What's the Big Idea? Inquiry Method Builds Understanding of Fractions

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Tracing Fraction Concept Development

Common Core State Standards in Mathematics

Grades 1 & 2	• Partition shapes into 2-4 equal shares & describe with fraction language. • Recognize that equal shares of identical wholes can have different shapes.
Grade 3	**Develop understanding of fractions as numbers.** • Understand connection between fraction notation and models , particularly area models and number lines. • Explain how a unit fraction is the building block of other fractions. • Recognize and generate simple equivalent fractions, and compare fractions by reasoning about their size.
Grade 4	• Extend understanding of fraction equivalence and ordering, including comparison of fractions with different numerators and denominators. • Utilize understanding of whole number operations and fractions as collections of unit fractions to 1) add and subtract fractions with like denominators and 2) multiply fractions by a whole number.
Grade 5	• Extend understanding of fraction operations (+ − × ÷)

Outline of Presentation

This presentation will elaborate:

- How **equal-sharing tasks** can be used to introduce fraction concepts
- How **multiple fraction models** can be used to deepen initial fraction understanding
- How **comparing fractions** can be taught with a focus on reasoning strategies

Instructional Framework:
A Inquiry-based Approach

Mathematics lessons:

- Focus on a **limited number of mathematical tasks** that students can engage in through **problem solving**
    - Methods for solving tasks are not prescribed
    - Multiple correct solution pathways often exist
    - Tasks can be solved by building on prior knowledge
- Utilize **discussion** of students' solutions to further students' knowledge of concepts and procedures
- Emphasize **explanation and justification** of ideas
- Mobilize students as a **community of learners**

Equal-Sharing Tasks

Building Knowledge of Fractions on the Foundation of Division and Real-life Sharing Situations

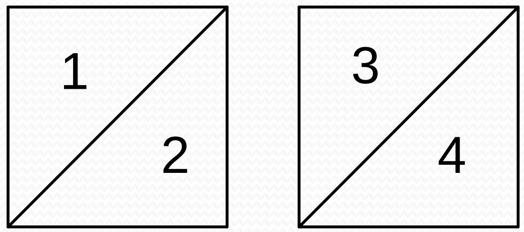
Examining a Sharing Task

How might children who have had no formal instruction on fractions approach the following problem?

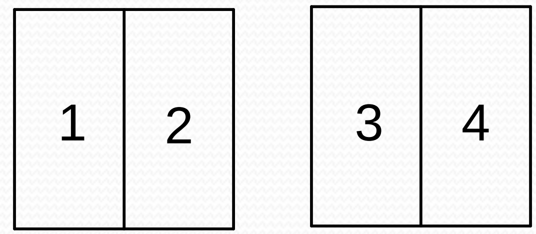
Four children want to share two (square) waffles so that each child gets the same amount. How much of a waffle should each child get?

Four children want to share two (square) waffles so that each child gets the same amount. How much waffle should each child get?

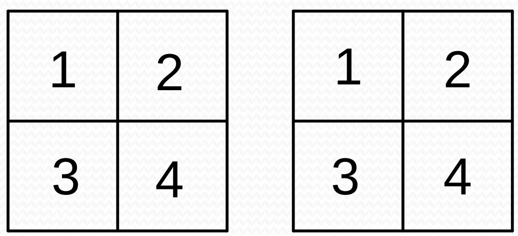
Solution A



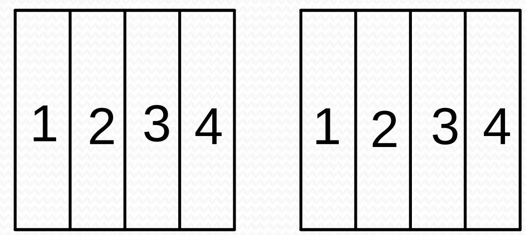
Solution B



Solution C



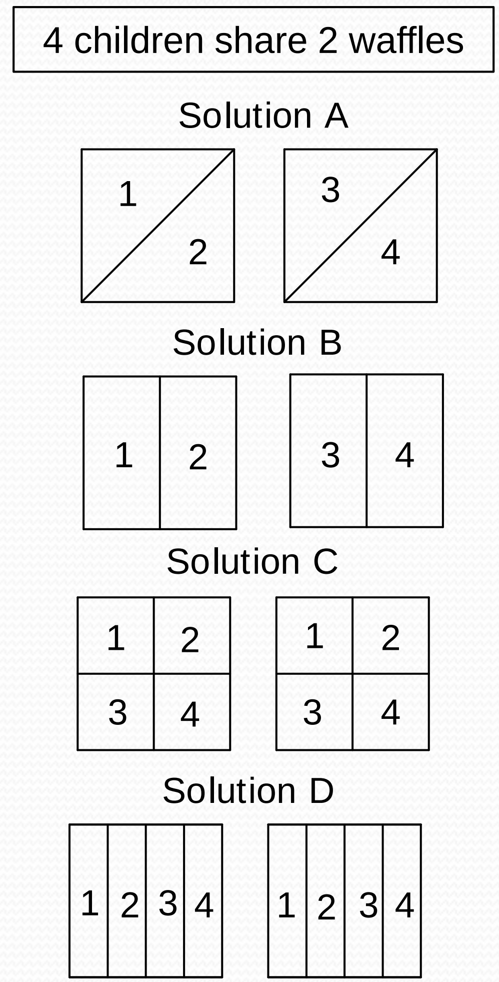
Solution D



What might children be able to learn about fractions from discussing and comparing their solutions to this problem?

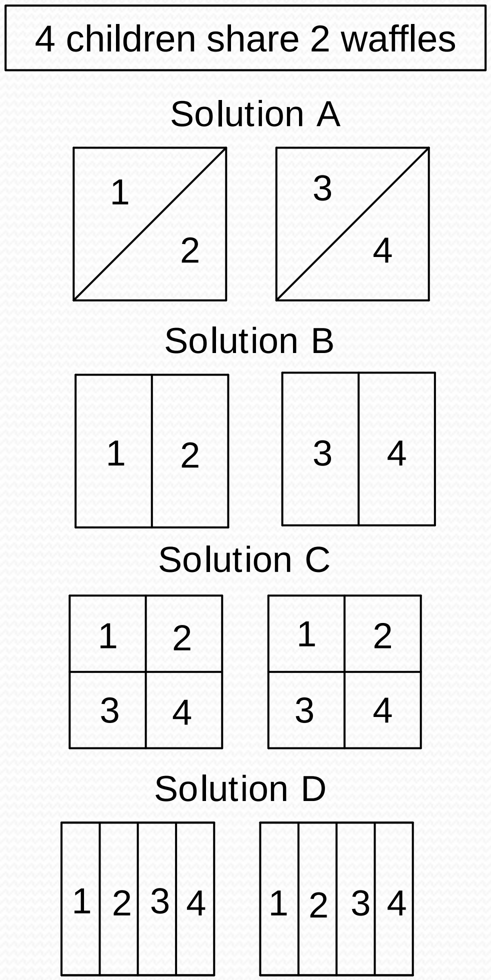
Teaching Opportunities

- Fraction names
  - *How much of a waffle is that piece? (What is the fraction name?) Why?*
- Fraction symbols (meaning of top and bottom number)
  - *For solution C, we decided that each child gets 2/4 of a waffle.*
  - *What do you think the bottom number – the 4 is telling about?*
  - *What do you think the top number – the 2 – is telling about?*



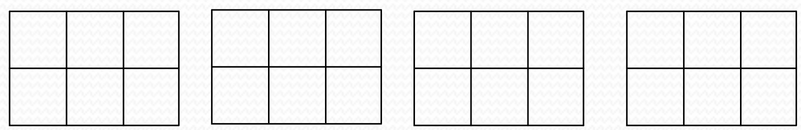
Teaching Opportunities

- Different shaped pieces can be the same fraction if they are the same size in relation to the whole
  - *Are the one-half size pieces for solutions A & B the same size or different sizes? How can we prove it?*
- Equivalent fractions
  - *We got two answers – ½ and 2/4. Are these fractions showing the same amount or different amounts of a waffle? How can we prove it?*



Another Sharing Problem…

Six children want to share these chocolate bars fairly.

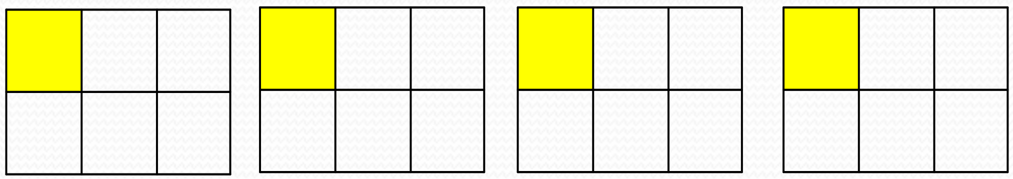


How much chocolate bar should each child get?

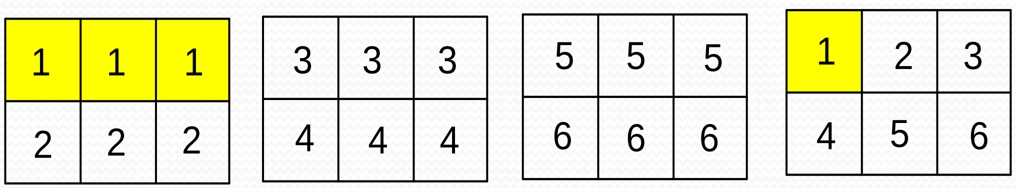
1. In what different ways might children with limited fraction knowledge approach this problem?
2. What might children be able to learn about fractions from discussing and comparing student-generated solutions to this problem?

Six children want to share these chocolate bars fairly. How much chocolate bar should each child get?

Solution A

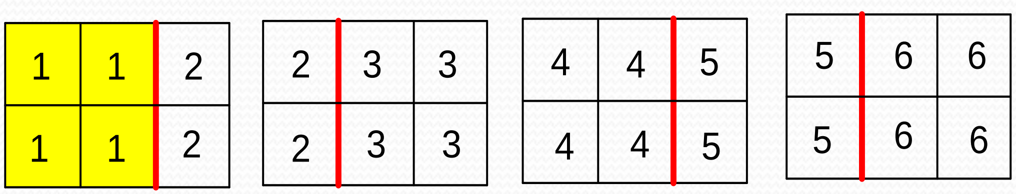


Solution B



How might you use the LEAST number of cuts?

Solution C



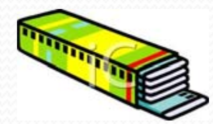
Teaching Opportunities:

- Meaning of 1/6, 3/6, 4/6, ½, and 2/3
- Attention to strategy of divide each object by # of people
- Equivalent fractions ½ & 3/6, 2/3 & 4/6
- Combining fractions informally: What is ½ and 1/6?

More Equal Sharing Problems

Rank by difficulty:

- Six children sharing 4 sticks of gum



- Three children sharing 10 cookies
- Three children sharing 5 brownies

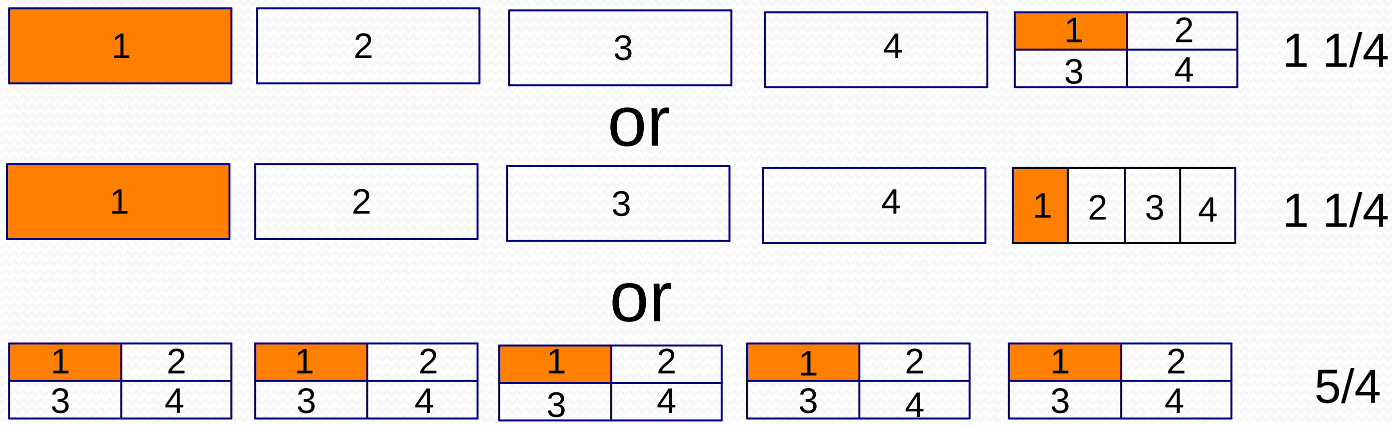
- Eight children sharing 2 pizzas
- Four children sharing 5 granola bars

- Four children sharing 6 candy bars

Level 1

Problems emphasizing unit fractions

Four children sharing 5 granola bars



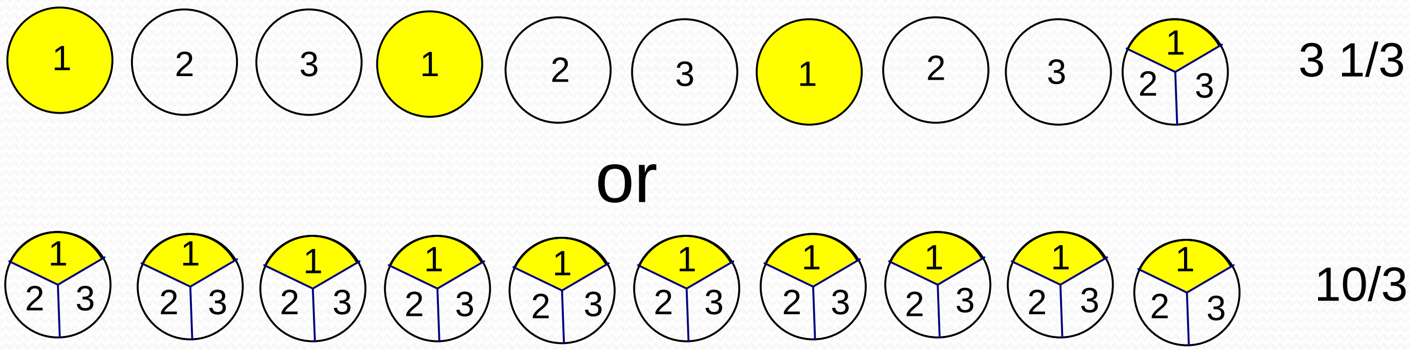
Level 1 problems provide the opportunity to…

- Stress the importance of equal-sized wholes & equal-sized parts
- Have students justify fraction names & symbols for fractional parts
- Explore different ways to represent the same fraction in a given region & justify how different representations are equivalent
- Introduce mixed numbers vs. improper fractions

Level 1

Problems emphasizing unit fractions

Three children sharing 10 cookies

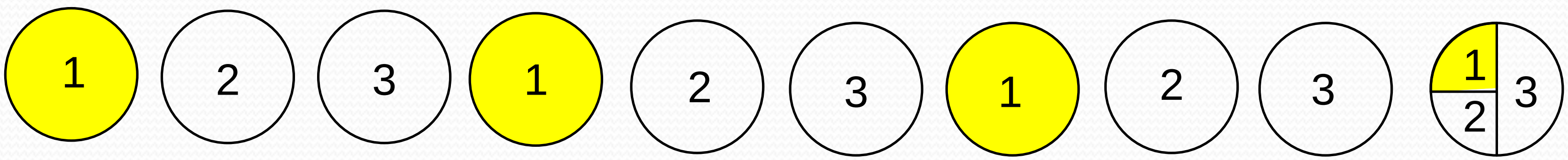


Level 1 problems also provide the opportunity to…

- Have students explore - as problem solvers - how to draw *difficult to draw fractions,* such as thirds and sixths

Confronting Misconceptions through Discussion of Student Errors

A common (flawed) solution
to 3 children sharing 10 cookies:



Key Questions:

• *In what ways is this solution correct? Where does it get off track?*

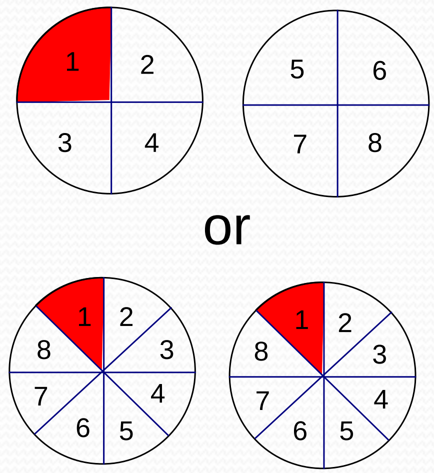
• *With the last cookie, what fraction of a cookie is each child getting (the way it is drawn now)?*

• *How can we revise this solution so that it is correct?*

Level 2

Problems Involving Non-unit Fractions for Which a “Cut in Half” Strategy Works

8 children sharing 2 pizzas



1/4

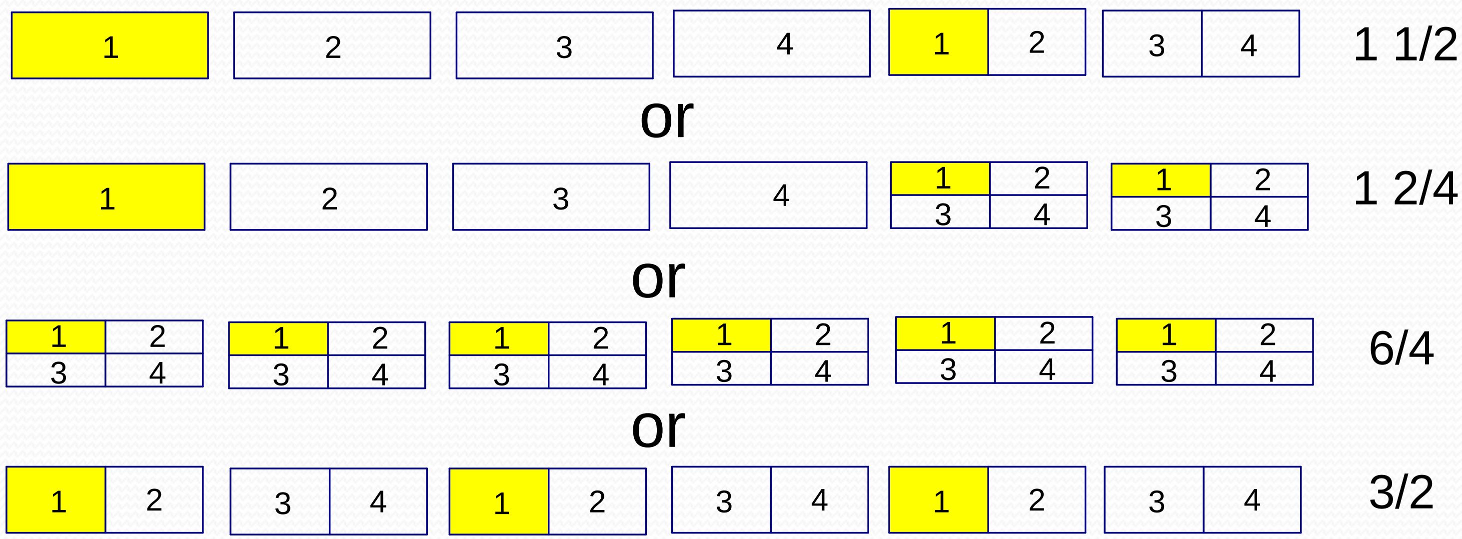
2/8

A typical strategy used by many children is to “cut in half” until that strategy does not work. Problems that can be solved utilizing the “cut in half” strategy tend to be easier than problems for which this strategy does not work.

Level 2

Problems Involving Non-unit Fractions for Which a "Cut in Half" Strategy Works

4 children sharing 6 candy bars

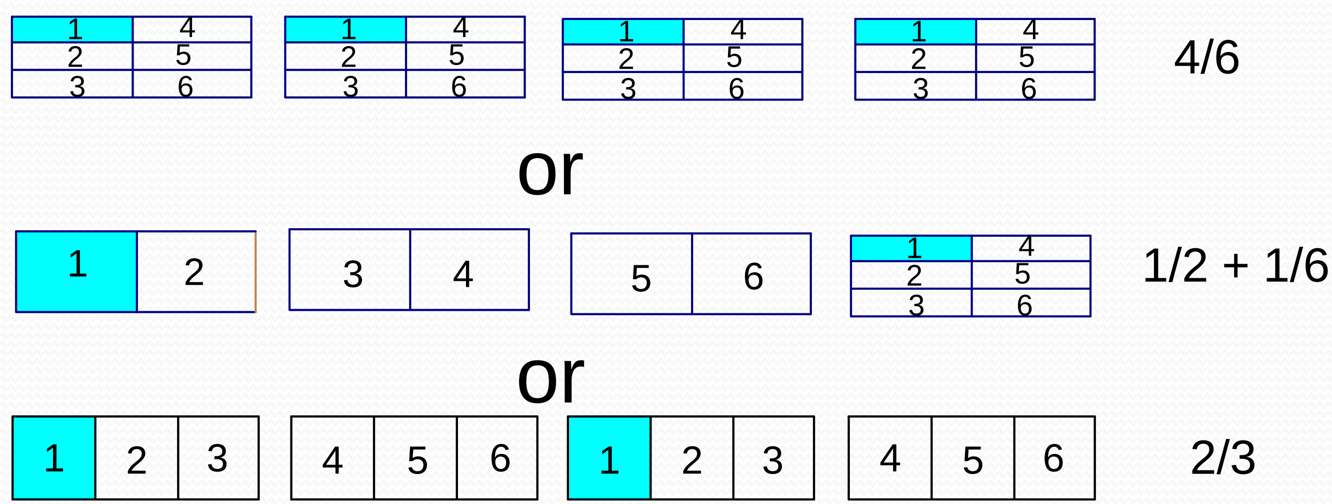


Level 2 problems provide the added opportunity to…
- Explore equivalency relationships involving non-unit fractions with fractions that are easy to draw (halves, fourths, eighths)

Level 3

Problems with potential to yield different-sized fractional parts that need to be combined

Six children sharing 4 sticks of gum

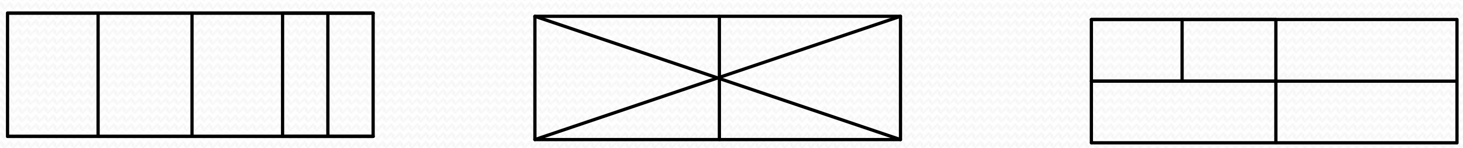


Key Questions:

- Is the combination of ½ of a piece of gum and 1/6 of a piece of gum the same as 4/6 of a piece of gum? How can we know?
- Is 4/6 of a piece of gum the same as 2/3? How can we prove it?

Confronting Misconceptions through Discussion of Student Errors

A common difficulty students have with the previous problem is drawing "sixths" correctly:



Key Questions:

• *How is this representation of a candy bar divided in sixths correct and incorrect?*

• *What fraction of a candy bar is each part (as drawn)?*

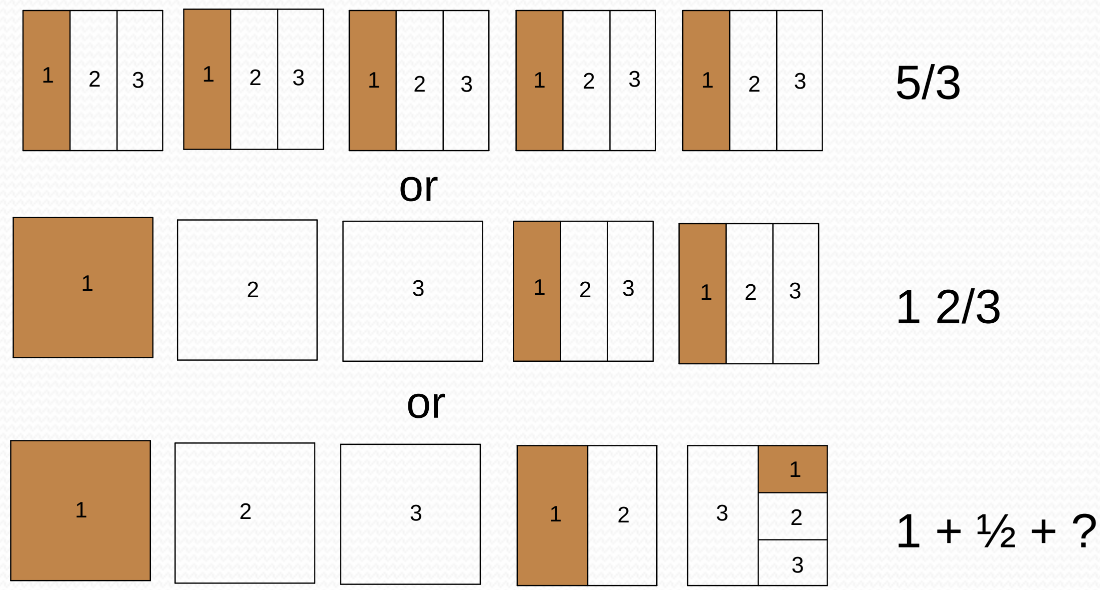
• *How can we make equal sixths if we start by cutting the candy bar in half?*

• *How can we make equal sixths if we start by cutting the candy bar in thirds?*

Level 3

Problems with potential to yield different-sized fractional parts that need to be combined

Three children sharing 5 brownies



Key Questions:

- *For the third solution, what fraction of a brownie is that little piece?*
- *(After establishing that the little piece is 1/6), how much of a whole brownie is ½ and 1/6 combined?*
- *How can we prove that 5/3 is the same as 1 2/3 and 1 4/6?*

Summing Up:

Why Introduce Fractions with Equal-sharing Tasks?

- Students can solve with minimal fraction knowledge
- Fraction knowledge can be built through discussion of students' varied solutions
  - How to name fractions
  - Meaning of fraction symbols (top and bottom numbers)
  - How fractions can be represented in different ways
  - Equivalent fractions
  - Improper fractions and mixed numbers
- Student drawings reveal common misconceptions about fractions, allowing these misconceptions to be confronted

Beyond Equal Sharing Tasks

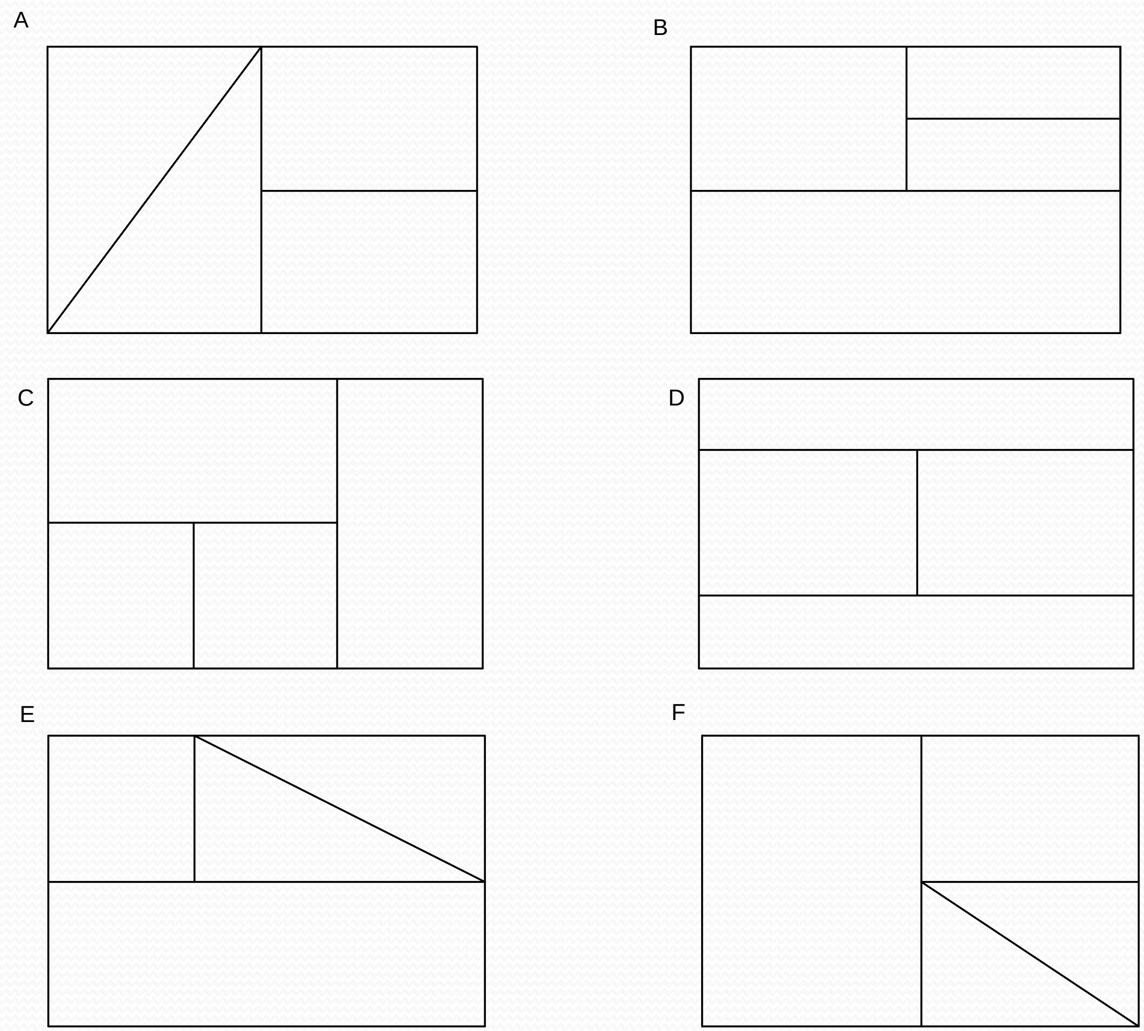
Exploring Part-Whole Relationships
through Multiple Fraction Models

Three Fraction Models

- Area Model, focus on part of a region
  - Circles (pizzas, pies, cookies)
  - Rectangles (candy bars, brownies)
  - Other shapes
- Set Model, focus on part of a set of objects
  - Bags of candy
  - Groups of people
- Linear Model, focus on part of a length or distance
  - Number Line
  - Ruler
  - Licorice, Rope, a Race Track
  - Fraction Strip or Bar

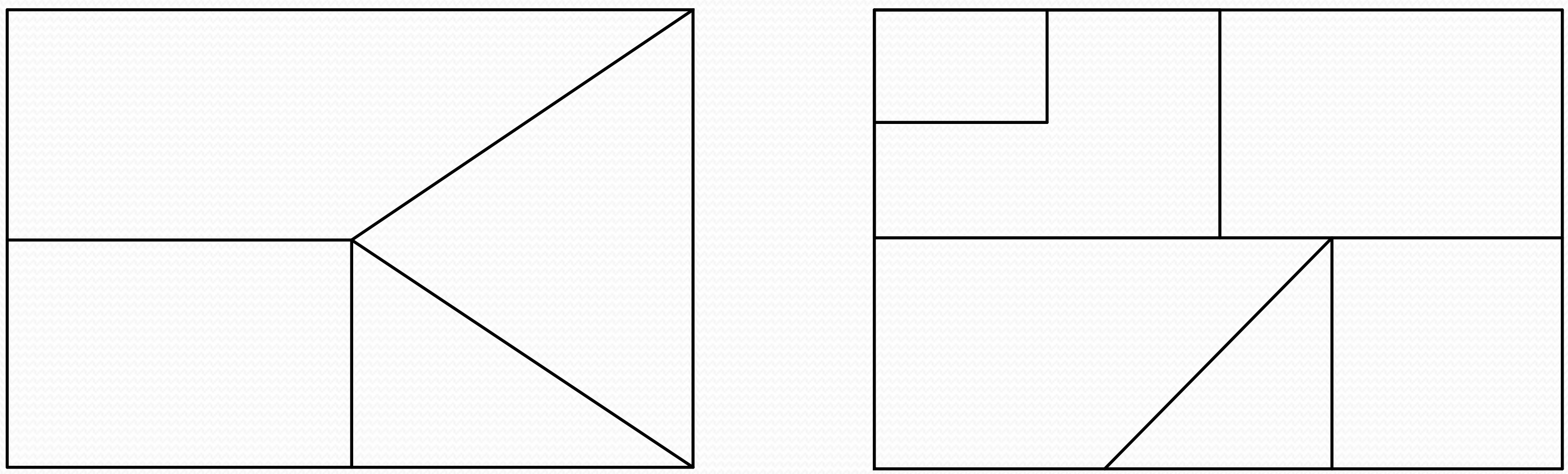
One Cake, Four Pieces

Each cake is cut into four pieces, but some of the cakes are not divided fairly into fourths. Using your knowledge of fractions, your task is to figure out how much of a whole cake each piece is. Be ready to explain your mathematical thinking to the class.



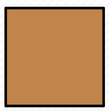
Challenge Cakes

What fraction of a cake is each piece? Be ready to justify your solutions.



Start with a Part

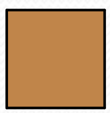
This is one-sixth of a candy bar.



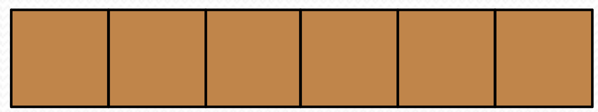
Draw a whole candy bar.

Start with a Part

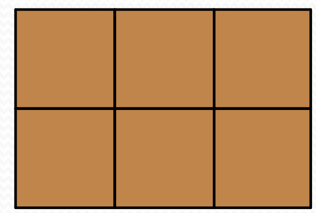
This is one-sixth of a candy bar.



Draw a whole candy bar.



or



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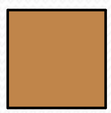
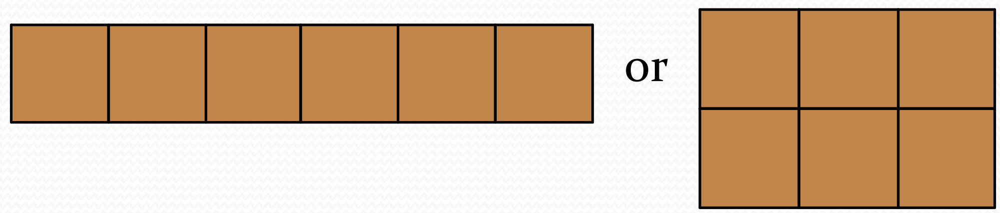
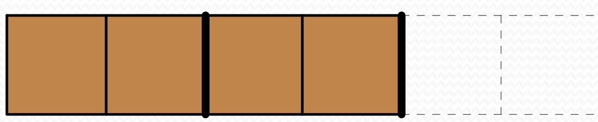


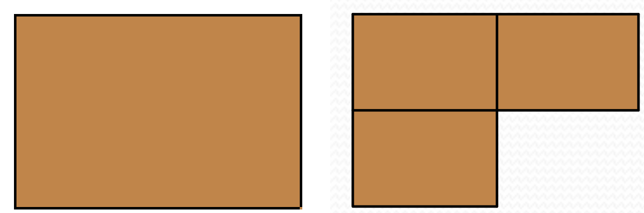
Draw a whole candy bar.



Draw two-thirds of a candy bar



Draw 1 ¾ candy bars.



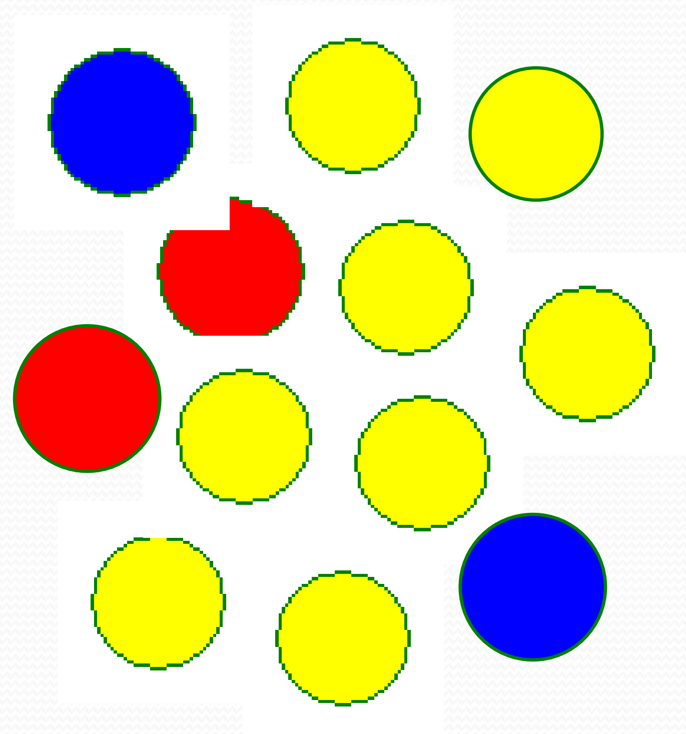
Fraction of a Whole (Set Model)

These are Emma's marbles.

What fraction of Emma's marbles are blue? red? yellow?

Can you name the fraction for each color in more than one way?

Explain how you know.



Fractions of a Whole (Set Model)

Jeremy has a bag of 24 candies. He has promised to give ½ of the candies in the bag to Sam. He has promised to give 1/3 of the candies in the bag to Joanne. If he keeps his promises, how many candies will he give each friend? What fraction of the bag of candies will he have left?

Our Class is a Set!

What fraction of our class brought a lunch today? What fraction of our class is eating a school lunch?

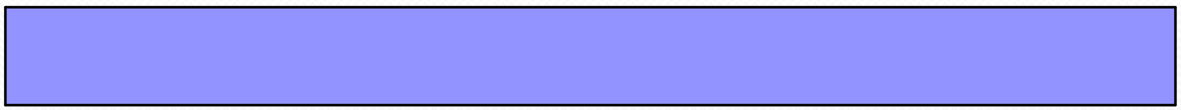
What fraction of our class is wearing sneakers today?

What fraction of our class likes cheese pizza the best?

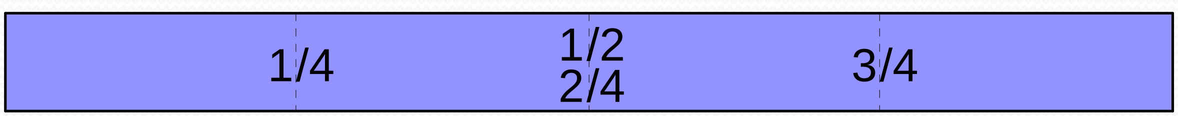
What common fractions are these fractions close to or the same as?
How do you know?

Folding Paper Strips (Linear Model)

Imagine this paper strip is a race track for bugs. One end is the starting line and the other end is the finish line.



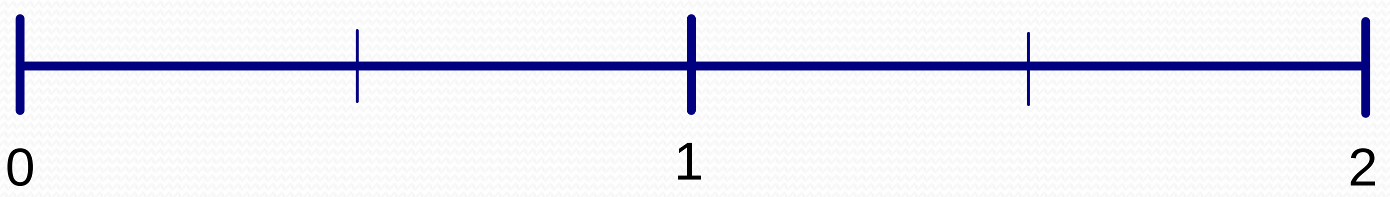
1. Estimate the halfway point in the race.
2. How can we use paper folding to identify the exact point that is ½ of the strip?
3. Estimate where a bug will be when it has gone ¼, 2/4, & ¾ of the race. Then use paper folding to find the exact points that are ¼, 2/4, & ¾ of the strip.



Locating Points on a Number Line

Estimate the location of the following fractions on the number line
Be ready to justify your estimates.

1 1/2   3/4   1/8   4/8   1 3/4   4/2



The Measuring Cup Challenge

Provide students with an 8-ounce cup with a strip of masking tape up the side and the 1/3 cup measure marked. Also provide an identical unmarked cup and water.

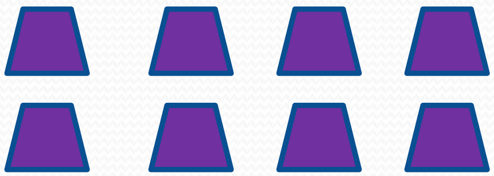
Challenge students to use water pouring and their knowledge of fractions to locate the following measures on the cup:

2/3, 1, ½, ¼, ¾, 1/8, 3/8, 5/8, 7/8

More Start with a Part

Set Model

This is 2/3 of a bag of candies:



How many candies are in a whole bag?

Linear Model

Erin's mom gave her a long string of strawberry licorice this morning. Erin ate some and has only 3/10 of the original licorice string. This is the piece of the licorice string Erin has left:

Draw the piece of licorice Erin had to start with.

Summing Up:

Why Explore Part-Whole Fraction Tasks with Multiple Models?

- Multiple models encourage students to think about wholes and parts in different ways
  - Area Model: Part of a region
  - Set Model: Part of a set of objects
  - Linear Model: Part of a length or distance
- Part-whole tasks provide opportunities to solidify basic fraction ideas by applying them to new contexts
- Part-whole tasks emphasize relationships among different fractions, particularly fraction "families" (i.e., halves, fourths, eighths) and relationships to benchmarks (i.e., 0, ½, and 1)

Comparing Fractions with Reasoning Strategies

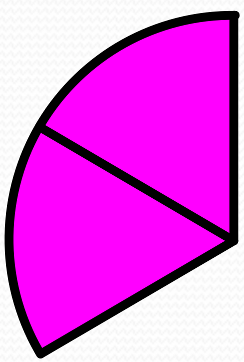
Reasoning Strategies for Comparing (and Ordering) Fractions

Strategy Types	**Sample Reasoning**
Common Numerator *# of parts is the same, focus on size of parts*	3/7 is greater than 3/8 because there are 3 "parts" in each fraction, and seventh-size pieces are bigger than eighth-size pieces.
Common Denominator *Size of parts are the same, focus of # of parts*	3/5 is less than 4/5 because both fractions have fifth-size pieces and 3/5 has fewer "pieces."
Relationships to benchmark fractions (*particularly ½ and 1*)	5/12 is less than 3/5 because 5/12 is less than ½ and 3/5 is a little more than ½.
Distance from benchmark fractions (*particularly ½ and 1*)	11/12 is greater than 9/10 because both fractions have one "piece" missing. The twelfth-size piece missing from 11/12 is smaller than the tenth-size piece missing from 9/10, so 11/12 is closer to 1.

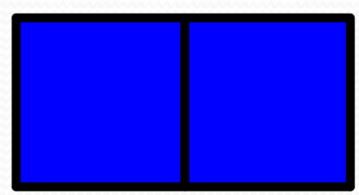
Fraction Comparison Strategies: Getting Started

- Start with FRACTION CIRCLE manipulatives!
- Fraction circles provide a single image of each fraction, which facilitates comparison through mental imagery.

Circle Model:
Always 2/6

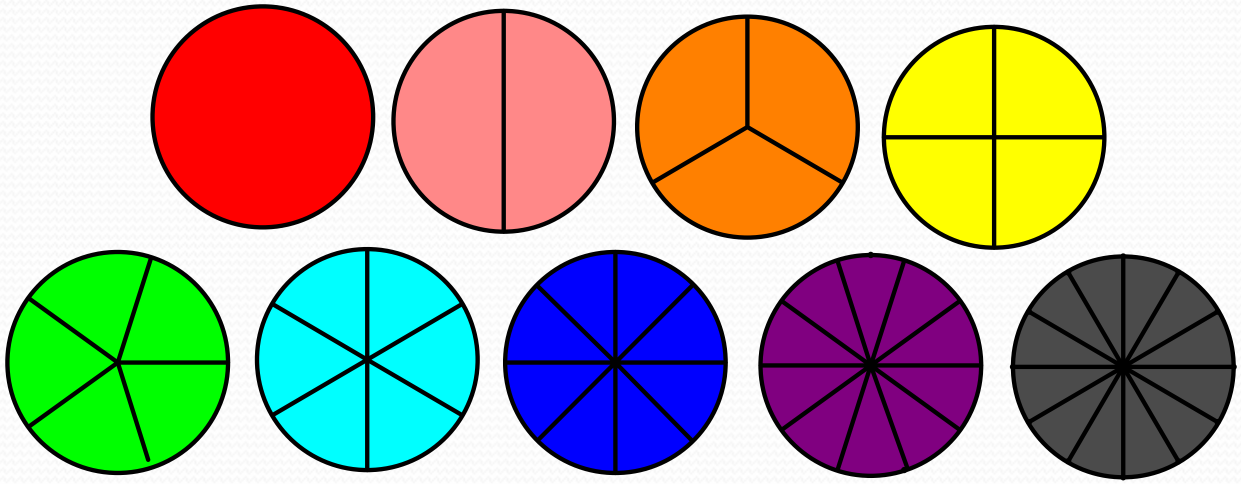


Bar or Rectangle Model:
Could be 2/6 or 2/4 or 2/2...
no way to know without the whole



Get to Know Fraction Circle Kits

- Free Exploration
  - Student observations: Names of pieces, equivalent fractions, & unequal parts that combine to make a whole or another part
- Guided Exploration
  - What fraction of a circle do 2 blues represent?
  - What pieces can you use to build fractions equivalent to 1/3?



Guide Students to Compare Fractions in Context: Pizza Party Problems

For each problem…

1. Use mental imagery and reasoning to make a **conjecture**.
2. **Verify** your solution with the fraction circles.
3. **Discuss** your thinking.

Try It!

CONJECTURE – VERIFY – DISCUSS

At a special pizza luncheon, Eduardo ate 1/6 of a large pepperoni pizza. David ate 1/10 of a large cheese pizza. Which boy ate more pizza? How do you know?

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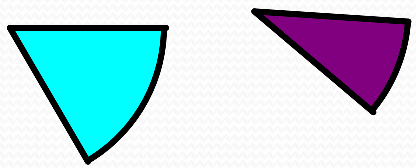


Reasoning Strategy: Common Numerator

Both boys have 1 piece of pizza. But Eduardo's sixth-size piece is bigger because it comes from a pizza cut in six pieces. David's piece is from a pizza cut in ten pieces, which would make smaller pieces. 1/6 is more than 1/10.

Key Questions:

- *(CONFRONTING MISCONCEPTION) Could someone who conjectured that 1/10 would be more please share why you thought that? Why did that turn out to be incorrect?*
- *How could you figure out that 1/6 of a pizza is greater than 1/10 without looking at the fraction circles?*

Follow-up Problems:
Common Numerator Strategy

CONJECTURE – VERIFY – DISCUSS

Focus on Other Unit Fractions:

- Who would have eaten more if Eduardo ate 1/10 of a large pizza and David ate 1/12 of a large pizza?
- Which is the greater fraction: 1/20 or 1/18?

Extend to Non-unit Fractions:

- Cristina ate 3/10 of a large cheese pizza. Maria ate 3/8 of a large mushroom pizza. Which girl ate more pizza?
- Which is the greater fraction: 13/15 or 13/20

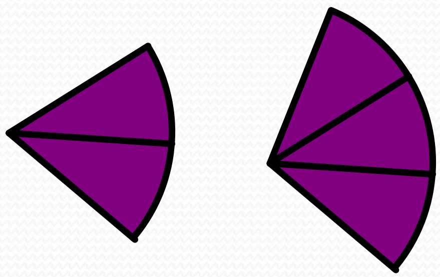
Key Understanding to Emphasize:
The reason one can focus on the size of the parts (denominator) is because the fractions being compared have the same number of parts (numerator).

How might students reason about this problem?

CONJECTURE – VERIFY – DISCUSS

Devyn ate 2/10 of a large cheese pizza and Hannah ate 3/10 of a large cheese pizza. Who ate more cheese pizza? How do you know?

Devyn ate 2/10 of a large cheese pizza and Hannah ate 3/10 of a large cheese pizza. Who ate more cheese pizza? How do you know?



(Correct) Reasoning Strategy: Common Denominator

Both girls had slices from a pizza cut in tenths, so the pieces are all the same size. Hannah ate more because she had 3 pieces and Devyn only had 2. 3/10 is more than 2/10.

Common Misconception:

Some students identify 2/10 as the larger fraction because they are mistakenly equating smaller numerators with larger pieces (based on experiences with common numerator problems). To overcome this misconception, students must mentally coordinate the meaning of the values of numerators (the # of pieces) and denominators (the size of pieces).

Recommendation:

Juxtapose problems that can be solved using the common numerator and common denominator strategies, and press students to explain their reasoning… to talk explicitly about the number of parts and the size of parts.

Key Questions:

How do the number of parts in these fractions compare?

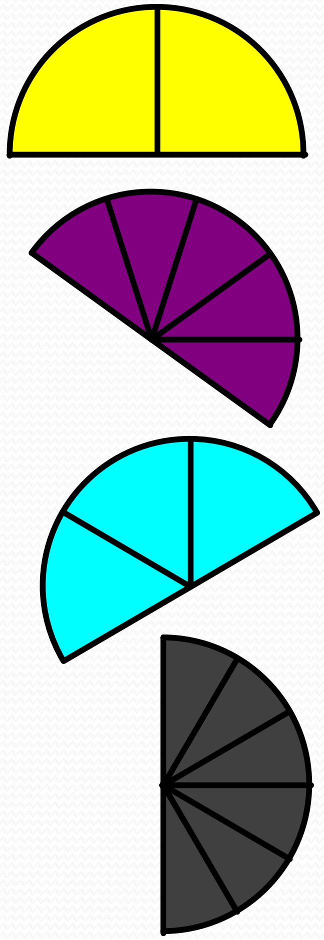
How do the size of the parts in these fractions compare?

Prepare to Compare to the Benchmark ½:

Explore Fractions Equivalent to 1/2

Model one-half of a circle with as many fraction circles as you can and look for patterns…

- Which kinds of pieces can make exactly half of a circle? Which cannot?
- What are some names of fractions that are equivalent to ½ that you can build with the pieces?
- Extend the pattern: What other fractions – ones that we don't have pieces for - are equivalent to ½? How do you know?



Problems to Evoke Fraction Comparison Strategy: Compare to the Benchmarks (½, 1)

CONJECTURE – VERIFY – DISCUSS

Compare to ½:

- Sophia ate 4/10 of a large cheese pizza. Erick ate ½ of a large mushroom pizza. Who ate more pizza? How do you know?
- Which is more? 1/2 or 7/12, 4/5 or 1/2, 1/2 or 9/20

Compare to a fraction equivalent to ½:

- Alan ate of a 6/12 of a large pepperoni pizza, and Jason ate 5/8 of a large cheese pizza. Who ate more pizza? How do you know?
- Which is more? 4/8 or 5/12, 7/12 or 5/10, 10/20 or 5/8

Problems to Evoke Fraction Comparison Strategy:

Compare to the Benchmarks (½, 1)

CONJECTURE – VERIFY – DISCUSS

Compare fractions on either side of ½:

- Sarah ate 4/10 of a large mushroom pizza, and Carson ate 3/4 of a large sausage pizza. Who ate more pizza? How do you know?
- Which is more? 5/12 or 4/6, 7/10 or 3/8, 10/16 or 11/25

Compare fractions on either side of 1:

- Michael and Will's families went to a diner to enjoy some apple pie. Michael's family ate 9/10 of a pie. Will's family ate 8/5 pies. Which family ate more pie? How do you know?
- Which is more? 9/6 or 11/12, 7/8 or 4/3, 23/20 or 12/15

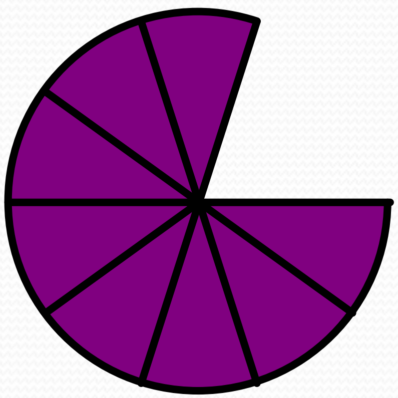
Challenge!

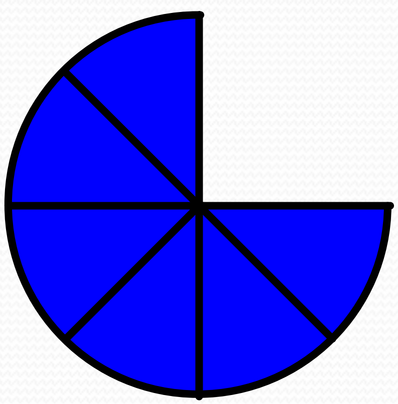
CONJECTURE – VERIFY – DISCUSS

1. At the end of the pizza party, 8/10 of a cheese pizza and 6/8 of a pepperoni pizza are left over. Which kind of pizza has the most left over? How do you know?

2. At the end of the pizza party, 5/8 of a mushroom pizza and 4/6 of a sausage pizza are left over. Which kind of pizza has the most left over? How do you know?

At the end of the pizza party, 8/10 of a cheese pizza and 6/8 of a pepperoni pizza are left over. Which kind of pizza has the most left over? How do you know?

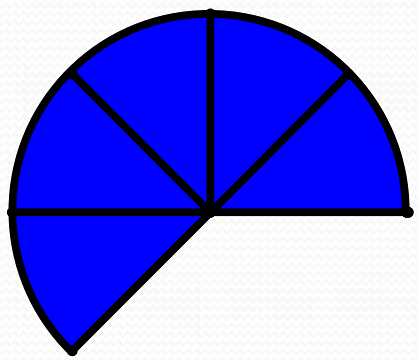


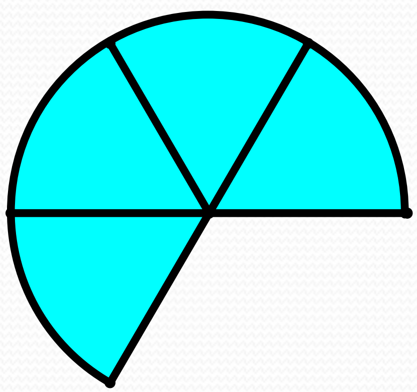


Reasoning Strategy:
Focus on Distance from the Benchmark 1

Both pizzas have 2 pieces missing. There is 2/10 missing from the cheese pizza and 2/8 missing from the pepperoni pizza. I know that 2/10 is smaller than 2/8, so the cheese pizza with 8/10 is closer to being a whole pizza. There is more cheese pizza left over.

At the end of the pizza party, 5/8 of a mushroom pizza and 4/6 of a sausage pizza are left over. Which kind of pizza has the most left over? How do you know?



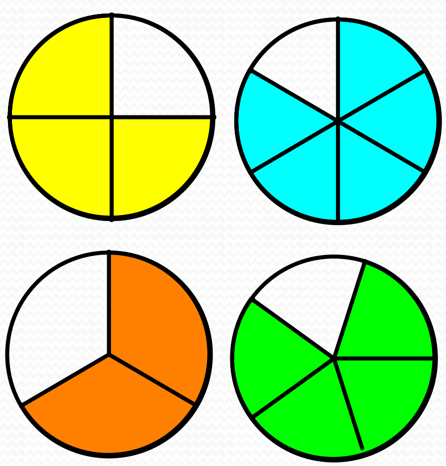


Reasoning Strategy:
Focus on Distance from Benchmark ½

The leftovers for both pizzas are exactly one piece more than ½. The mushroom pizza has 1/8 more than ½, and the sausage pizza has 1/6 more than ½. Since 1/6 is greater than 1/8, then 4/6 is greater than 5/8. The sausage pizza has the most left over.

Problem to Explore Fraction Comparison Strategy: Distance from the Benchmark 1

Imagine a pie case at the bakery with pies cut into different equal-sized parts, just like your fraction circles…one in halves, one in thirds, one in fourths, etc. Now imagine that a customer buys one piece of each kind of pie.



Conjecture – Verify – Discuss:

- What fraction of each pie is left? How do you know?
- Which of the pies has the most remaining? Which has the least remaining? What patterns do you notice?
- How can we compare fractions that have "one part missing" from the whole? What if there were two parts missing?

Foster Connections:

Explore Fraction Comparison in Non-Circle Contexts

Linear Contexts:

- In art class, Ella used 5/8 of a tube of red paint. Jeremy used 5/9 of a tube of red paint. Which student used more red paint? How do you know?
- Angela and Kendra are wrapping presents. Angela used 3 2/3 feet of tape. Kendra used 3 5/12 feet of tape. Which girl used more tape. How do you know?

Non-Circle Area Context:

- Eddie's used 2 3/4 large sheets of paper to make his art project. Sam's art project used 2 7/8 large sheets. Which boy used more paper on their art project? How do you know?

Summing Up:

Why Emphasize use of Reasoning Strategies for Fraction Comparison (and Ordering) Tasks?

- Alternate strategies have significant limitations
  - Pictures are difficult to draw accurately
  - Converting to common denominators is often poorly understood and can be cumbersome with uncommon fractions
- Reasoning strategies encourage children to mentally coordinate conceptual information about the meaning of numerators and denominators
- Reasoning strategies develop students' abilities to compare fractions while also developing their *fraction number sense*, meaning a sense of how fractions relate to other fractions

A "Formula" for Fraction Success

Equal-sharing Tasks to Introduce Fractions

+

Part-Whole Tasks with Multiple Models to Extend Concepts

+

Emphasis on Reasoning Strategies for Fraction Comparison Tasks

→

A Solid Foundation of Fraction Concepts

Recommended Resources

Books:

- *Teaching Student Centered Mathematics: Grades 3 – 5* or *Elementary and Middle School Mathematics: Teaching Developmentally.* Both titles are by **John A. Van de Walle** (and others, depending on title and edition). See chapter on "Developing Fraction Concepts."

- *Lessons for Introducing Fractions: Grades 4-5* and *Lessons for Extending Fractions: Grade 5*. Both titles are by **Marilyn Burns** and part of the Teaching Arithmetic series of teacher resource books. I believe these lessons are appropriate for grades 3 - 5.

Recommended Resources

Web-based Resource:

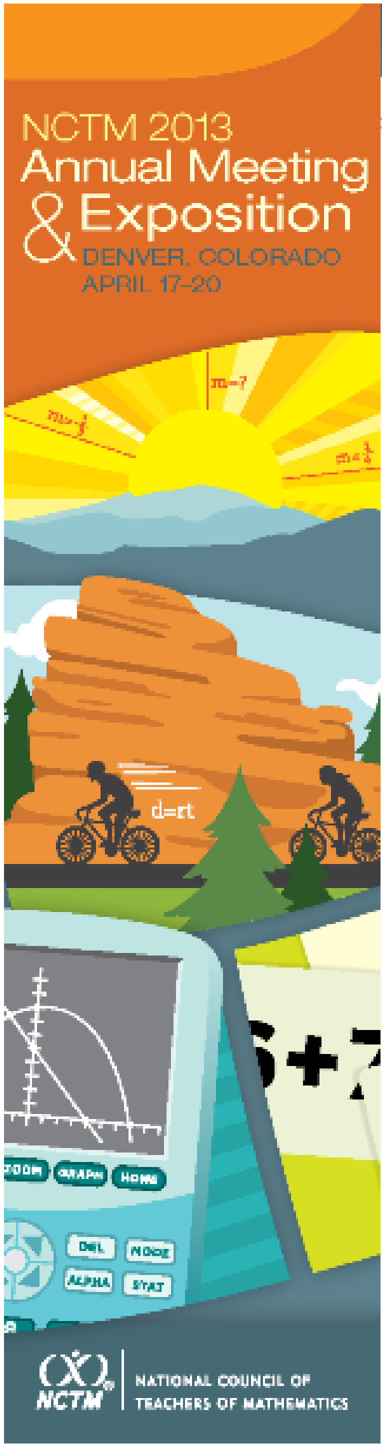
- *Rational Number Project: Initial Fraction Ideas*. Authors: Kathleen Cramer, Merlyn Behr, Thomas Post, & Richard Lesh. http://www.cehd.umn.edu/rationalnumberproject/rnp1-09.html

  The Rational Number Project is an on-going NSF funded research project that develops curriculum on rational number concepts. The website referenced provides materials that can be downloaded for free, including templates for fraction circle manipulatives.

Recommended Resources

Articles:

- Bray, W. S. (2013). How to leverage the potential of mathematical errors. *Teaching Children Mathematics*, *19*(7), 424 - 431.
- Bray, W. S. & Abreu-Sanchez, L. (2010). Comparing fractions with number sense: From models to reasoning. *Teaching Children Mathematics, 17*(2), 90-97.
- Empson, S. B. (1995). Using sharing situations to help children learn fractions. *Teaching Children Mathematics,* 2(2), 110-114.
- Empson, S. B. (2001). Equal sharing and the roots of fraction equivalence. *Teaching Children Mathematics, 7*(7), 421-425.



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