Geometry Is Not Just for Squares!

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Assessment has been defined by the NCTM as the "process of gathering evidence about a student's knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes" (Assessment Standards, 1995, p. 3)

The NCTM *Assessment Standards* (1995) advocate that:

- assessment be an integral part of instruction
- assessment be aligned with instruction
- multiple sources be used to assess student understandings
- various types of mathematical knowledge be assessed
- assessment should help students learn more mathematics
- assessment should lend itself to making valid inferences about the mathematics that students have learned.

Assessment has as its goal not only to inform teachers of student understandings but also help students learn significant mathematics.

"In order to develop mathematical power in *all* students, assessment needs to support the continued mathematics learning of *each* student. This is the central goal of assessment in school mathematics." (Assessment Standards, 1995, p. 6)

Every mathematics lesson is an assessment opportunity. It need not be an interruption of the mathematical learning process as often is the case.

Research on Geometric Reasoning

The Van Hiele Model

The van Hiele model states that there are five discrete levels of geometric reasoning.

The van Hiele levels are characterized as follows.

- At **Level 1**, students identify, name, and compare geometric shapes based on their appearance alone. For example, a student at this level might say an object is a rectangle because it resembles a door.
- At **Level 2**, students look at geometric figures based on their components and can identify their various properties.

  They can also discover, using concrete means, a figure's various properties. For example, students would identify a rhombus as having four equal sides.
- At **Level 3**, students are able to logically relate, in an informal way, properties and rules previously discovered. A student at this level would understand that a square is a rectangle and a rhombus.
- At **Level 4**, students understand the role of postulates, axioms, and theorems and are able to prove theorems both formally and deductively. Traditionally, high school geometry is taught at level 4.
- At **Level 5**, students are able to establish theorems in geometric systems that are based on postulates different from those of Euclidean geometry (Clements & Battista, 1992).

Research on the Van Hiele Theory

Generally supported by the research is:

- The model's ability to describe students' thinking.
- The important idea for educators that advancement in level is dependent on instruction and not on age.
- The hierarchic nature of the levels.
- The existence of a pre-recognition level 0.
- The ideas that students may reason at one level in one geometric topic and at another level in a different geometric topics.

The discrete nature of the levels is generally not supported by research.

The van Hiele model hypothesizes that one cannot advance from one level of reasoning to another without having mastered the level of reasoning in the previous level.

In addition, the model hypothesizes that the process of advancing from one level to another is dependent on instruction and not on age.

Another characteristic of this model is that each level has its own language and that one speaking in the language of a higher level will not be understood by those reasoning at lower levels. \

Recent Research on Geometric Reasoning

As part of an on-going research program in Cognitively-Guided Instruction in Geometry at the University of Wisconsin, Madison, students' reasoning about shape at the lower levels has been further refined (NCRSME, 1994).

Their preliminary findings are as follows:

Geometric Reasoning by **Resemblance**

The first level at which children reason about shape is termed resemblance and roughly corresponds to van Hiele level 1.

- Students at this level classify objects based on their resemblance to other shapes, and often rely on irrelevant characteristics of the figure.
- Students may classify unfamiliar geometric figures according to a shape well known to them in spite of the fact that the known shape may share little in common with the unfamiliar shape.

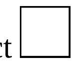
Reasoning based on resemblance can be further subdivided into either direct or indirect resemblance.

For example, a students whose reasoning is based on direct resemblance would classify a chevron as a triangle because it resembles a 'pushed in' triangle.

Also characteristic of direct resemblance would be a reliance on visual stereotypes.

- For example, a student may not classify a 'thin' rectangle as a rectangle because it is 'too thin' in relation to the students' rectangle prototype of a door.

In addition, a student relying on visual prototypes may classify a given figure in two different categories depending on its orientation.

- For example, the student may classify this object  as a square, but may classify the same object rotated 45 degrees as a 'diamond'.

Students can also classify object based on *indirect resemblance* by modifying in their mind the figure into another figure that is better known to the student.

- For example, a chevron could be considered a triangle by indirect resemblance by 'pulling back' two of its sides until it looks like a triangle.

Geometric reasoning by **Attributes**

Here students directly refer to a shape's attributes, perhaps using very informal language.

However, students at this level may not understand the relationships between a figure's attributes.

- For example, students who do not understand the relationship between the angles and the number of sides of a figure would have to count its angles even though they know how many sides it contains.

Students who understand such relationships reason at a higher level than those who do not.

At this level, students also understand that some attributes of a figure are not changed by certain other action.

- For example, the student would understand that all rectangles have four sides and four angles even if it is 'long and skinny' or 'short and fat.'

Reasoning of this type roughly corresponds to van Hiele level 2.

Geometric Reasoning Based on **Properties**

- Students see a figure in terms of its properties and the intricate relationship between a figure and its properties.
- They understand that to remove a critical property from a given shape changes the shape's classification and that removing a 'non-critical' attribute, such as size, would not change its classification.

A more mature level of geometric reasoning is exhibited by a student understands the relationships between the properties of a shape.

Reasoning of this type roughly corresponds to reasoning at van Hiele level 3.

Such an understanding helps students to develop understandings about the relationships between classes of figures.

- For example at this level, a student would know that a *square* is also a *parallelogram* because it is a quadrilateral with both pairs of opposite sides parallel.

Geometric Assessment Tasks

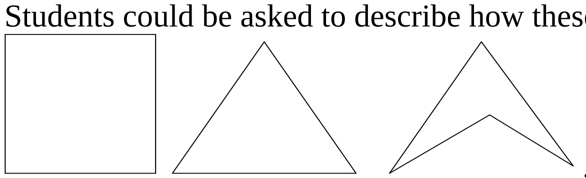
Described here are some assessment tasks which teachers can use to assess and extend students' understandings in geometry.

Polygon Comparison

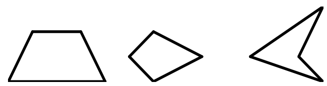
[Lehrer et al, 1993]

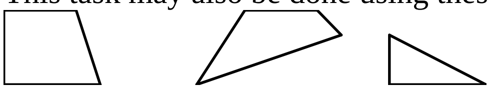
Task Description. This task requires students to compare and contrast 3 different polygons.

Students could be asked to describe how these three figures are alike and how they are different

.

This task is helpful in determining if students are using a figure's appearance or its parts and properties in this comparison.

This task may also be done using these two trios:  and

 .

Assessment Results

I asked three students to look at these three objects  and to describe any similarities and differences among the three figures.

One of the students responded in a manner characteristic of *direct resemblance*: "They're different because...this one [triangle]...doesn't come in. This one is just a plain triangle....This one [chevron] comes in for a triangle...this [first] one is just a rectangle."

Overall, all 3 students classified the polygons based on their *appearance* or *resemblance to known shapes*.

Polygon Sort

[Fuys, Geddes, & Tischler (1988)]

<u>Task Description</u>. Given a set of cardboard polygons, students are asked to sort the figures in a meaningful way and to give a rationale for their sort.

If students are unsure of how to do this, ask them to determine how the various polygons are alike and to sort the figures on these likenesses.

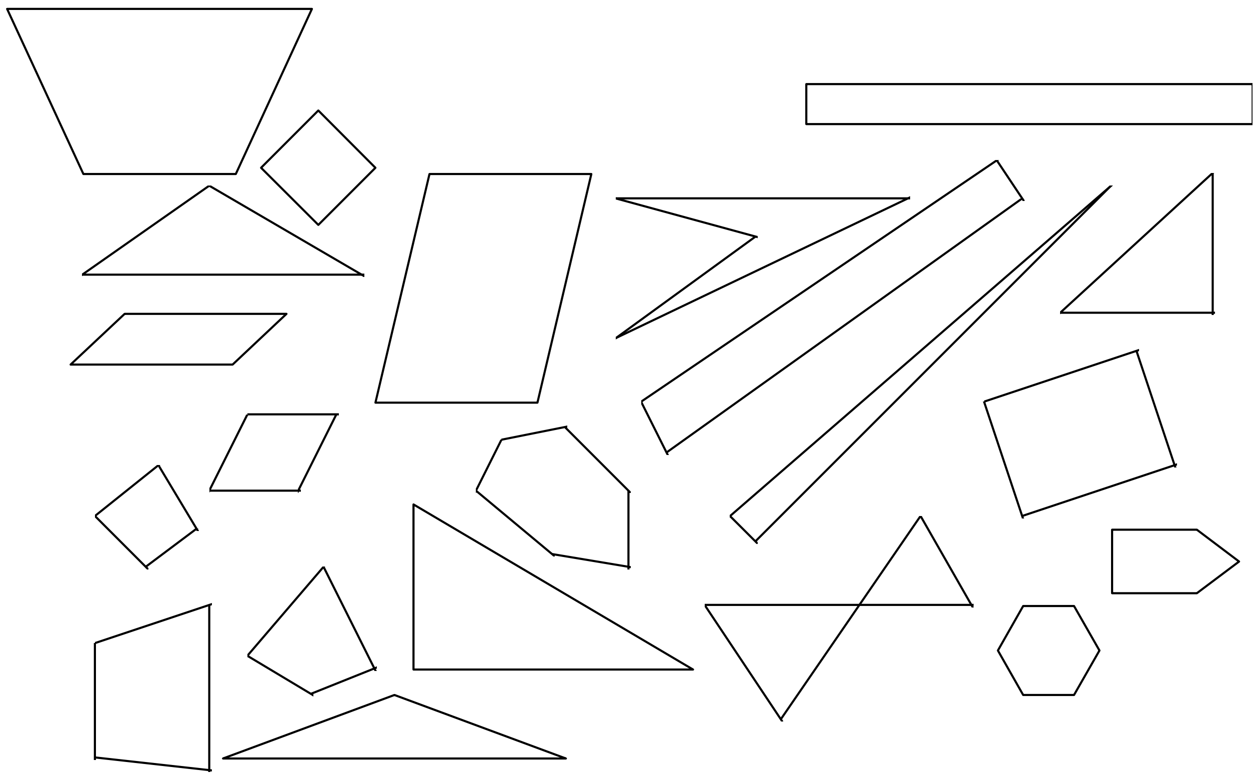
This task can help reveal if students are reasoning based on appearance, attributes, or properties and if that reasoning is stable in various contexts.

The *materials* required for this task would include a wide variety of polygons cut out of cardboard.

The greater the diversity of polygon cut-outs, the more potential there is to assess more fully students' geometric reasoning.

To help best determine the level of students' knowledge about various subclasses of a particular polygon:

- Examples of all the various subclasses of triangles and quadrilaterals should also be included.
- Include both examples of 'typical' (like those found in texts) and 'atypical' (for example, a 'thin' triangle) examples of polygons.



Polygon Definitions

Task Description. In this task students give a definition for a specific type of polygon and a rationale for that definition.

The task can be varied by changing the figure to be defined.

To help students' clarify their thinking, the teacher can draw the figure which matches the students' definition.

- Often students modify an incorrect definition when they see that the teacher's drawing matching their definition doesn't match up with that of the figure they are trying to define.

This task is important because it asks students to focus on:

- a figure's attributes and properties
- on the minimum amount of information necessary to define a figure.

Assessment Results

The students were asked to define a rectangle.

John defined it: "It has four sides. It's a polygon...and all four sides would be...right angles and has two sets of parallel lines and...the only difference is two lines are like shorter than two other lines."

His definition is based, in part, on prototype rectangles as was exhibited by his addition of properties not necessarily held by all rectangles.

Furthermore, he showed a possible lack of understanding of the relationships between the rectangles and squares.

Concept Card

[Geddes and Fortunato (1993)]

Task Description. Another task involves the use of a concept card which asks students to study examples and non-examples of a geometric figure that is given a fictitious name, such as 'quips'.

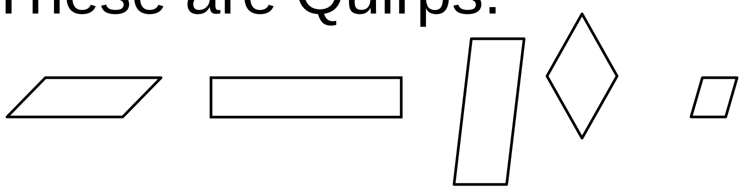
This activity is an "effective way to familiarize students with a concept and to make them aware of its distinguishing characteristics" (Geddes & Fortunato, 1993, p. 204).

This task also does not allow students to rely on rotely acquired facts about standard geometric figures.

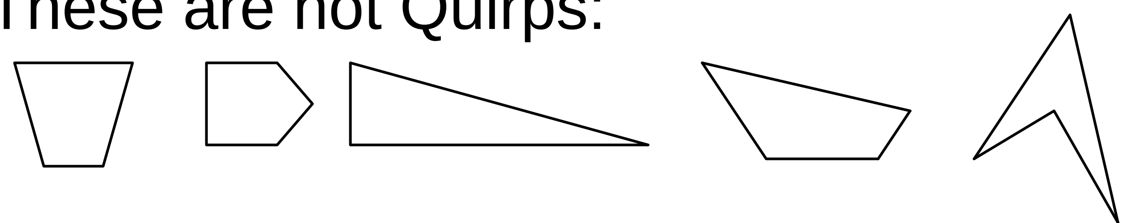
Concept Card

(adapted from Geddes, and Fortunato,1993)

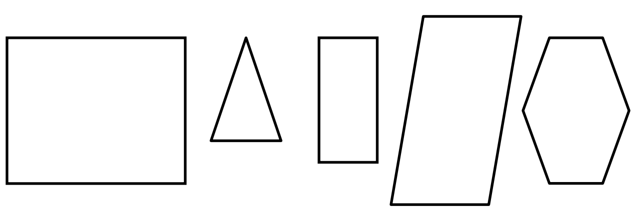
These are Quirps:



These are not Quirps:



Which of these are Quirps?



Draw some quirps.

Draw some non-quirps.

What is a quirp?

Assessment Results

During the assessment, the students studied a concept card involving 'larps' (which were actually trapezoids) taken from Geddes and Fortunato (1993, p. 208).

After considering this, John seemed to have the best notion of what a larp is:

"I think the differences are that larps have four sides and they're parallel and that non-larps, they can have any number of sides but . . . they can have parallel lines but if they don't have four sides, then they're not a larp . . . They just have maybe one set of parallel lines . . . the requirements are four sides and at least one set of parallel lines."

John classified larps based on their parts and properties of parts and was thinking deductively.

Conclusions

The Assessment Standards for School Mathematics (NCTM, 1995) call for assessment that not only helps teachers know what students understand but for assessment tasks that help students extend their mathematical understanding. The nature of these assessment tasks and the group dynamics also helped students learn significant geometry.

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