Reasoning and Proof on the SAT

NCTM Annual Meeting
and Exposition

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SAT Mathematics Content Areas

- Number and Operations (20–25%)
- Algebra and Functions (35–40%)
- Geometry and Measurement (25–30%)
- Data Analysis, Statistics, and Probability (10–15%)

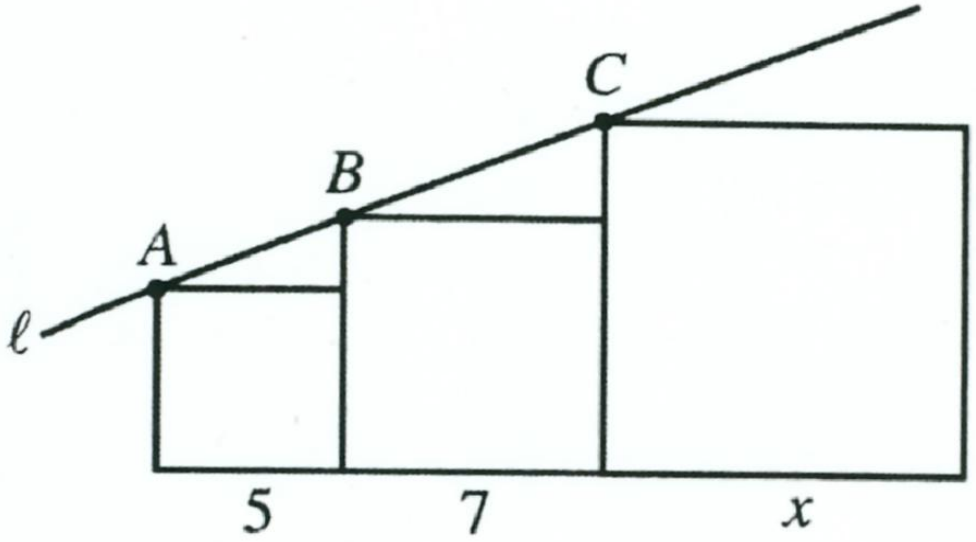
Multiple Choice

If the function f is defined by $f(x) = 3x + 4$, then $2f(x) + 4 =$

(A) $5x + 4$
(B) $5x + 8$
(C) $6x + 4$
(D) $6x + 8$
(E) $6x + 12$

Key = E

Student-Produced Response

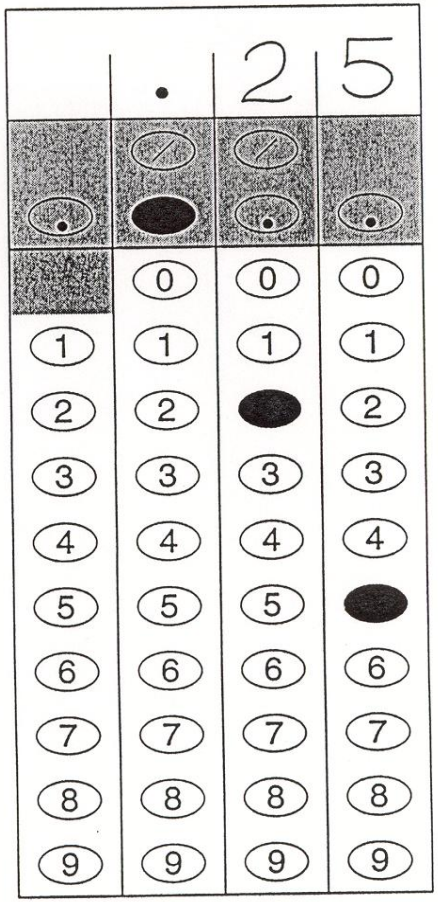


The figure above shows three squares with sides of length 5, 7, and x, respectively. If A, B, and C lie on line ℓ, what is the value of x ?

Key = 9.8, 49/5

SPR Answer Grid

Student-Produced Responses



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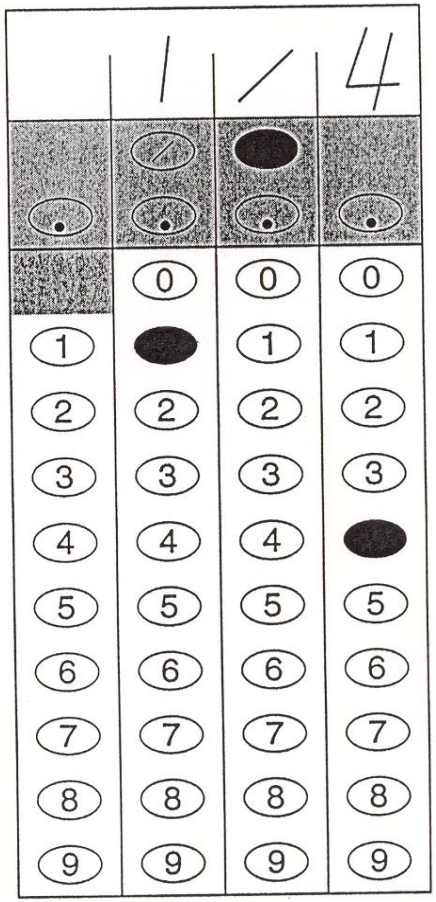


Standards for Mathematical Practice of the CCSSM

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics

Standards for Mathematical Practice of the CCSSM

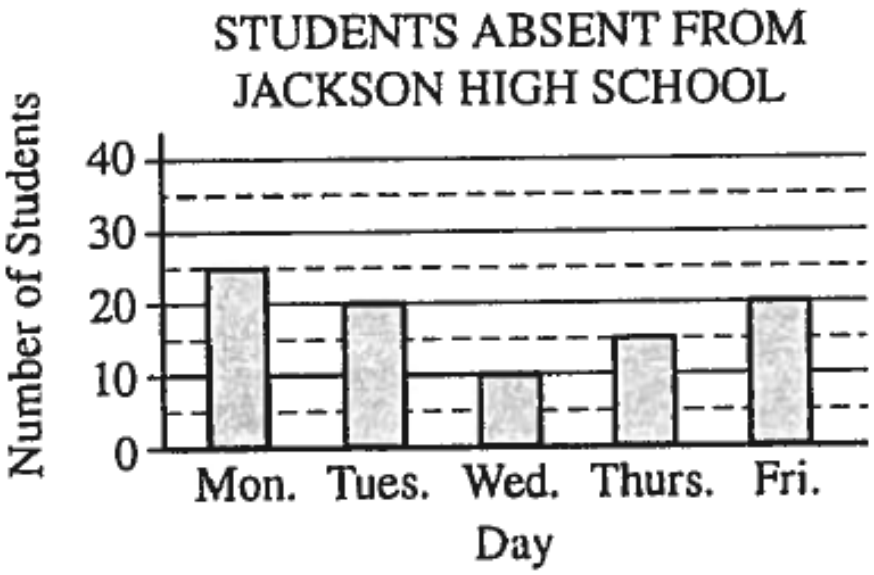
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Make sense of problems and persevere in solving them

Make sense of problems and persevere in solving them

Mathematically proficient students:

- **Explain the meaning of a problem**
- **Look for entry points to a solution**
- **Plan a pathway instead of simply "jumping in"**
- **Consider analogous problems and try special cases and simpler forms of the problem**
- **Monitor and evaluate their progress**
- **Change course if necessary**



The bar graph above shows the number of students who were absent from Jackson High School last week. Of those students, 8 were absent 2 days each, 1 was absent 3 days, and no students were absent more than 3 days. If 5 percent of the students in the school were absent at least 1 day last week, how many students are enrolled in Jackson High School?

Key = 1600 ; P+ = 0.60

What is one possible number of gallons of oil that can be added to 40 gallons of gasoline so that the ratio of oil to gasoline in the resulting mixture is at least 1 to 8 but not more than 1 to 5 ?

Key: $5 \leq x \leq 8$; P+ = 0.57

Tank A and tank B each have a capacity of 1000 gallons. There are 900 gallons of water in tank A and 200 gallons of water in tank B. Water is to be pumped out of tank A into tank B at the rate of 2.5 gallons per minute. How many minutes after pumping begins will tank A contain 100 gallons less than tank B ?

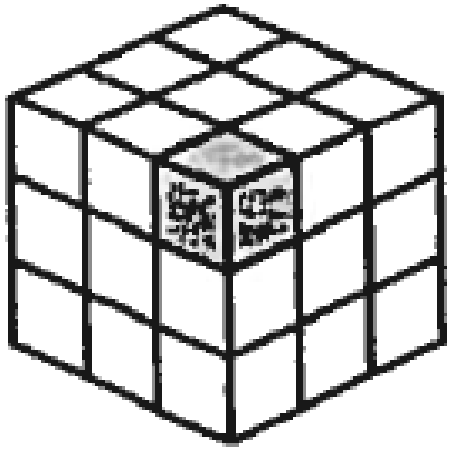
Key: 160 ; P+ = 0.21

Reason abstractly and quantitatively

Reason abstractly and quantitatively

Mathematically proficient students:

- **Make sense of quantities in their relationships in problem situations**
- **Can *decontextualize*: abstract a situation, represent it symbolically, manipulate representing symbols**
- **Can *contextualize*: pause as needed to probe into the referents of the symbols involved**
- **Create a coherent representation of the problem at hand**



The cube shown in the figure above is made from 27 small cubes, each with an edge of length 1. If the shaded cube is removed, what will be the surface area of the remaining solid?

Key: 54 ; P+ = 0.14

The top of a round table with a circumference of 8π feet is to be completely covered by 4 square pieces of cloth, each with sides of length 4 feet. If none of the pieces overlap, what is the total area of cloth, in square feet, that will hang over the edge of the table top?

(A) $32\pi - 8$
(B) $16\pi - 32$
(C) $64 - 8\pi$
(D) $64 - 16\pi$
(E) $16 - 4\pi$

Key: D ; P+ = 0.23

The lengths of the sides of a triangle are $3x$, $5x$, and $7x$, and the perimeter of the triangle is 45. How much longer is the longest side of the triangle than the shortest side?

(A) 3
(B) 4
(C) 8
(D) 9
(E) 12

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Key: E ; P+ = 0.72

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Construct viable arguments and critique the reasoning of others

Construct viable arguments and critique the reasoning of others

Mathematically proficient students:

- **Use stated assumptions, definitions, and previously established results in constructing arguments**
- **Break situations into cases**
- **Recognize and use counterexamples**
- **Justify conclusions and respond to the arguments of others**
- **Distinguish correct from flawed logic**
- **Determine the domain of an argument**

If $|ax - 1| \leq 1$, where a is a positive even integer, which of the following CANNOT be a value of x^2 ?

(A) 0

(B) $\frac{1}{4}$

(C) $\frac{1}{2}$

(D) 1

(E) 4

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Key: E ; P+ = 0.48

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$$\frac{A}{8} + \frac{2}{9} = \frac{B}{72}$$

In the equation above, if A represents a positive integer, which of the following could be the value of B ?

(A) 7
(B) 16
(C) 18
(D) 26
(E) 34

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Key: E ; P+ = 0.34

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If a number is in list X, it is also in list Y.

If the statement above is true, which of the following statements must also be true?

(A) If a number is in list Y, it is also in list X.
(B) If a number is in list Y, it is not in list X.
(C) If a number is not in list Y, it is not in list X.
(D) If a number is not in list Y, it is in list X.
(E) If a number is not in list X, it is not in list Y.

Key: C ; P+ = 0.24

The median age of a group of 15 students is 17 years. If the oldest student in the group is 20 years old, which of the following could be the number of 17 year olds in the group?

I. 3
II. 10
III. 14

(A) I only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II, and III

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Key: E ; P+ = 0.25

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If $3 < a < 7 < b$, which of the following must be greater than 20 ?

(A) a^2
(B) $2b$
(C) ab
(D) $b - a$
(E) $b + a$

Key: C ; P+ = 0.74

If $1 < a < b < 10$ and $x = \dfrac{a \times 10^5}{b \times 10^{-3}}$, which of the following inequalities is true?

(A) $10^0 < x < 10^1$

(B) $10^1 < x < 10^2$

(C) $10^2 < x < 10^3$

(D) $10^7 < x < 10^8$

(E) $10^8 < x < 10^9$

Key: D ; P+ = 0.13

An integer that is a multiple of 12 must also be a multiple of 4 and 8.

Which of the following integers disproves the statement in the box above?

(A) 24
(B) 32
(C) 40
(D) 48
(E) 60

Key: E ; P+ = 0.55

If $xy = x + y$ and $y > 2$, which of the following must be true about x ?

(A) $x = y$

(B) $x = 0$

(C) $x = 1$

(D) $0 < x < 1$

(E) $1 < x < 2$

Key: E ; P+ = 0.31

For all x, let the function f be defined by $f(x) = a(x-h)^2 + k$, where a, h, and k are constants. If a and k are positive, which of the following CANNOT be true?

(A) $f(10) = 1$

(B) $f(0) = -5$

(C) $f(0) = 5$

(D) $f(1) = -h$

(E) $f(-1) = h$

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Key: B ; P+ = 0.23

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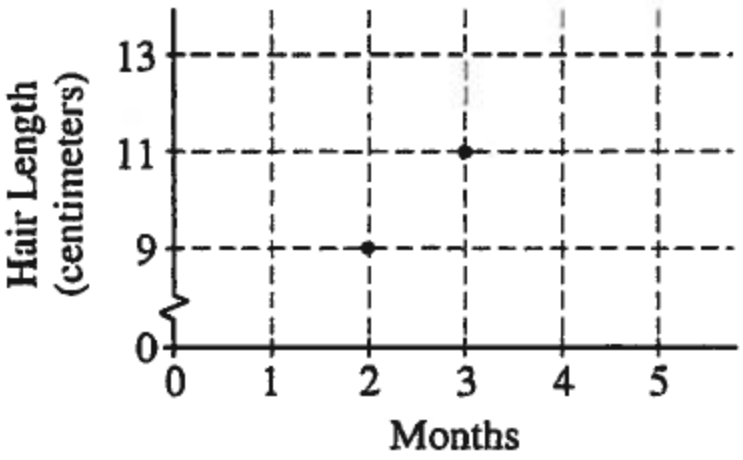
Model with mathematics

Model with mathematics

Mathematically proficient students:

- **Apply the mathematics they know to solve real-world problems**
- **Use geometry to solve a design problem**
- **Use a function to describe how one quantity of interest depends on another**
- **Make assumptions and estimations**
- **Simplify a complicated situation**
- **Apply their results in the original context**



Sarah decided not to cut her hair for six months. The graph above shows the length of her hair 2 months and 3 months after her decision not to cut it. If the length of her hair for the six months is graphed, the points all lie on a line. What was the length of Sarah's hair, in centimeters, the day she decided not to cut it?

(A) 5
(B) 5.5
(C) 6
(D) 6.5
(E) 7

Key: A ; P+ = 0.50

The population of a certain country was 20 million at the end of 1990 and 24 million at the end of 2000. At the end of each year from 1991 to 2000, the population was x percent greater than it had been at the end of the previous year. Which of the following equations must x satisfy?

(A) $(20)(10)\left(\frac{x}{100}\right) = 24$

(B) $20 + \left(\frac{x}{100}\right)^{10} = 24$

(C) $20 + 10\left(\frac{x}{100}\right) = 24$

(D) $20 + 10\left(1 + \frac{x}{100}\right) = 24$

(E) $20\left(1 + \frac{x}{100}\right)^{10} = 24$

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Key: E ; P+ = 0.10

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A colony of bacteria grows so that t days after the start of an experiment, the number of bacteria is $n \cdot 2^{\frac{t}{2}}$, where n is the number of bacteria at the start of the experiment. If there are 10,000 bacteria 6 days after the experiment's start, what is the value of n ?

(A) 1,250
(B) 1,667
(C) 2,500
(D) 3,333
(E) 5,000

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Key: A ; P+ = 0.57

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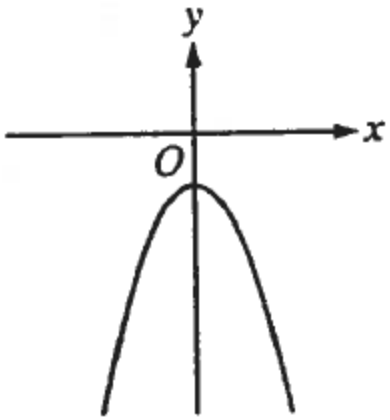
Use appropriate tools strategically

Use appropriate tools strategically

Mathematically proficient students:

- **Consider the appropriate tools when solving a mathematical problem**
- **Use pencil and paper, ruler, calculator**
- **Make sound decisions about when each of these tools might be helpful**
- **Strategically use estimation and other mathematical knowledge**
- **Analyze graphs of functions with graphing calculators**



In the figure above, the parabola is the graph of the function f defined by $f(x) = ax^2 + b$, where a and b are constants. Which of the following could be the value of $a + b$?

I. -3
II. 0
III. 3

(A) I only
(B) II only
(C) III only
(D) I and III
(E) II and III

Key: A ; P+ = 0.34

$$a = x^2$$

$$b = (x + 1)(x - 1)$$

$$c = (x + 1)^2$$

If $x \geq 0$ in the three equations above, what is the ordering of a, b, and c ?

(A) $a < b < c$
(B) $b < a < c$
(C) $b < c < a$
(D) $c < a < b$
(E) $c < b < a$

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Key: B ; P+ = 0.53

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If $ab > 10^5$, then the value of $\dfrac{a + \dfrac{1}{b}}{5a}$ is closest to which of the following?

(A) 0.1
(B) 0.2
(C) 0.3
(D) 0.4
(E) 0.5

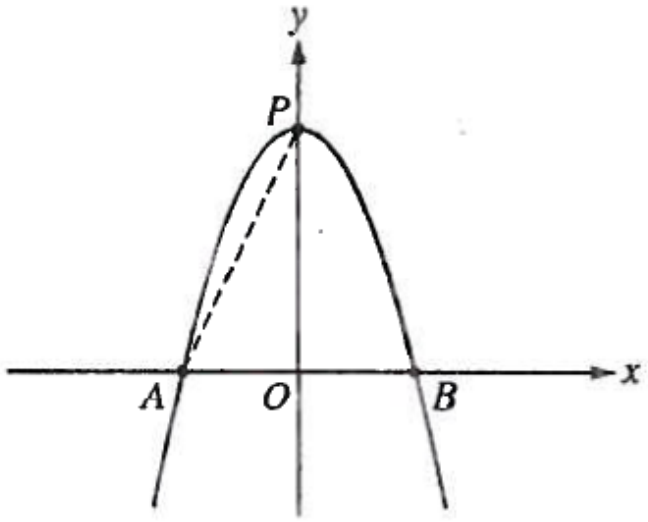
Key: B ; P+ = 0.22

Attend to precision

Attend to precision

Mathematically proficient students:

- **Communicate precisely to others**
- **State the meaning of symbols they choose**
- **Carefully specify units of measure**
- **Label coordinate axes clearly**
- **Calculate accurately and efficiently**
- **Express numerical answers to a degree of precision appropriate to the problem context**



The parabola above is the graph of $y = -x^2 + k$, where k is a constant. If $AB = 10$, what is the slope of $\overline{AP}$?

(A) 2

(B) $\frac{5}{2}$

(C) 5

(D) 10

(E) 20

Key: C ; P+ = 0.21

An integer k is a "half square" if $2k$ is the square of a positive integer. For example, 18 is a half square because $2 \cdot 18 = 36 = 6^2$. What is the smallest half square that is greater than 100 ?

Key: 128 ; P+ = 0.26

An art school offers a painting class if at least 10 students enroll. The school charges a tuition t that is a function of the number of students s who enroll in the class. Tuition, in dollars, for each student is $t(s) = \frac{1000 + 20s}{s - 1}$. What is the total tuition amount, in dollars, that the school should collect if 21 students enroll? (Disregard the dollar sign when gridding your answer.)

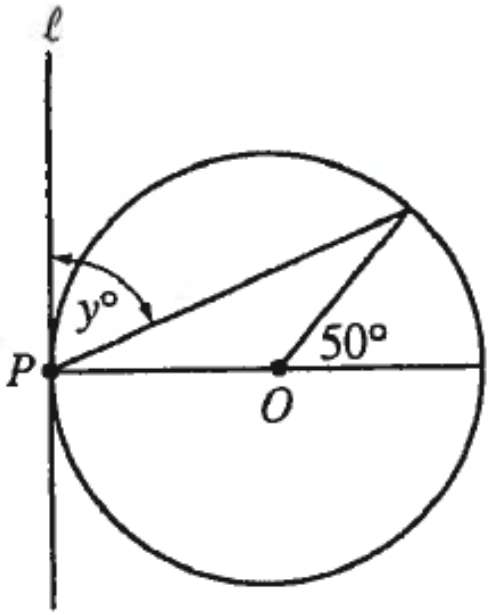
Key: 1491 ; P+ = 0.11

Look for and make use of structure

Look for and make use of structure

Mathematically proficient students:

- **Look closely to discern a pattern or structure**
- **Recognize the significance of an existing line in a geometric figure**
- **Step back for an overview and shift perspective**
- **See complicated things as single objects or as being composed of several objects**



In the figure above, line ℓ is tangent to the circle at point P, and point O is the center of the circle. What is the value of y?

(A) 50
(B) 55
(C) 60
(D) 65
(E) 75

Key: D ; P+ = 0.40

If $u + w = 9$, what is the sum of u, w, and -12 ?

(A) -21

(B) -3

(C) 3

(D) 4

(E) 21

Key: B ; P+ = 0.88

In a 2-digit integer, the digits are represented by K and V. If $10 \cdot K + V = 16$, which of the following equals 61 ?

(A) $10 \cdot V + K$

(B) $\frac{10}{V} + K$

(C) $\frac{10}{K} + V$

(D) $\frac{K \cdot V}{10}$

(E) $\frac{10}{K \cdot V}$

Key: A ; P+ = 0.35

If $k = 11^3 \times 25^3 \times 49^3$, what is the least prime divisor of k?

(A) 3
(B) 5
(C) 7
(D) 9
(E) 11

Key: B ; P+ = 0.29

Look for and express regularity in repeated reasoning

Look for and express regularity in repeated reasoning

Mathematically proficient students:

- **Notice if calculations are repeated and draw appropriate conclusions from this**
- **Look for general methods and for shortcuts**
- **Maintain oversight of the process as they work to solve a problem, while attending to the details**
- **Continually evaluate the reasonableness of their intermediate results**

3, 8, 13, 18, …

The first term in the sequence above is 3, and each term after the first term is 5 more than the preceding term. Which of the following numbers is a term in the sequence?

(A) 998
(B) 999
(C) 1,000
(D) 1,001
(E) 1,002

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Key: A ; P+ = 0.79

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Your Questions ????

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