Three Models to Preview Calculus Concepts
Grades 6 to 11

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Intro questions you need to discuss first:
Questions for "There's Something about Mary" (distance vs. time):
Problem:
Mary is standing by the swing set in the playground when she decides to walk to the slide which is 12 feet away. She walks at a steady pace and arrives at the slide in 3 seconds. She stands near the slide for 10 seconds, then returns to the swing set in 2 seconds.
Sketch a graph of this problem (Hint: you should get a trapezoid)

Questions:
What are the units of the rate of change of the segment in the first 3 seconds?
(The units of the rate of change are feet per second.)
What does the rate of change of the three segments tell you about the problem?
(The rate of change tells us the speed that Mary is moving at.)
When was Mary moving the fastest?
(Mary was moving fastest between 13 & 15 seconds, when rate of change is steepest.)
What was her speed then?
(She was moving at 6 feet per second.)
What was the rate of change between 3 and 13 seconds and what does that tell you?
(The rate of change is 0, which tells you she was standing still.)
Are her speeds at the start and end reasonable speeds
(Was she walking, or running? What was her speed in mph?)
(In the 1st 3 seconds she was walking at 2.727mph.(4 ft/sec) In the last two seconds, she jogged at 4.09 mph. (6 ft/sec). From 3 to 13 seconds she was standing still. (0 ft/sec.))
What was her average speed in ft/sec over the first 10 seconds?
(Her average speed was 1.2 feet per second.)
What was her speed at 14 seconds?
(Her speed at 14 seconds was 6 ft./sec.)
What was her speed in feet per seconds at exactly 3 seconds? (Justify your answer.)
(The speed can not be determined. It is 4 ft/sec no matter how close you get before 3 seconds, but 0 ft/sec no matter how close to 3 seconds, but larger than 3)

Yellow Camaro problem:

You are driving to Las Vegas at 75 mph on I-10 when you are passed by a Chevy Camaro Z-28 Yellow convertible. You maintain this speed for 3 hours.

Sketch a graph of this situation

Yellow Camaro (simple speed vs time) questions:

What are the units of the sides of the rectangle that represents this graph?
(horizontal units are hours, vertical are Miles per hour.)
What is the area of the rectangle, including units?
(The area is 225 miles… we talk about why the units are miles, some people usually are unsure)
What calculus concept is this showing middle school students?
(That the area under a curve is a useful value. They have just found the integral $\int_0^3 75x = 225$)
What are the units of the rate of change in this problem? What real world thing does this relate to? Where is the rate of change the greatest?
(Units are feet per second per second or ft/sec^2. This is Pat's acceleration. Her acceleration (deceleration) is greatest in the last second ($-$ $4ft/sec^2$))
How far has she travelled in the first 5 seconds? While stopping?
(She travelled 16 feet in the first five second, 2 feet while coming to a stop)

The Pat and Chris problem: (speed vs. time)

- Pat, a student in your class, is standing near the red school door, when her bus arrives. From a stop, Pat speeds up evenly to a rate of 4 feet per second (this takes her 2 seconds). Pat walks at this pace for 8 seconds, then stops 5 feet from the bus door to talk with a friend, Chris. The stopping occurs in 1 second.
- Sketch a graph of this problem (Hint: you should get another trapezoid)

Pat and Chris questions:

How far did Pat walk from her starting point to where he stopped to talk to Chris?
(The area of the trapezoid is 38 feet, so that is how far she moved)
What was her average speed over the 11 seconds?
(Pat's average speed = $\frac{38}{11}$ft/sec which is $3\frac{7}{11}$ft/sec)
How far was she from the school bus door before she moved?
(She was 33 feet from the bus **if** Chris was between her and the bus. Or, she was 43 feet from the bus if Chris was past the bus. One could argue for any number in between those as well, but most often people say 33 feet.)
What gender are the 2 students? Justify your answer.
(Pat is a girl, I use "she' and "her" in the problem. Chris is whichever gender you want as long as you give a reason in school appropriate language: i.e. Chris is a girl because all the boys are goofs, and Pat does not like any of them. Or, Chris is Pat's boyfriend, so he is male)
What information in the problem is not needed?
(You do not need the color of the door, nor the fact that Pat is in your class.)

Cannon Ball Problem:

To generate publicity when the circus comes to town, they will shoot a human cannon baller off the top of a roof, 40 feet above the street. The person is shot out at a 56 degree angle, with an initial velocity of 60 feet per second. Sketch a graph of the person's height vs. time

Questions:

Traditional:

What is the max height? When does it occur?

When is the cannonballer at a height of 50 feet?

How high is the person after 2 seconds?

How long until he gets to the net?

TW questions:

Is the initial velocity a reasonable velocity? (a research questions for students)

What is not finished with the sketch (or my graph)?

What are the units of the rate of change?

What does the rate of change tell you about the situation?

When is the person moving the fastest?

What is the biggest velocity? Is that a reasonable velocity? (a research questions for students)

When is the person going up at a speed of 20 feet per second? Down at 20 feet per second?

Can you find a place where the downward vertical velocity is equal to the upward vertical velocity when the person exited the cannon? When does that happern?

What is the average speed from start to the net?

Does the person ever go exactly that vertical velocity?

Find some values for the rate of change and create a scatter diagram of them.

- What kind of relationship seems to be true?
- Find the line of best fit using Modify on PRIZM or Transfrm App on TI.
- What do the slope and Y-intercept of the line of best fit tell you about the situation?

Can your Algebra 1 students find the graph of the derivative of a parabolic graph? We just did!!

Ferris Wheel Problem:

You go on a Ferris wheel that has a 40 foot diameter and takes 30 seconds to complete one revolution. Sketch your height vs. time.

Here are some questions I might ask, in no particular order of importance:

1. What does the Y-intercept tell you about the ride?
2. When do you get to the top of the Ferris wheel?
3. When are you going up and when are you going down?
4. What are the units of the rate of change of the graph?
5. Where are you on the ride when the vertical rate of change the largest? (at what times during the ride does this occur?)
6. Where are you on the ride when the vertical rate of change is equal to zero? (at what time during the ride does this occur?)
7. When is the vertical rate of change increasing?
8. Are there times when the vertical rate of change is positive, but the changes are getting smaller? (larger?)
9. What is the average vertical speed from when you are at the "bottom" of the ride until you are at the "top" of the ride?
10. What was your height above ground 5 seconds before the ride?
11. How many people got on the ride after you did? (I assume one thing about the graph most of you have drawn.)
12. Does your sketch show that you got off the ride or that you are still on it?

These next questions require a graphing calculator, and probably work best in Algebra 1 and beyond:

13. When are you at the same height as the center of the Ferris wheel?
14. How high are you off the ground 3 seconds after your ride began?
15. What is your vertical speed 3 seconds after you got on the ride?
16. When is your vertical velocity –3 ft/sec?
17. What is the value of the greatest vertical velocity? Where does this occur?
18. Find the time where your increasing vertical velocity starts to decrease.
19. What is your average vertical velocity between
    a. 2 and 3 seconds?
    b. 14 and 16 seconds?

These next questions would be for Algebra 2/ Trigonometry classes or Pre-Calculus classes, after students had learned about trigonometry function graphs and transformations of the graphs.

20. Find the equation of your ride on the Ferris wheel from the time you got on the last open car for two complete revolutions. Graph this equation.
21. Sketch the tangent line using your PRIZM and find where the slope of the tangent line is positive, negative, and zero.
22. Are there places on the graph where your vertical speed is positive (or negative) and the change in your speed is also positive?
23. What about places where the vertical speed is positive (or negative) and the change in your speed is negative?
24. Find enough values of your vertical velocity in order to find a regression equation for your velocity. Graph your vertical velocity on the same axes as the graph.
25. Move the graph up so that the center line of the velocity graph is on the same Y= value as the center line of the height graph.
26. Compare the two graphs:
    a. Periods
    b. Amplitudes
    c. Trig function

Here is a way to alter the question in order to have students analyze the situation in a different way, especially if your students have worked on the Ferris wheel idea before, or if you have a strong Honors or gifted class or once they get to a class where they have seen the Ferris wheel several times already:

27. Sketch a graph of your horizontal distance from the start of the ride.
    a. How does this change your graph?
    b. You have a choice of how your graph is drawn, which did you choose?

Note, if you have students work this problem in Geometry, there are many other questions you could have them explore:

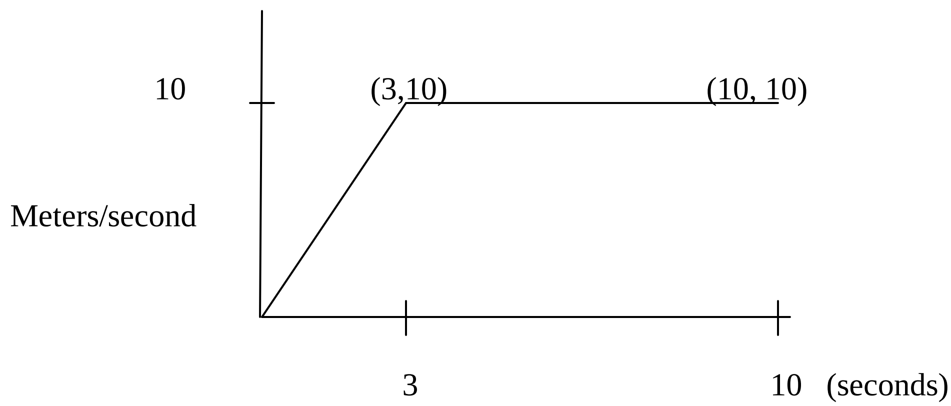
28. How far do you go on your ride?
29. What is your circular velocity on the Ferris wheel?
30. If there is a bug halfway between you and the center of the Ferris wheel, is the bug's circular velocity half of yours? Justify your answer.

In Calculus or Pre-Calc, ask students to sketch the vertical velocity vs. time graph from the sketch of height vs. time, before they do the analysis or write the equation for height vs. time.

Roller Coaster… Plot points on the PRIZM and investigate to see which hill the roller coaster is on.

2 Runners in a race. An AP AB Calculus released question from the 2000 exam.

Using a graph of the two runners speeds (in meters per second) vs. time, answer these questions:



Two runners, A and B, race on a straight track. The graph which consists of two line segments shows the velocity in meters per second of Runner A. The velocity of Runner B is given by $v(t) = \frac{24t}{2t+3}$.

a. Find the velocity of Runner A and the velocity of runner b at time t = 2 seconds.
b. Find the acceleration of Runner A and Runner B at time t = 2 seconds.
c. Find the total distance run by both runners over the course of 10 seconds. Indicate units of measure

Additional questions developed by MS and HS teachers in Hawaii in the Spring of 2012 plus TW ideas:

The graphs cross at two places. What is happening when the graphs intersect?
Who is ahead in the race at the two times when the graphs intersect?
Who wins a 100 meter race? By what distance do they win?
Will the person who won the 100 meter race continue to lead or will they be caught?
If they are caught, when will that occur? What happens after that time, if the model is valid for a full minute?

Filling an auditorium problem.

Released AP AB Calculus exam question from 2009 Calc AB test

Here is the actual question:

2. The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditoriumat time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditoriumafter time $t = 1$.

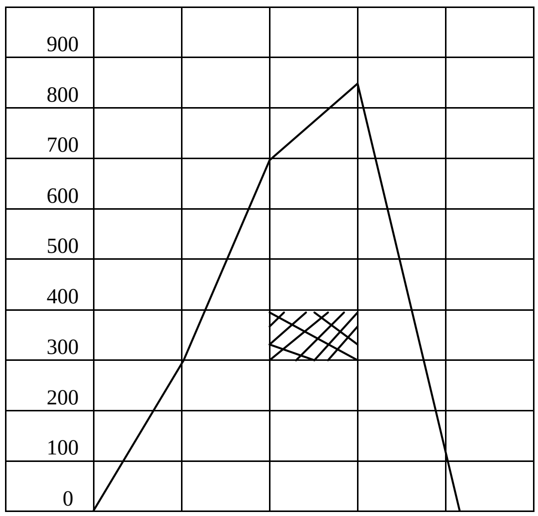
(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all peoplewho enter the auditorium after the doors open, and use the model for total wait time from part (c).

The bold questions can be answered, as can others related to the problem. See the MS adaptation on the next few pages. For the MS problem, I made the graph a shape made by line segments to simple finding the areas needed. In HS, students can use Reimann Sums to find the areas needed.

Middle School adaptations for the Calculus Problem on P. 31 (graph on P. 49)

This graph shows the number of people entering an auditorium for a rock concert. The vertical axis unit is the number of people entering per hour and the horizontal axis marks the time that has passed. We are showing what happens for each half hour. There is no one in the auditorium 2 hours before the concert starts, and the doors are closed when the concert starts at time = 2 hours.

Hundreds of people entering the auditorium per hour



0 0.5 1.0 1.5 2.0 2.5

Hours (the concert begins at time = 2)

1. How many people do you think attended the concert?
2. What is happening at 45 minutes?
3. What geometric concept does the shaded part of the graph represent?
4. In the context of the problem and using the correct units, how would you interpret the meaning of the shaded region of the graph?
5. What does the area between the graph and the horizontal axis represent?
6. Calculate the "areas" for the horizontal intervals $0 \leq t \leq 0.5$;
   $0.5 \leq t \leq 1.0$;
   $1.0 \leq t \leq 1.5$; and
   $1.5 \leq t \leq 2.0$.

What does the sum of all of these values equal? What does this value represent?   .

7. What does the area after 2.0 tell you? What is its value?

8. Where is the maximum value of the graph? A student says this is when the auditorium is the most full. Is the student correct? Explain your reasoning.

9. When is the auditorium the most full? Justify your answer.

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10. The total wait time for the wait time for all of the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The total wait time for all those who enter in the first hour is 373 minutes. The total wait time for all those who enter during the second hour is 388 hours. What is the average wait time in minutes for the people in each of these two groups?

11. Is it reasonable to determine the total wait time for everyone who arrived before the concert started by finding the average of the total wait time in order to find the average wait time for all the people who got to the concert on time? Justify your answer.

12. On average, how long does a person who arrived before the start of the concert wait for the concert to begin?

^^^^^^^^^^^^^^^^   Additional questions you might consider ^^^^^^^^^^^^^^^^^^^^^

13. What does the rate of change of the line segments tell you about the situation?

14. When would be the worst crowding at the doors to enter the auditorium?

15. At 1.5 hours, is the number of people entering the auditorium increasing or decreasing?

16. What is the rate of change of people entering the auditorium at t = 1 hour? Justify your answer.

17. During what time period is the rate of the number of people entering the auditorium changing the fastest?

For both of the first two web URL's scroll down to where you find the tests listed in order to look at the questions:

http://apcentral.collegeboard.com/apc/members/exam/exam_information/1997.html

Gets you to the page in AP Central with the different AB Calculus test questions.

http://apcentral.collegeboard.com/apc/members/exam/exam_information/8357.html

Gets you to the page in AP Central with the different Statistics test questions.

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FYI: A ***really great*** page to check out:

Here is a URL for a web page written by Dixie Ross. Scroll down and you will find links to 8 different adaptations of AP Calculus and AP Statistics questions for Middle School, Algebra 1 and 2, Geometry, and Pre-Calculus students:

http://apcentral.collegeboard.com/apc/members/courses/teachers_corner/29924.html

If you scroll to the bottom of this web page, you will see a link to the answers . Below the answer link are other links back to interesting parts of the AP Central web pages.

In my opinion: If you are using these ideas, make sure to tell the students that the questions they worked on were ones that you adapted from an AP question. Also, tell the other teachers in your building and your district know you are doing this, so they may use them too.