Learn It First, then Prove It

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The Quadratic Formula in Algebra

Traditional approach:
"Today you will prove the quadratic formula."

A better approach:
"This is the quadratic formula:"

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Objectives:
Be able to *pronounce* it and *write* it correctly.

- x equals
- the *opposite* of b
- plus or minus
- the square root of
- the *quantity* b squared minus $4ac$
- *all* divided by $2a$.

(Learn the *meaning* when you are "older.")

To make you "older":

(1) Two-step equations: Solve $5x + 3 = 17$

- $5x = -3 + 17 \Rightarrow 5x = 14 \Rightarrow x = 2.8$

(2) Abs. value equations: Solve $|5x + 3| = 17$

- $5x + 3 = \pm 17$ (Look! A "±" sign!)
- $5x = -3 \pm 17$ (Opposite of 3, ± number!)
- $5x = 14$ or $5x = -20$
- $x = 2.8$ or $x = -4$

(3) Equations with squares: Solve $(5x + 3)^2 = 17$

- $\sqrt{(5x+3)^2} = \sqrt{17}$ (Take *pos*. sq. rt.)
- $|5x + 3| = \sqrt{17}$ $\sqrt{\text{number}^2} = |\text{number}|$
- $5x + 3 = \pm\sqrt{17}$ (Same steps as #2)
- $5x = -3 \pm \sqrt{17}$ (Opposite of 3, ± sq root...)
- $x = \dfrac{-3 \pm \sqrt{17}}{5}$ (... divided by a number!)
- By calc., $x = 0.2246...$ or $x = -1.4246...$

(4) Solve: $25x^2 + 30x + 9 = 17$

- $(5x + 3)^2 = 17$ ("The pot is on the floor!") (This is now an *old* problem.)
- $x = 0.2246...$ or $x = -1.4246...$ (See #3.)
- Check:

$25(0.2246...)^2 + 30(0.2246...) + 9 \stackrel{?}{=} 17$

$8 + 9 \stackrel{?}{=} 17$

$17 = 17$, which checks!

(5) Solve: $25x^2 + 30x - 8 = 0$

- $25x^2 + 30x - 8 + 17 = 0 + 17$, etc. (This step completes the square.)
- $(5x + 3)^2 = 17$ (Again, the pot is on the floor.)
- $x = 0.2246...$ or $x = -1.4246...$ (See #4.)

Now for the meaning!

You recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$25x^2 + 30x - 8 = 0$ is a *quadratic* equation. From Example (5) you recall:

- $x = 0.2246...$ or $x = -1.4246...$

Let $a = 25$, let $b = 30$, and let $c = -8$. Substitute:

$$\frac{-30 \pm \sqrt{30^2 - 4(25)(-8)}}{2(25)}$$

$$= \frac{-30 \pm \sqrt{900 + 800}}{50}$$

$$= \frac{-30 \pm \sqrt{1700}}{50}$$

$= 0.2246...$ or $-1.4246...$

These are the solutions for x!

>>>---> The quadratic formula gives you solutions of a quadratic equation, as long as it is in the form

$$ax^2 + bx + c = 0$$

So the *complete* quadratic formula is:

- If $ax^2 + bx + c = 0$, (the *hypothesis*)
- then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (the *conclusion*)

The formula has an "If" part and a "then" part.

The formal proof can be done or not done. But the students now realize what the quadratic formula is, and how it can be used to find solutions of a quadratic equation.

There are *reasons* to use the quadratic formula.

Archery Problem: Ann Archer shoots an arrow into the air. At time t seconds, the arrow is $d(t)$ meters above the ground, where

$$d(t) = -5t^2 + 13t + 6$$

Finding the time(s) t when the arrow is, say, 33 m high leads to a quadratic equation,

$$33 = -5t^2 + 13t + 6$$

A Linear Combination of Sinusoids in Trig

Students learn properties of the trig functions.

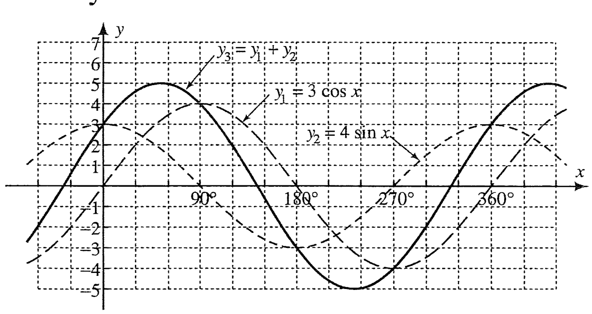
Traditional approach:
"Today you will prove the
linear combination property:

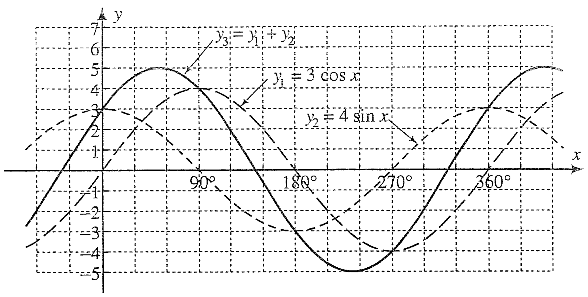
$$b \cos x + c \sin x = A \cos (x - D)$$

A better approach:
" On your grapher, plot (degree mode):

$$y_1 = 3 \cos x$$
$$y_2 = 4 \sin x$$
$$y_3 = 3 \cos x + 4 \sin x$$

What do you observe about the results?





Observations:

- y_1 and y_2: Sinusoids with equal periods, 360°
- y_1 and y_2: Different amplitudes, 3 and 4
- So y_3 is a **linear combination** of two sinusoids with equal periods.
- y_3: Appears to be a sinusoid with period 360°, amplitude about 5, and phase displacement about 50°.

Conjecture:

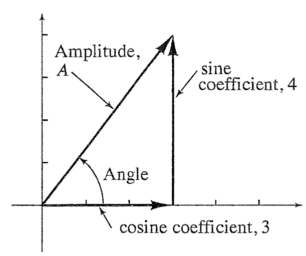
- A linear combination of two sinusoids with equal periods is another sinusoid with the same period, but a different amplitude and a phase displacement.

Assuming that the conjecture is true,

$$3 \cos x + 4 \sin x = A \cos (x - D)$$

How could you calculate the amplitude, $A \approx 5$, and the phase displacement, $D \approx 50°$?

Draw an auxiliary diagram,



- Students rather quickly find that $A^2 = 3^2 + 4^2$
- In 50 years of teaching, I have had only three students who, on their own, came up with the fact that D is the *angle* in the aux. diagram.
- $\tan D = \dfrac{4}{3}$
- $D = \arctan (4/3) = 53.1301...°$ or $233.1301...°$
- Pick the 53.1301...° (Why?)
- $\therefore\ 3 \cos x + 4 \sin x = 5 \cos (x - 53.1301...°)$

Check to see if the conjecture is reasonable.

- Plot $y_4 = 5\cos(x - 53.1301...^\circ)$
  The graph seems to coincide with y_3.
- More convincing: Store $D = 53.1301...^\circ$ without rounding and make a table.

x	y_3	y_4
10°	3.6490...	3.6490...
20°	4.1871...	4.1871...
30°	4.5980...	4.5980...
40°	4.8692...	4.8692...
50°	4.9925...	4.9925...

Then have students write various linear combinations as single sinusoids, such as:

(1) $y = 12\cos x + \sin x$
(2) $y = -7\cos x + 24\sin x$
(3) $y = -8\cos x - 11\sin x$
(4) $y = 6\cos x - 6\sin x$
(5) $y = (\sqrt{6} + \sqrt{2})\cos x + (\sqrt{6} - \sqrt{2})\sin x$

The first four have phase displacements terminating in four different quadrants.

The fifth has a surprise phase displacement!

Two reasons for learning this property:

- The *good* reason:

If a musical note is played simultaneously by several instruments, the sound waves add to form a note with the *same* period. But the loudness (amplitude) is *not* simply the sum of the individual loudnesses.

- The *real* reason:

The property is needed later on for such things as understanding the solutions to certain differential equations. But this reason is not convincing to students who are just studying trigonometry.

Once students know what the property says and how it can be used, the proof can lead (some) students to remark, "So *that's* why it works!"

Prove that the sum of two sinusoids with equal periods is a sinusoid with the same period and a different amplitude and phase displacement. That is, for constants A, b, c, and D, prove that

$$b\cos x + c\sin x = A\cos(x - D)$$

Proof: (Why write this?)

$A\cos(x - D)$
[Write one side of the desired equation.]

$= A(\cos x\cos D + \sin x\sin D)$
[by the composite argument property]

$= (A\cos D)\cos x + (A\sin D)\sin x$
[Distribute, commute, and associate.]

$= b\cos x + c\sin x,$
where $b = A\cos D$ and $c = A\sin D$

$\therefore\ b\cos x + c\sin x = A\cos(x - D)$, Q.E.D.
[by the transitive and symmetric properties]

Note: $b^2 = A^2\cos^2 D$ and $c^2 = A^2\sin^2 D$

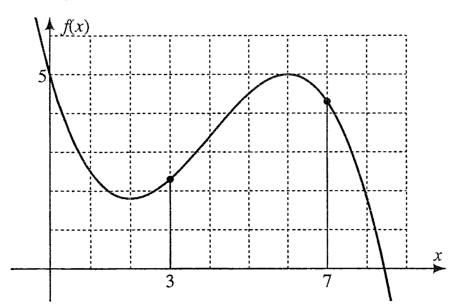
So $b^2 + c^2 = A^2\cos^2 D + A^2\sin^2 D$
$b^2 + c^2 = A^2(\cos^2 D + \sin^2 D) = A^2 \cdot 1 = A^2$

and $\dfrac{c}{b} = \dfrac{A\sin D}{A\cos D} = \tan D \Rightarrow D = \arctan(c/b)$

The Mean Value Theorem in Calculus

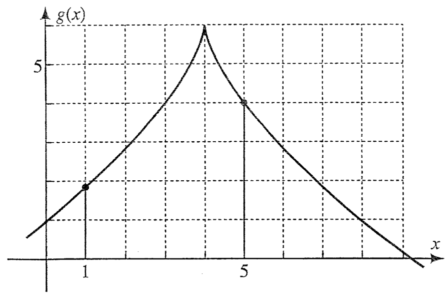
Under "suitable conditions" for a function f, there is a point $x = c$ in the interval (a, b) where the tangent line parallels the secant line connecting the endpoints $(a, f(a))$ and $(b, f(b))$. "Is paralled to" means "has the same slope $f'(c)$"

1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$



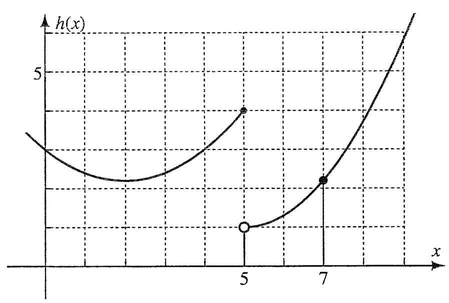
- Draw secant line from $(3, f(3))$ to $(7, f(7))$.
- Draw tangent at $x = c$ between 3 and 7 that is *parallel* to the secant line.
- From graph, $c \approx$ __________
- Is f differentiable on *open* interval $(3, 7)$? ___
- Is f continuous on *closed* interval $[3, 7]$? ___

2. For $g(x) = 6 - 2(x-4)^{2/3}$



- Draw secant line from $(1, g(1))$ to $(5, g(5))$.
- Is g differentiable on *open* interval $(1, 5)$? __
- Is g continuous on *closed* interval $[1, 5]$? ___
- Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.

3. For $h(x) = \begin{cases} 2.2 + 0.2(x-2)^2, & \text{if } x \le 5 \\ 1 + 0.3(x-5)^2, & \text{if } x > 5 \end{cases}$



- Draw secant line from $(5, h(5))$ to $(7, h(7))$.
- Is g differentiable on *open* interval $(5, 7)$? __
- Is g continuous on *closed* interval $[5, 7]$? ___
- Tell why there is *no* value of $x = c$ between $x = 5$ and $x = 7$ at which $h'(c)$ equals the slope of the secant line.

The "suitable conditions" mentioned above are:

f is differentiable on the *open* interval (a, b) and f is continuous on the *closed* interval $[a, b]$.

Students understand *why* these hypotheses are *sufficient* (although not *necessary*).

The above three examples are excerpts from the seven examples in Exploration 5-5a (attached) that accompanies Foerster; *Calculus: Concepts and Applications;* © 1998, 2005, 2010 by Key Curriculum Press and now published by Kendall Hunt Publishing Co.

The formal proof of the mean value theorem is usually done as a corollary of Rolle's theorem, which in its historical form states that for polynomial functions, there is a point between consecutive zeros at which there is a horizontal tangent line.

Two reasons for learning mean value theorem:

- Good reason:

The theorem explains why on a car trip, there is a time between the start and the finish at which your *instantaneous* speed is exactly equal to your *average* ("mean") speed for the trip.

- Real reason:

The mean value theorem can be used as a lemma in the proof of the fundamental theorem of calculus. (But this reason is not convincing to students at this point in the calculus course.)

Conclusions

- Sometimes it is advantageous for students to *memorize* something without understanding, then discover the meaning over time. e.g.
  * The quadratic formula in algebra.
  * The definition of limit in calculus.
- Sometimes it is advantageous for students to *use* a property before they have *proved* it. That way they are sure what the property *means,* and thus are more willing to do the proof. e.g.
  * The linear combination property in trig
- Sometimes it is advantageous for students to be led to *discover* a property via a set of carefully-chosen examples. e.g.
  * The mean value theorem in calculus.
  * The fundamental theorem of calculus.

Final Exam!

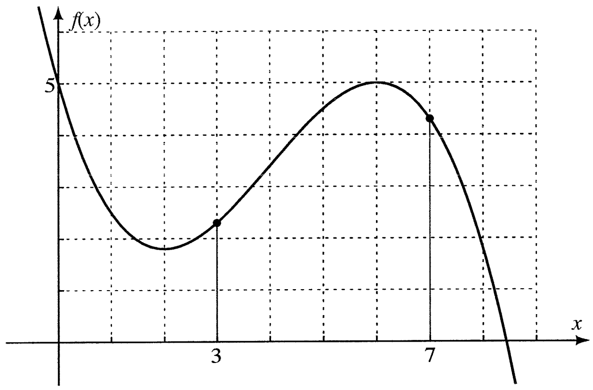
On the index card, please write the *one* most important thing you learned as a result of attending this session.

Name ______________________________ Group _________________________ Date ____________________

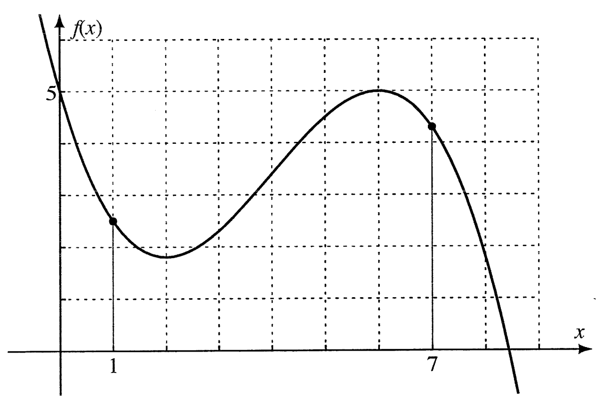
Exploration 5-5a: The Mean Value Theorem

Objective: Without looking at the text, discover the hypotheses and conclusion of the mean value theorem.

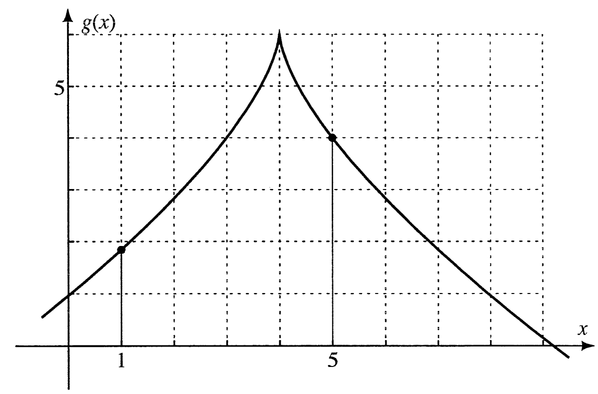
1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$, graphed below, there is a value of $x = c$ between 3 and 7 at which the tangent to the graph is parallel to the secant line through $(3, f(3))$ and $(7, f(7))$.
   - Draw the secant line and the tangent line.
   - From the graph, $c \approx$ ________
   - Is f differentiable on he open interval $(3, 7)$? ____
   - Is f continuous on the closed interval $[3, 7]$? ____



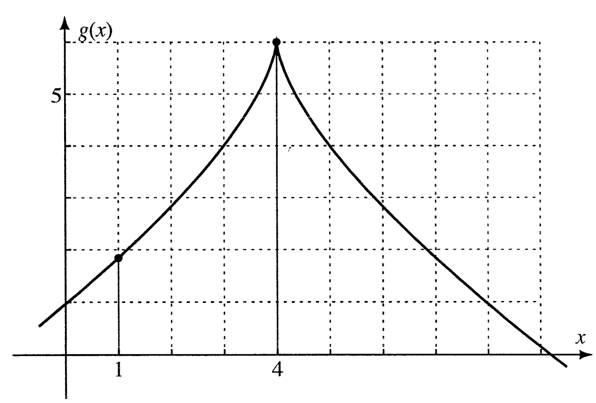
2. Function f from Problem 1 has *two* values of $x = c$ between $x = 1$ and $x = 7$ at which $f'(c)$ equals the slope of the corresponding secant line. (That is, the tangent line parallels the secant line.)
   - Draw the secant and tangents on the graph below.
   - From the graph, $c \approx$ ______ and $c \approx$ _______
   - Is f differentiable on the open interval$(1, 7)$? ____
   - Is f continuous on the closed interval $[1, 7]$? ____



3. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,
   - Draw a secant line through $(1, g(1))$ and $(5, g(5))$
   - Is g differentiable on the open interval $(1, 5)$? ___
   - Is g continuous on the closed interval $[1, 5]$? ___
   - Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.



4. Function g from Problem 3, *does* have a value $x = c$ in $(1, 4)$ for which $g'(c)$ equals the slope of the secant line through $(1, g(1))$ and $(4, g(4))$.
   - Draw the secant line and tangent line, below.
   - From the graph, $c \approx$ ________
   - Is g differentiable on the open interval $(1, 4)$? ___
   - Is g continuous on the closed interval $[1, 4]$? ___

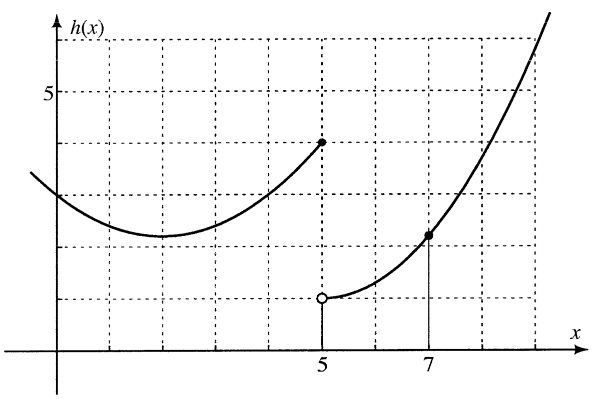


(Other side.)

5. Piecewise function h is defined by

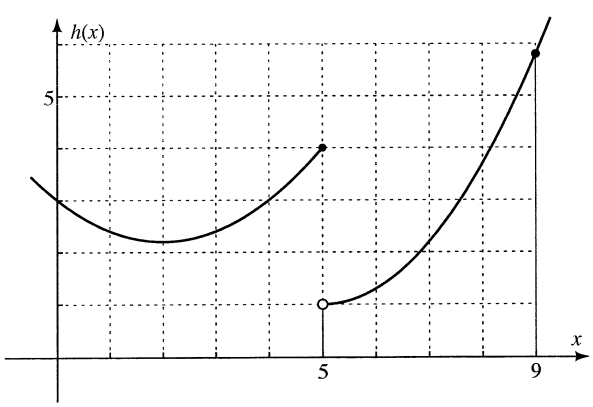
$$h(x) = \begin{cases} 2.2 + 0.2(x-2)^2 & \text{if } x \le 5 \\ 1 + 0.3(x-5)^2 & \text{if } x > 5 \end{cases}$$

- Draw a secant line through $(5, h(5))$ and $(7, h(7))$
- Is h differentiable on the open interval $(5, 7)$? ___
- Is h continuous on the closed interval $[5, 7]$? ___
- Why is there *no* value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line?



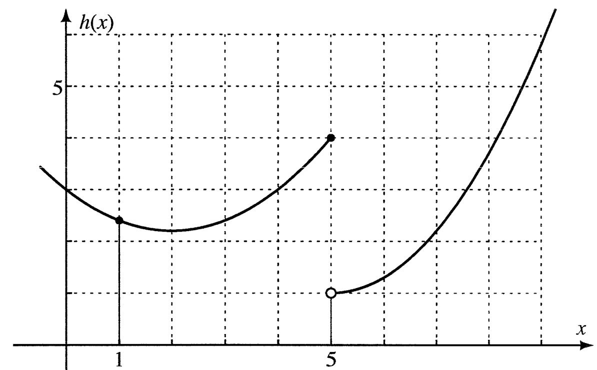
6. The graph below is function h from Problem 5.

- Draw a secant line through $(5, h(5))$ and $(9, h(9))$
- Is h differentiable on the open interval $(5, 9)$? ___
- Is h continuous on the closed interval $[5, 9]$? ___
- There *is* a point $x = c$ in $(5, 9)$ where $h'(c)$ equals the slope of the secant line. Draw the tangent line. Estimate the value of c. _______



7. The graph below is function h from Problem 5

- Draw a secant line through $(1, h(1))$ and $(5, h(5))$
- Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line.
- Is h differentiable on the open interval $(1, 5)$? ___
- Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.



8. The mean value theorem states:

If f is differentiable on (a, b) and
f is continuous on $[a, b]$,
then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ i.e., the secant's slope

For which problem(s) are

- the hypotheses and conclusion true? __________
- the hypotheses and conclusion not true? _______
- the conclusion true, but not the hypotheses? ____

9. The number c is an x-value where the *instantaneous* rate of change equals the *average* ("mean") rate of change. Explain why the hypotheses are **sufficient** conditions for the conclusion, but *not* **necessary** conditions.

10. What did you learn as a result of working this Exploration that you did not know before?

Name ______________________ Group ______________________ Date ______________________

Solutions, Exploration 5-5a: The Mean Value Theorem

Objective: Without looking at the text, discover the hypotheses and conclusion of the mean value theorem.

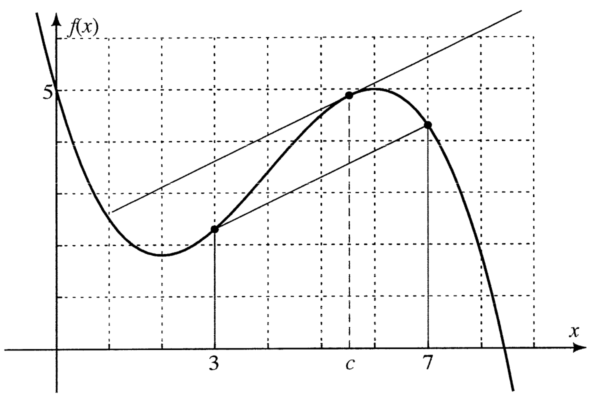
1. For $f(x) = -0.1x^3 + 1.2x^2 - 3.6x + 5$, graphed below, there is a value of $x = c$ between 3 and 7 at which the tangent to the graph is parallel to the secant line through $(3, f(3))$ and $(7, f(7))$.

   Draw the secant line and the tangent line. • graph

   From the graph, $c \approx$ 5.5

   Is f differentiable on he open interval $(3, 7)$? • Yes

   Is f continuous on the closed interval $[3, 7]$? • Yes



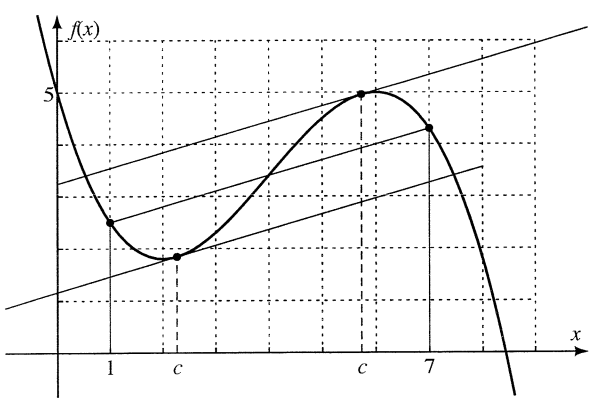
2. Function f from Problem 1 has *two* values of $x = c$ between $x = 1$ and $x = 7$ at which $f'(c)$ equals the slope of the corresponding secant line. (That is, the tangent line parallels the secant line.)

   Draw the secant and tangents on the graph below.

   From the graph, $c \approx$ • 2.3 and $c \approx$ • 5.7

   Is f differentiable on the open interval$(1, 7)$? • Yes

   Is f continuous on the closed interval $[1, 7]$? • Yes



3. For $g(x) = 6 - 2(x - 4)^{2/3}$, graphed below,

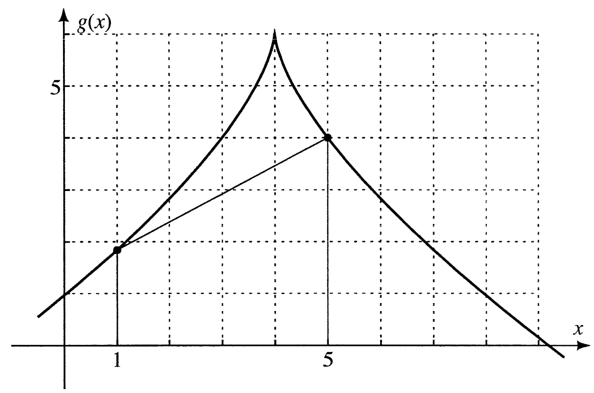
   Draw a secant line through $(1, g(1))$ and $(5, g(5))$

   Is g differentiable on the open interval $(1, 5)$? • No

   Is g continuous on the closed interval $[1, 5]$? • Yes

   Tell why there is *no* value of $x = c$ between $x = 1$ and $x = 5$ at which $g'(c)$ equals the slope of the secant line.

   • The slope of the secant line is positive. For x between 1 and 4 the tangent lines are too steep. For x between 4 and 5 the tangent lines have negative slope. (At $x = 4$ the tangent line is vertical, and thus has no slope.)



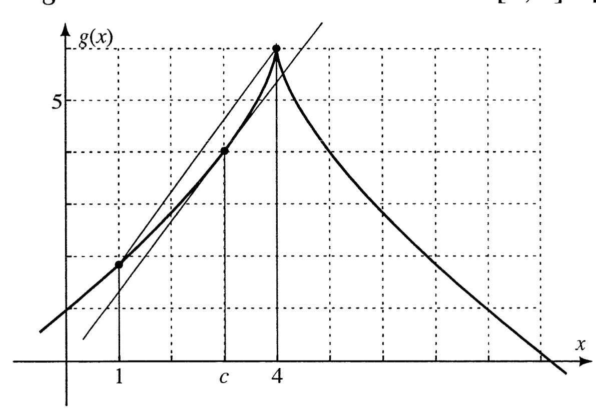
4. Function g from Problem 3, *does* have a value $x = c$ in $(1, 4)$ for which $g'(c)$ equals the slope of the secant line through $(1, g(1))$ and $(4, g(4))$.

   Draw the secant line and tangent line, below.

   From the graph, $c \approx$ • 3.0

   Is g differentiable on the open interval $(1, 4)$? • yes

   Is g continuous on the closed interval $[1, 4]$? • no



(Other side.)

5. Piecewise function h is defined by

$$h(x) = \begin{cases} 2.2 + 0.2(x-2)^2 & \text{if } x \le 5 \\ 1 + 0.3(x-5)^2 & \text{if } x > 5 \end{cases}$$

Draw a secant line through $(5, h(5))$ and $(7, h(7))$

Is h differentiable on the open interval $(5, 7)$? • yes

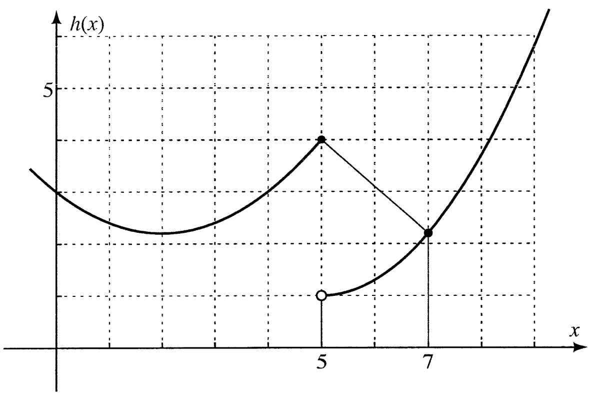
Is h continuous on the closed interval $[5, 7]$? • no

Why is there *no* value $x = c$ in $(5, 7)$ for which $h'(c)$ equals the slope of the secant line?

• The secant line has slope $\dfrac{2.2 - 4}{7 - 2} = -0.9$.

• For x between 5 and 7, the tangent lines have positive slope

• Thus no tangent line has slope –0.9.



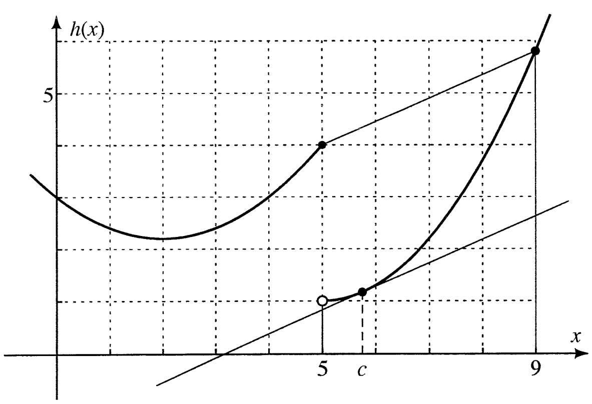
6. The graph below is function h from Problem 5.

Draw a secant line through $(5, h(5))$ and $(9, h(9))$

Is h differentiable on the open interval $(5, 9)$? • yes

Is h continuous on the closed interval $[5, 9]$? • no

There *is* a point $x = c$ in $(5, 9)$ where $h'(c)$ equals the slope of the secant line. Draw the tangent line. Estimate the value of c. • $c \approx 5.75$



7. The graph below is function h from Problem 5

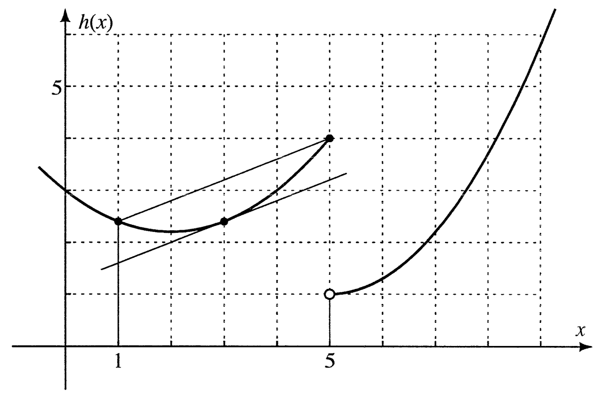
Draw a secant line through $(1, h(1))$ and $(5, h(5))$

Show that there is a point $x = c$ in $(1, 5)$ where $h'(c)$ equals the slope of the secant line. • $c \approx 3$

Is h differentiable on the open interval $(1, 5)$? • yes

Explain why h is continuous on $[1, 5]$, even though there is a step discontinuity at $x = 5$.

• For a closed interval, one-sided continuity at the end points from inside the interval is sufficient for continuity on that interval. And $\lim_{x \to 5^-} h(x) = h(5) = 4$.



8. The mean value theorem states:

If f is differentiable on (a, b) and
f is continuous on $[a, b]$,
then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ i.e., the secant's slope

For which problem(s) are

the hypotheses and conclusion true? • 1, 2, 4, 7

he hypotheses and conclusion not true? • 3, 5

the conclusion true, but not the hypotheses? • 6

9. The number c is an x-value where the *instantaneous* rate of change equals the *average* ("mean") rate of change. Explain why the hypotheses are **sufficient** conditions for the conclusion, but *not* **necessary** conditions.

• If the hypotheses are true, then the conclusion is true. So the hypotheses "do the job" (sufficient).

• But the conclusion can be true (as in Problem 6) even if the hypotheses are not true. So the hypotheses are not "necessary" to do the job.

10. What did you learn as a result of working this Exploration that you did not know before?

• Answers will vary.