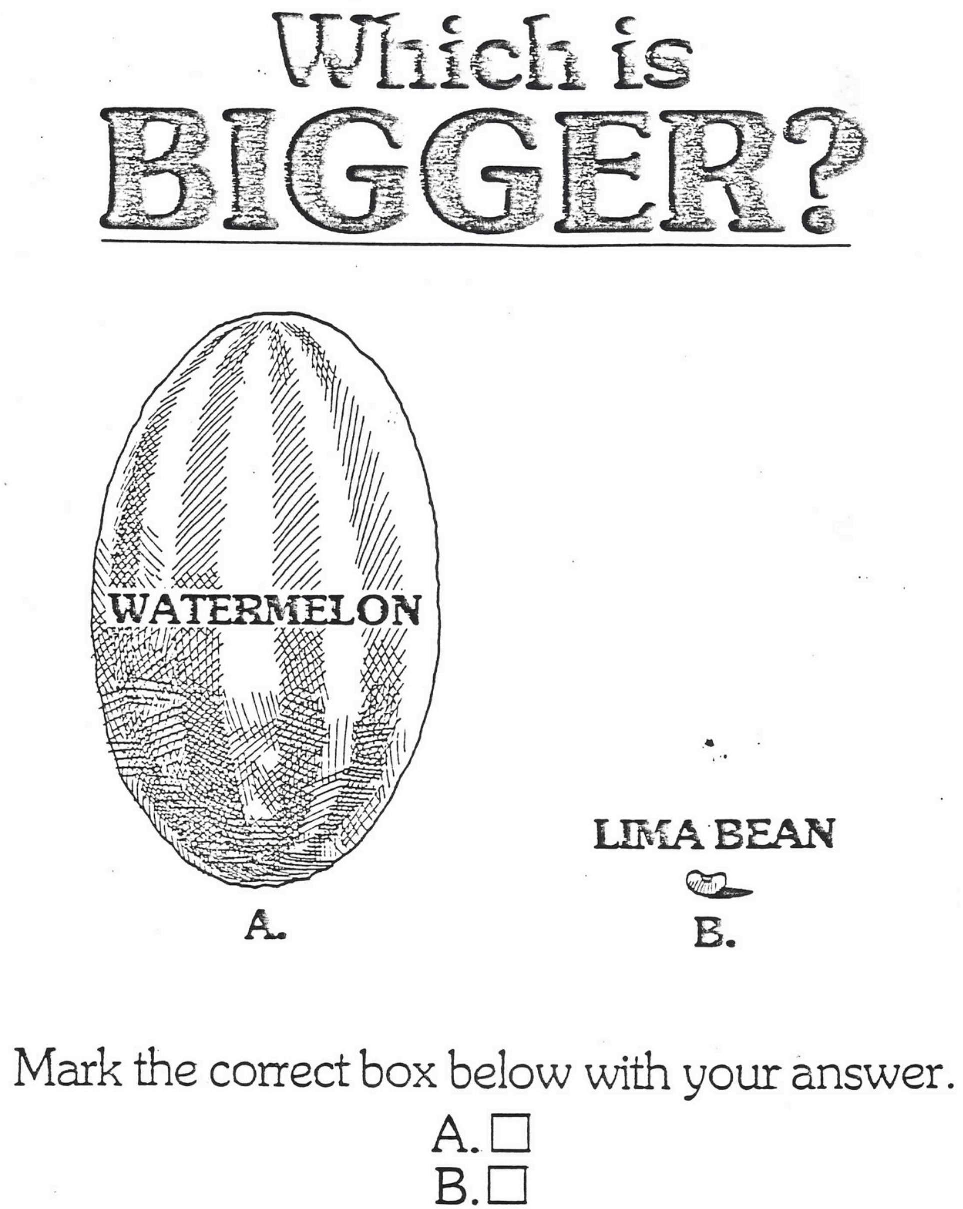


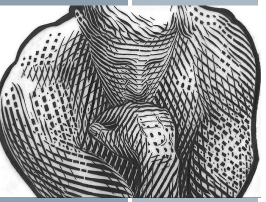
We have seen one test sample. Are all good?

- *Can you show reasoning on a multiple choice test?*

- *Consider the next slide.*

A Multiple Choice Examination Question—Reasoning?

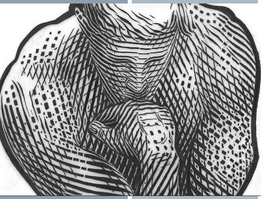




A Multiple Choice Examination Question—Reasoning?

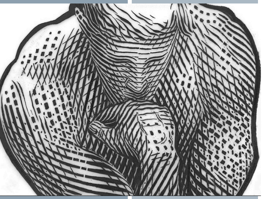
NCTM has published *Reasoning and Sense Making.*

What is "reasonable and sense making" about the last item?



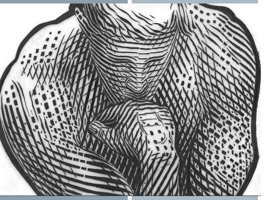
René Descartes wrote

The two operations of our understanding, intuition and deduction, on which alone we have said we must rely in the acquisition of knowledge.



René Descartes wrote

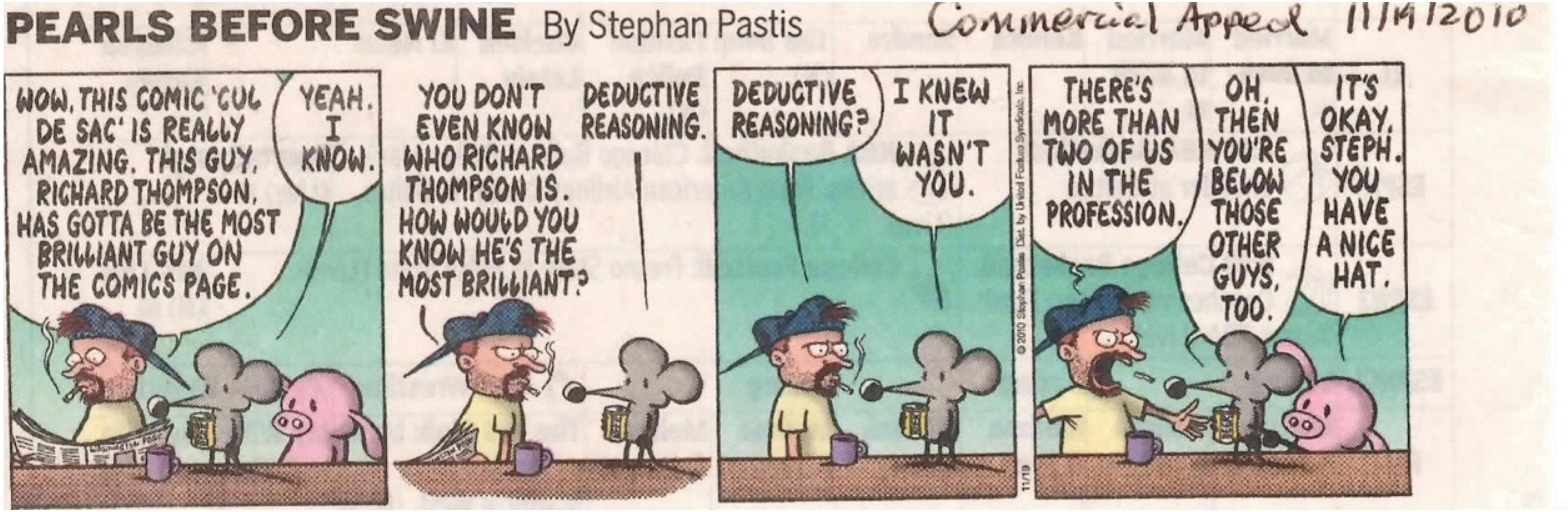
Some years ago I was struck by the large number of falsehoods that I had accepted as true in my childhood, and by the highly doubtful nature of the whole edifice that I had subsequently based on them. I realized that it was necessary, once in the course of my life, to demolish everything completely and start again right from the foundations if I wanted to establish anything at all in the sciences that was stable and likely to last.



When René Descartes wrote the last quotation, what might be necessary to start working on in deduction (and proof) and logic in mathematics?

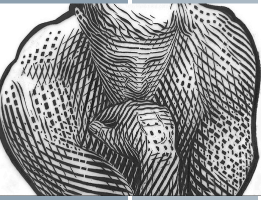
Could it be inferences from logic? How far do we go with students? Consider the following:

Is logic necessary?



Use reasoning and avoid mind sets, too small sample spaces, etc.!

Is logic necessary?



Translation of Sarge’s quotation

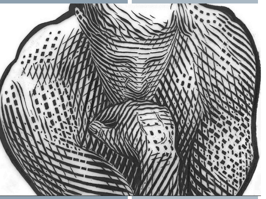
If I don’t have a nail, I lose a shoe.

If I lose a shoe, I lose a horse.

I I lose a horse, I lose a rider.

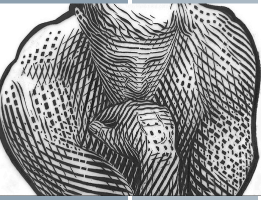
If I lose a rider, I lose a battle.

Zero went to buy nails. What type of logic was he using? Is this type of logic acceptable for proof?



Does Zero’s reasoning always work?

If line a is parallel to line b, and line b is parallel to line c, what is your conclusion?



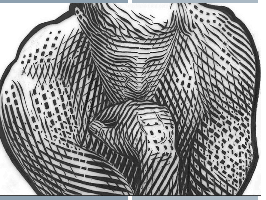
Does Zero's reasoning always work for a proof?

If line a is parallel to line b, and line b is parallel to line c, what is your conclusion?

I think that your conclusion is line a is parallel to line c.

Now consider the following:

If line a is parallel to line b, and line b is parallel to line a, what is your conclusion?



Does your reasoning always work for proof?

If line a is parallel to line b, and line b is parallel to line a, I think that your conclusion is line a is parallel to line a.

Now consider the following:

What is your definition of parallelism?

Can a line be parallel to itself?

Follow Descartes's lead and examine everything.

What reasoning would you use?

"I wonder, sir, if you would indulge me in a rather unusual request?"

What reasoning would you use?

"I wonder, sir, if you would indulge me in a rather unusual request?"

I expect that you see a pattern and that the people are in order of decreasing height from left to right. Do you always follow patterns for "proof"?

Does your curriculum allow you to question your students' reasoning?

Do you have the time needed to do that?

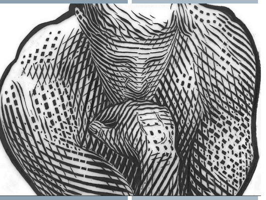
Reasoning Problem for Middle Schoolers

Three boxes of fruit are on a high shelf. You can't see into the boxes. One box contains all apples; one contains all oranges; and one contains apples and oranges. Each box is labeled BUT all the labels are incorrect. Choose one box; reach in and grab one fruit. Look at it. Now label all the boxes correctly. How can you do it? (Billstein, et al, 1988)

Apples **Oranges** **Apples and Oranges**

Reasoning Problem for Middle Schoolers

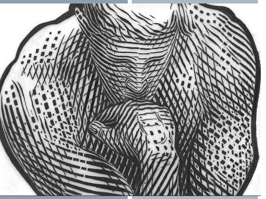
- What are you thinking?
- How are you reasoning?
- Explain it to a colleague!
- Is it a proof?



Reasoning Problem for Fourth Graders

- Complete each problem. Put one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

```
      3 □ 7 □
  X         9
  -----------
    □ □ 1 □ 4
```



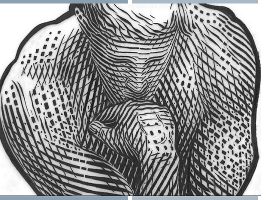
What reasoning did you use? Is there more than one answer?

Complete each problem. Put one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9 in each box to make the calculations correct. You may use a digit more than once.

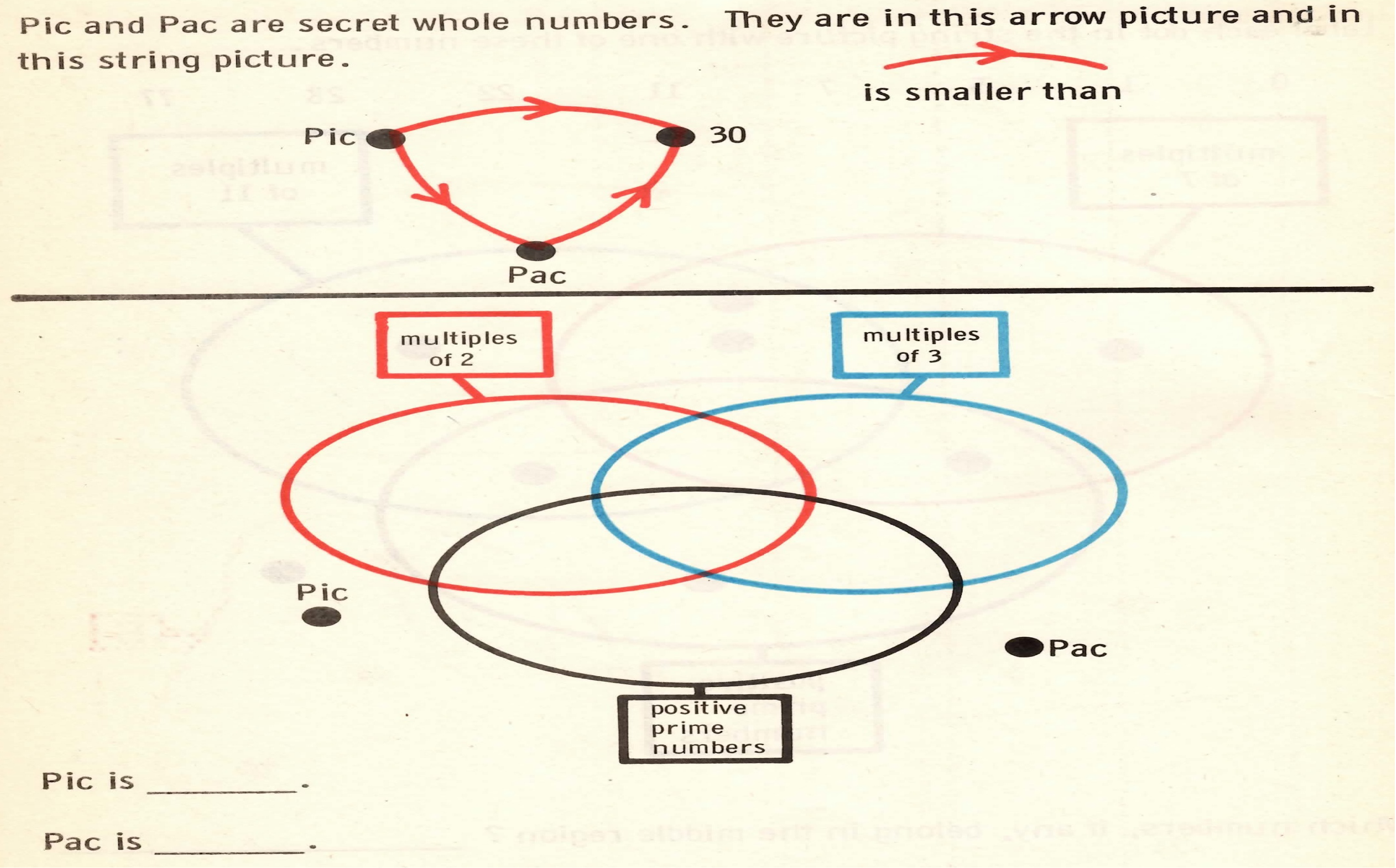
```
    3 □ 7 □
  X       9
  ---------
  □ □ 1 □ 4
```

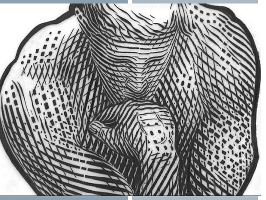
More Divine Intervention?

Here you have likely used all possible cases. Would this constitute a proof if you explained your reasoning?

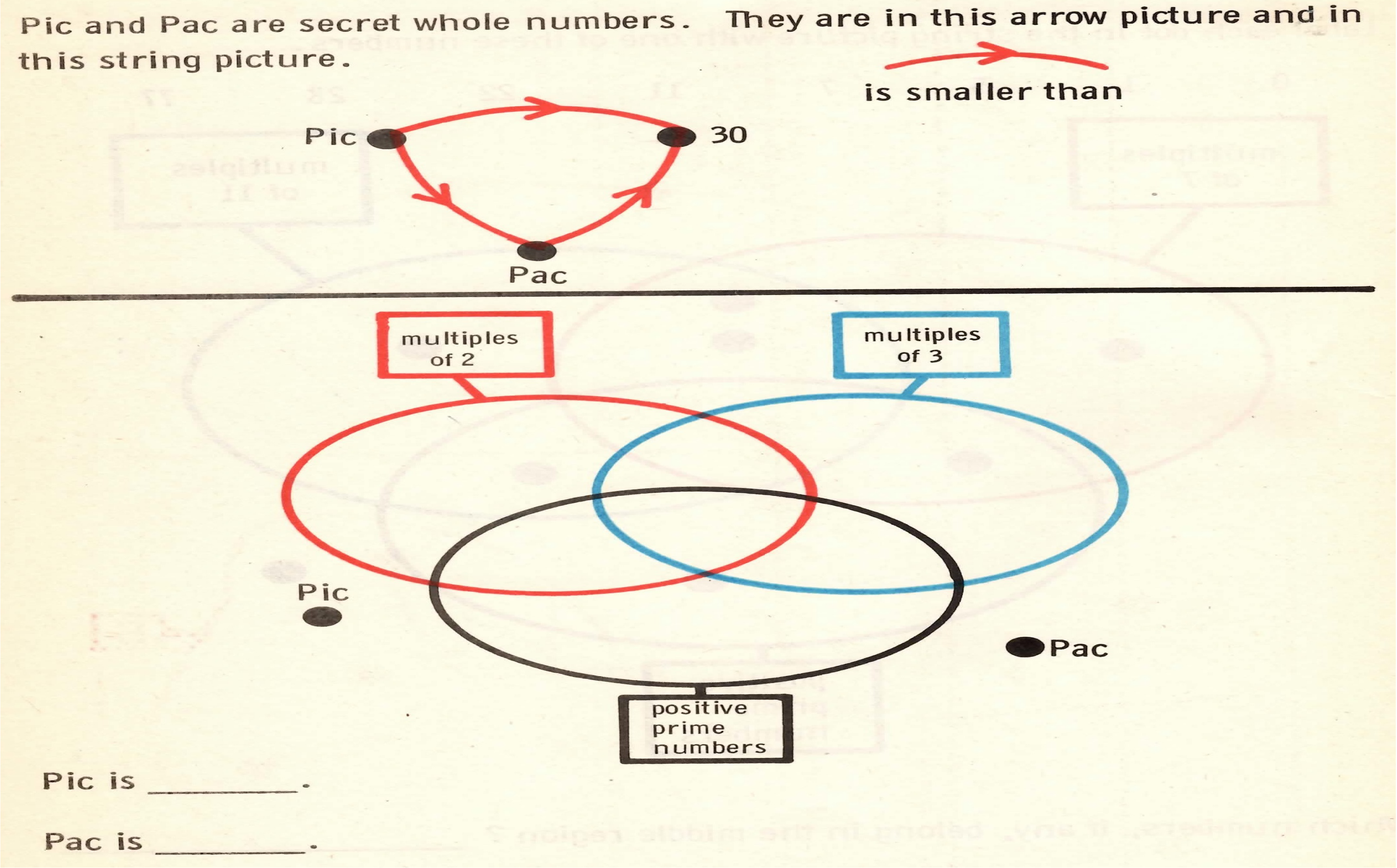


Frederique Papy had young children reasoning and proving long ago in the following:





So what are Pic and Pac? Explain your reasoning to prove your answers.


Pic and Pac are secret whole numbers. They are in this arrow picture and in this string picture.

Pic is ________.

Pac is ________.

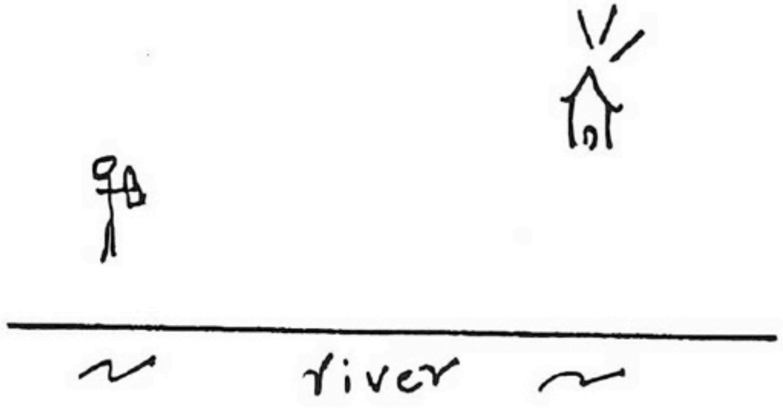
How would you reason here?

Should the policeman give her a ticket?

Do you believe the driver?

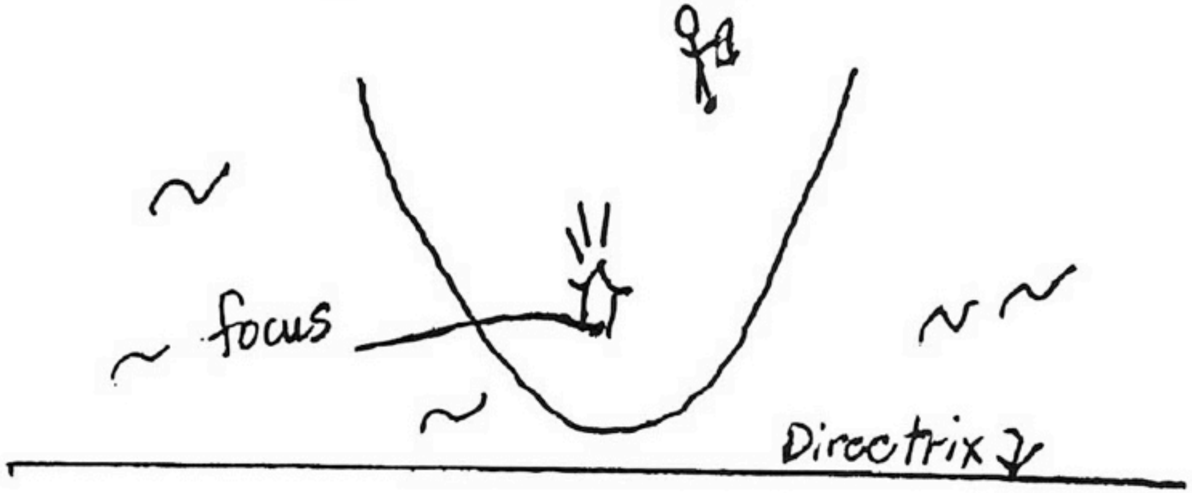
Secondary Problems: A Series—Part 1

A hiker, carrying a bucket, sees that his tent is on fire. To what point on the bank of the river should the hiker run to fill his bucket in order to make his trip to the tent as short as possible. (Lott and Smith)



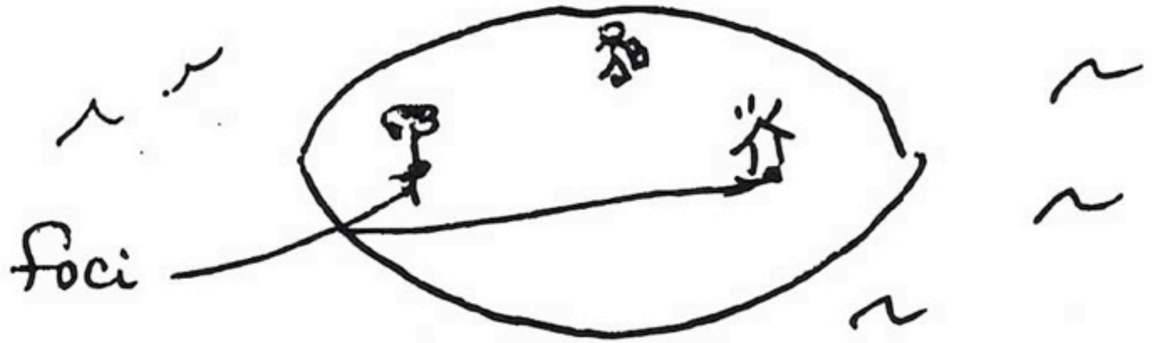
Secondary Problems: A Series—Part 2

Suppose the hiker is on Parabolic Peninsula as pictured below, and the same question is posed. (Lott and Smith)



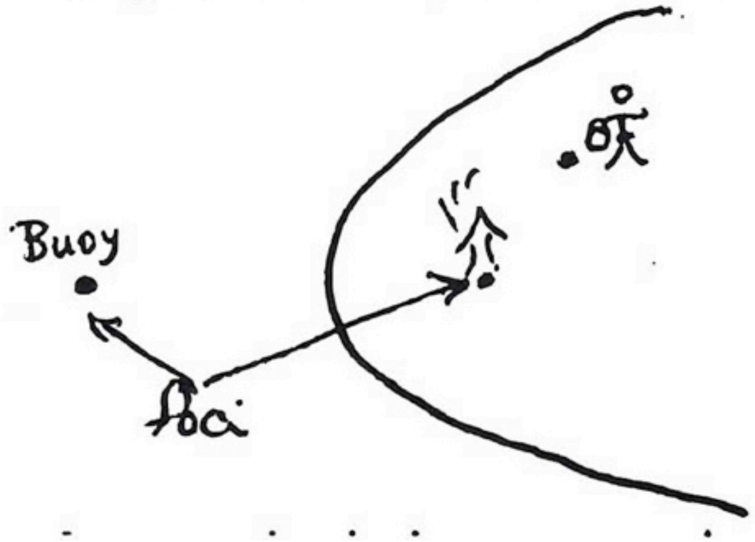
Secondary Problems: A Series—Part 3

Suppose the hiker is on Elliptic Isle as pictured below and the same question is posed. (Lott and Smith)



Secondary Problems: A Series—Part 4

Suppose the hiker is on Hyperbolic Bay as pictured below and the same question is posed. (Lott and Smith)

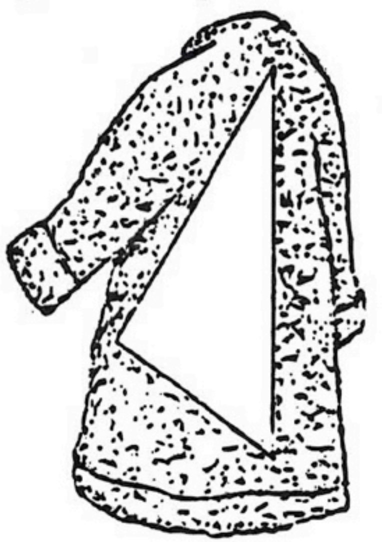


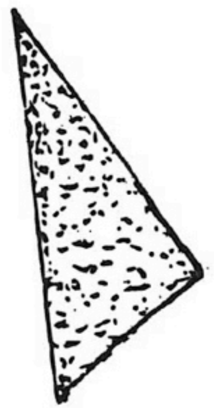
Secondary Problems: The Last

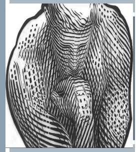
Arthur Engel wrote a problem entitled "The Tragic Mistake of the Poor Tailor of Sikinia."

A dog tore a triangular hole in a mink coat. The mink fur grows is only on one side. The tailor cut a patch to fit the hole but it fit only on the wrong side. How can you help him?

The triangular piece can be cut and the and re-sewn and it takes only one cut. Can you help him?



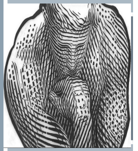




Lynn Steen wrote:

Research [about reasoning] does support a few general conclusions.

- successful learners are mathematically active [Anderson, Reder, & Simon, 1996].
- Passive strategies (memorization, drill, templates) are much less likely than active tasks (discussion, projects, teamwork) to produce either lasting skills or deep understanding.
- successful mathematics learners are more likely to engage in reflective (or "metacognitive") activity [Resnick, 1987].



Lynn Steen wrote:

- Experienced teachers know that knowledge and performance are not reliable indicators of either reasoning or understanding.
- Deep understanding must be well-connected.
- Persons with well-connected understanding attach importance to different patterns and are better able to engage in mathematical reasoning.
- Students with different levels of skills may be equally able to address tasks requiring more sophisticated mathematical reasoning [Cai, 1996].

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