

Reasoning and Sense Making with CAS: Generalizing Patterns in Algebra

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Common Core State Standards for Mathematics Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision.
7. Look for and make use of structure.
8. Look for express regularity in repeated reasoning.

Standards for Mathematical Practice #5: Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, *a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software*. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Mathematics | High School—Geometry

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Mathematics | High School—Number and Quantity

Numbers and Number Systems. ...

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. They can be used to generate data for numerical experiments, to help understand the workings of matrix, vector.

Mathematics | High School—Algebra

Expressions. ... An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. ...

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Arithmetic with Polynomials and Rational Expressions A-APR

Understand solving equations as a process of reasoning and explain the reasoning

6. Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$,

where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Mathematics | High School—Functions

Functions describe situations where one quantity determines another. ...

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, *technology* is valuable for varying assumptions, exploring consequences, and comparing predictions with data. ...

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

CAS Capabilities

Explore how CAS on the TI-Nspire handles the following calculations.

1. $100!$
2. $\frac{1}{\sqrt{3}}$
3. $4 \cdot \sqrt[3]{2} + \sqrt[3]{54}$
4. $\frac{\sqrt{5} + \sqrt{2}}{2\sqrt{2} - \sqrt{5}}$
5. i^{67}
6. $(6+i)(3-2i)$
7. $\frac{2-i}{4+3i}$
8. $\log_4 8 + \log_3 \sqrt{3}$
9. $e^{-\infty}$
10. $e^{\pi i}$
11. $\sin\left(\frac{5\pi}{6}\right)$
12. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
13. $\cos\left(\frac{\pi}{12}\right)$
14. $\cos^{-1}\left(\frac{\sqrt{3}+1 \cdot \sqrt{2}}{4}\right)$
15. $\tan\left(\frac{\pi}{2}\right)$
16. $\tan^{-1} \infty$
17. $\sin \sin^{-1} x$
18. $\sin^{-1} \sin x$
19. $\sin \cos^{-1} x$
20. $\sin \sec^{-1} x$
21. $\tan \csc^{-1} x$
22. $\sqrt[3]{-8^2}$
23. $-8^{\frac{2}{3}}$
24. $\sqrt{6} \sqrt{-18} + 2\sqrt{3}$
25. $\ln -1$
26. $\sin^{-1} -2$
27. $\sqrt[3]{-8^2} \text{ \& } -8^{\frac{2}{3}}$
28. Solve $x^2 + 1 = 0$
29. Solve $4\sin^2 3\theta = 1$ with $0 \leq \theta < 2\pi$
30. Factor the polynomial $x^4 + x^2 - 12$.

Comments?

Using CAS to Search for Patterns and Develop Formulas

1. Have the TI-Nspire CAS calculate each of the following:

a. $\ln(2) + \ln(3)$

b. $\ln(6) + \ln(y)$

c. $\ln(x) + \ln(10)$

Based on these results, what conjectures might be made about the value of $\ln(x) + \ln(y)$? Have your CAS calculator test your conjecture. Did it work? Why or why not? What modifications might need to be made so that you get the expected result? How might our conjecture be proven?

2. CAS provides wonderful opportunities for dialog like this during the exploration of various standard algebraic rules. Play around with the calculator to see what you might do to explore patterns for the following algebraic rules.

$$\ln x^y$$

$$a^x \cdot a^y$$

$$a \cdot b^x$$

$$a^{x^y}$$

$$a^{-x}$$

$$\sqrt{x} \cdot \sqrt{y}$$

3. What is the value of each of the following?

a. ${}_6C_2$

b. ${}_6C_3$

c. ${}_6C_5$

d. ${}_9C_3$

e. ${}_9C_5$

f. ${}_9C_0$

d. ${}_{12}C_4$

e. ${}_{12}C_7$

f. ${}_{12}C_8$

4. In the expansion of $(x + y)^6$, what is the coefficient of each of the following terms? (And I'm thinking now would be a great time to put TI-Nspire to use!)

a. x^4y^2

b. x^3y^3

c. xy^5

5. In the expansion of $(x + y)^9$, what is the coefficient of each of the following terms?

a. x^6y^3

b. x^4y^5

c. x^9

6. In the expansion of $(x + y)^{12}$, what is the coefficient of each of the following terms?

a. x^8y^4

b. x^5y^7

c. x^4y^8

7. Based on the results of problems #3-56 (plus any other examples you wish to try), what is the coefficient of each of the following terms in the expansion of $(x + y)^n$, where n is a positive integers greater than 3.

a. $x^{n-1}y^1$

b. x^2y^{n-2}

c. $x^{n-3}y^3$

8. **Conjecture** – In the expansion of $(x + y)^n$, if n and r are positive integers with $n \geq r$, then the coefficient of the $x^{n-r}y^r$ term is:

Is it any wonder why ${}_nC_r$ or $\binom{n}{r}$, is called a binomial coefficient?

Note that many formulas in calculus involving derivatives and antiderivatives can be explored and developed through pattern recognition with the aid of CAS. For example, use the TI-Nspire CAS to find the derivative of $y = x + \sin x$; of $y = x^2 + \sin x$; and of $y = x^3 + \sin x$. Based on your observations, predict the derivative of $y = x^4 + \sin x$; of $y = f(x) + g(x)$. (*You can do the same with subtraction, multiplication, division and composition of functions!*).

Here's another example. Consider the Fundamental Theorem of Calculus, $\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$.

Students often have difficulty recognizing how to handle problems such as $\frac{d}{dx} \left(\int_0^{\sin x} e^{t^2} dt \right)$ where the upper bound of the integral, "x," is replaced by a function of x. Have the students use a CAS calculator to do a variety of these problems and then have them predict the solution to the general case, $\frac{d}{dx} \left(\int_a^{g(x)} f(t) dt \right)$. Have the students check their prediction using the TI-Nspire CAS. The result is quite interesting.

The image shows a TI-Nspire CAS screen with three rows of calculations. Each row contains a derivative expression on the left and its simplified result on the right.

- Row 1: $\frac{d}{dx} \left(\int_1^{\sin(x)} e^{t^2} dt \right)$ results in $\cos(x) \cdot e^{(\sin(x))^2}$.
- Row 2: $\frac{d}{dx} \left(\int_a^{g(x)} f(t) dt \right)$ results in $\frac{d}{dx}(g(x)) \cdot \lim_{t \rightarrow g(x)} (f(t))$.
- Row 3: A blank input field for a user-defined problem.

The bottom right corner of the screen shows the page number 2/99.

Other CAS-Active Activities:

Here are other problems that make use of CAS along with the other graphing calculator capabilities.

- Graph each of the following rational functions. Then using the **Proper Fraction** command, write the partial fraction decomposition of each rational function. Describe what, if any, information the partial fraction decomposition gives you about the graph.

a. $y = \frac{x^2}{x^2 + x - 2}$ b. $y = \frac{x^2}{2x^2 + 3x - 2}$ c. $y = \frac{3x^2 - x - 2}{x^2 + x - 2}$ d. $y = \frac{3x^2 - x - 2}{2x^2 + 3x - 2}$

Based on these examples, and others you might wish to try, what conjectures might be made about the graph of a rational function when the degree of the numerator and the degree of the denominator of the rational function are equal? Repeat this activity with rational functions where the degree of the numerator is less than the degree of the denominator, is one greater than the degree of the denominator, is two greater than the degree of the denominator, and so on.

- Graph the function $f(x) = x^2 + bx + 2$ for at least five values of b . For each, estimate the coordinates of the minima of the graph. Use these points determined (and any other minima you may have estimated) to find a function $f^*(x)$ such that if (x, y) is a minimum point of the graph of $f(x)$ for some value of b , then (x, y) is also a point on the graph of $f^*(x)$. Use analytically methods to justify your choice of $f^*(x)$. Extensions: Try the same type of activity for the local extrema for functions like $f(x) = ax^2 + x + 2$ or $f(x) = x^3 + bx^2 - x + 1$.

3. Make a conjecture as to how to simplify the following expressions. Then use CAS to verify (or not verify) your conjecture. (Do you see the possibilities here with regards to correct and incorrect rules of algebra!)

$$a + b + c =$$

$$a + d + e + b + c =$$

$$a + b \cdot c =$$

$$a \cdot b^2 =$$

$$\sqrt{a^2 b^2} =$$

$$a + d + b + c =$$

$$a + d + e + b + c + f =$$

$$a \cdot d + b \cdot c =$$

$$a + b^2 =$$

$$\sqrt{a^2 + b^2} =$$

4. Calculate the following summations of binomial coefficients. The results of these summations have something to do with other binomial coefficients. The pattern here can be extended to form one of the neater relationships in Pascal's Triangle.

a. $\sum_{k=2}^n \binom{k}{2}$

b. $\sum_{k=3}^n \binom{k}{3}$

c. $\sum_{k=4}^n \binom{k}{4}$

d. $\sum_{k=5}^n \binom{k}{5}$

5. Try a few of the summations shown below and then discuss the implications of hand-held technology that can perform these types of calculations (such as how such capabilities might be used to help students better understand certain types of mathematical induction proofs!).

$$\sum_{k=1}^n k$$

$$\sum_{k=1}^n k^3$$

$$\sum_{k=1}^n k(k+1)$$

$$\sum_{k=1}^n a + (k-1)d$$

$$\sum_{k=1}^{\infty} ar^{k-1}$$

$$\sum_{k=1}^n ar^{k-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k^2}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$$

6. Solve the following system of equations: $a_{11}x + a_{12}y = b_1$ $a_{21}x + a_{22}y = b_2$
(Does this look familiar? Can it be done with a general system for 3 equations and 3 unknowns?)
7. Calculus Activity 1: Prove or Disprove: If a cubic polynomial has three real roots, then the tangent line to the graph of this polynomial at a point with x -coordinate that is the average of two of the roots will intersect the graph at the third root. [See if it works for a specific case such as the function $f(x) = (x-2)(x+3)(x-4)$.]
8. Calculus Activity 2: Draw a tangent line to any cubic polynomial at (almost) any point (say $x = a$). Determine the second point (say $x = b$) where this tangent intersects the cubic. Determine the area of the region bounded by the tangent and the cubic. Next, draw the tangent line to the cubic at $x = b$ and determine second point (say $x = c$) where this tangent intersects the cubic. Determine the area of the region bounded by this second tangent and the cubic. Determine the ratio of the larger area to the smaller area? Prove this result.

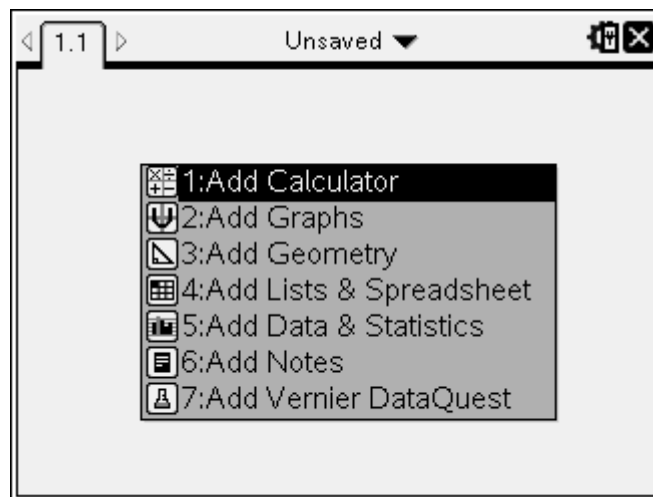
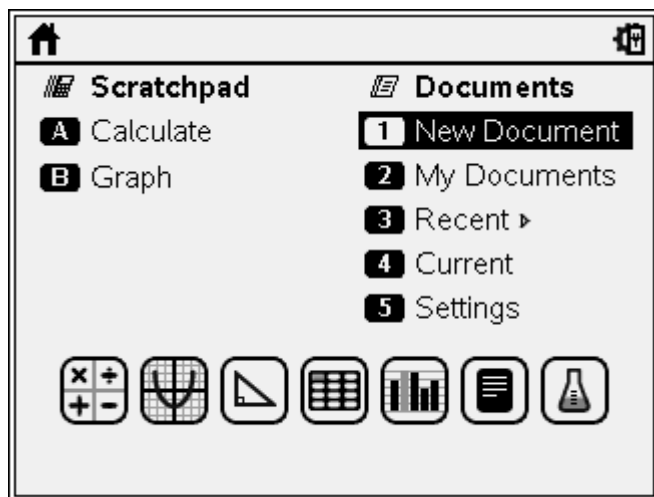
CAS and a Single "Nspired" Mathematics Problem

Consider the following problem situation:

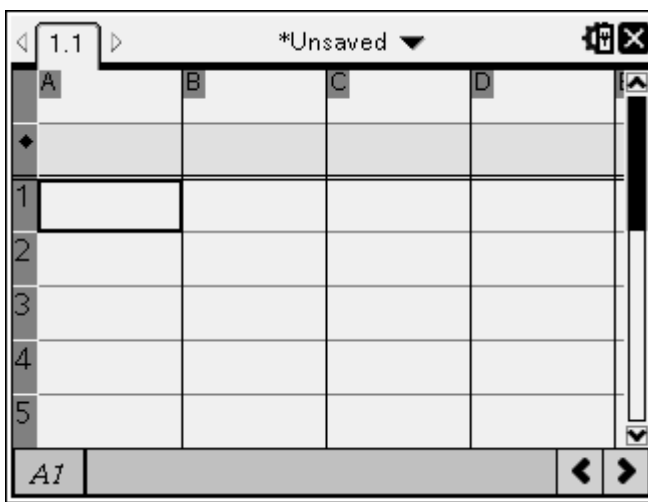
An airline transports approximately 1700 people per week between Chicago and London. A round-trip ticket for this flight costs \$514. The company is doing market research to determine whether it should increase the fare on this flight. Research indicates that for each \$10 increase in price, 23 fewer passengers will take the flight. What ticket price should the airline charge?

In solving this problem, the biggest hurdle for most students is coming up with an equation to model the problem situation. Even the students who do come up with the equation often do not know exactly what the equation represents or why the equation models the problem situation. The statistical and graphical capabilities of the TI-Nspire CAS can be used to solve the problem and to derive an equation that models the problem situation. The symbolic algebra capabilities of the TI-Nspire CAS then provides students with a means for discovering how to derive an equation that represents this and similar problem situations *without* using data or graph.

To begin, press **ON**, select **1:New Document**, and then **4:Add Lists & Spreadsheet**.



You can adjust the column sizes by pressing **menu**, selecting **1:Actions** and then **2:Resize**. Then click on the right and left arrows on the *NavPad* to stretch or shrink the size of the column. When you are finished with a column, press **ENTER**. You can repeat this process for the other columns, as necessary.



Now you are ready to enter data. In this example, column **A** will contain different values for the number of \$10 price increases the airline could impose of the ticket price. To enter a name for this column, which assigns a variable name to the entries in that column, press the up arrow on the *NavPad* until the box containing **A** is highlighted (unless you resized one or more of the columns, in which case you will have to press the appropriate arrows until the box containing **A** is highlighted). Then type in a word, such as *incr*, that is representative of the values in that column (in this case, number of \$10 price *increases*). Now press the down arrow on the *NavPad* to access the data list and enter about 20 values between 0 and 50, including 0 and 50.

A screenshot of a spreadsheet application window titled "1.1" with a status bar indicating "*Unsaved". The spreadsheet has four columns labeled A, B, C, and D. Column A is highlighted, and the cell A1 contains the text "incr". The row numbers 1 through 5 are visible on the left side of the grid.

A screenshot of the same spreadsheet application window. Column A is still highlighted. The cells A19 through A23 contain the values 43, 45, 48, and 50, respectively. The status bar at the bottom shows "A23".

Columns **B** and **C** will contain the number of people who will still take this flight and the ticket price corresponding to the number of \$10 price increases found in *incr* (column **A**). Name these columns *pass* and *price*, respectively. Finally, column **D** will contain the revenue generated by ticket sales corresponding to the number of \$10 price increases found in *incr* (column **A**). This column can be named *rev*.

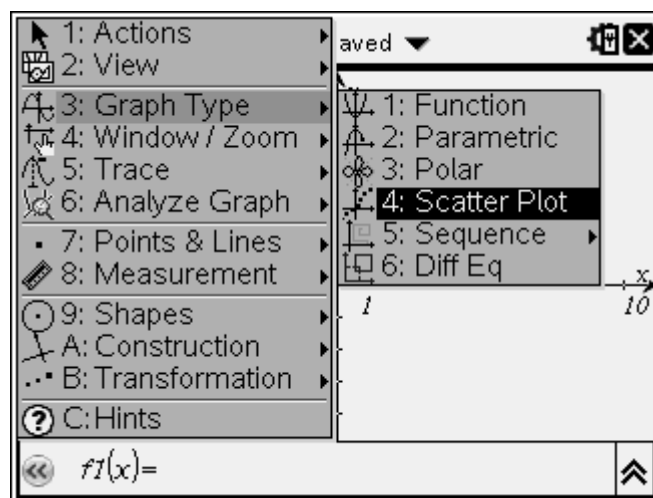
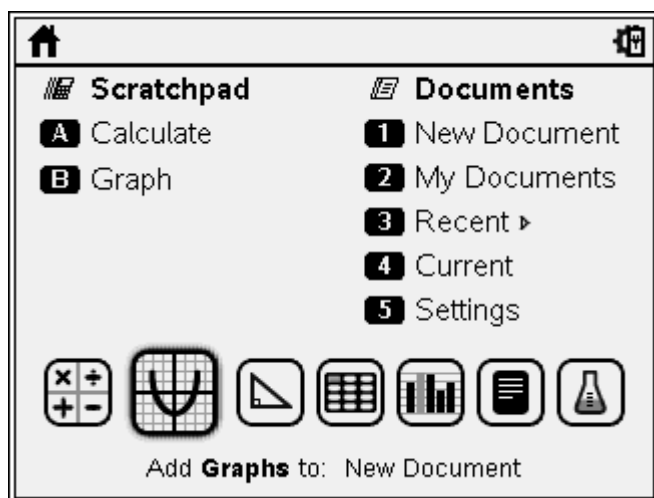
A screenshot of the spreadsheet application window showing columns A, B, C, and D. Column A is named "incr", B is named "pass", C is named "price", and D is named "rev". The data in column B for rows 1 through 5 is 0, 1, 2, 5, and 8. The status bar at the bottom shows "D rev".

The data for *pass* and *price* (columns **B** and **C**) is generated by having students calculate the number of people who would fly, along with the corresponding ticket price and revenue, for specific \$10 price increases that have been entered in *inc* (column **A**). The data for in *rev* (column **D**) can then be generated by multiplying entries from the same row of *pass* and *price* (columns **B** and **C**).

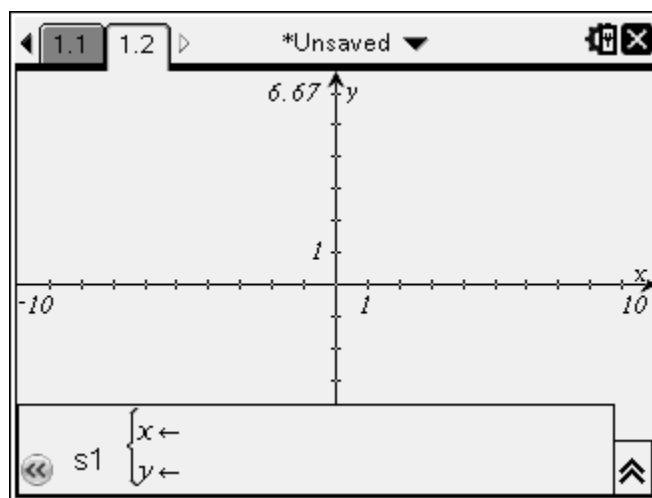
During the process of generating data for *pass* and *price*, students may recognize the patterns in the data being input and suggest that entries in each column could be “globally” defined (like a spreadsheet). For example, *rev* (column **D**) can be defined as the product of *pass* and *price* (columns **B** and **C**). What students may not recognize, at least initially, is that these “global definitions” form the basis for an equation that represents the problem situation. The screenshot at the left shows the data for the problem entered using formula rather than as student generated data.

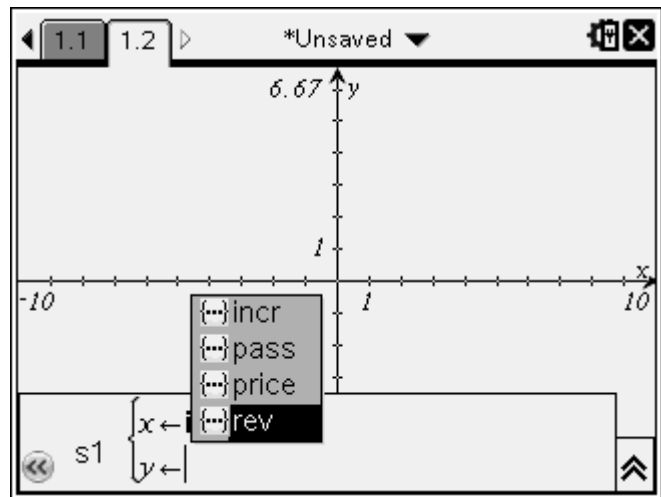
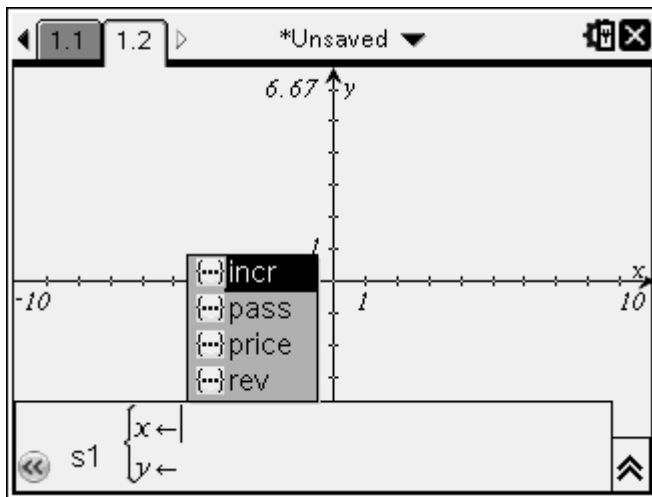
	A incr	B pass	C price	D rev
				=1700-23* =514+10*a=b[*]c[*]
1		0	1700	514 873800
2		1	1677	524 878748
3		2	1654	534 883236
4		5	1585	564 893940
5		8	1516	594 900504

Once the data has been entered, a scatter plot of *incr* (column **A**) versus *rev* (column **D**) can be generated. To graph the data, press **HOME** and select **Graphs**. This opens a 2nd “page,” which contains a graph screen. Press **menu**, and select **3:Graph Type**, followed by **4:Scatter Plot**.

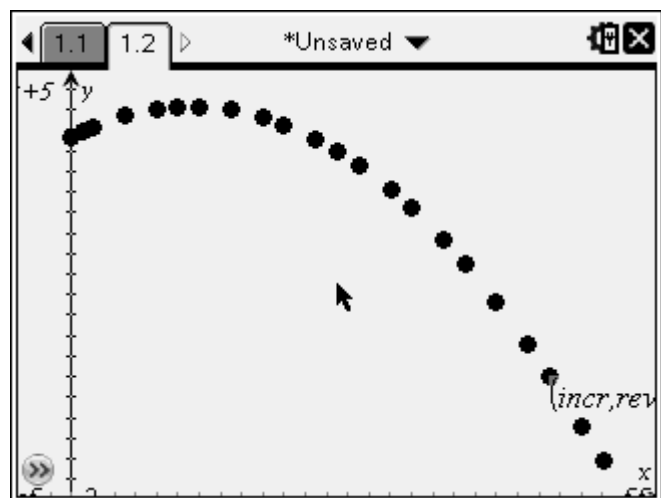
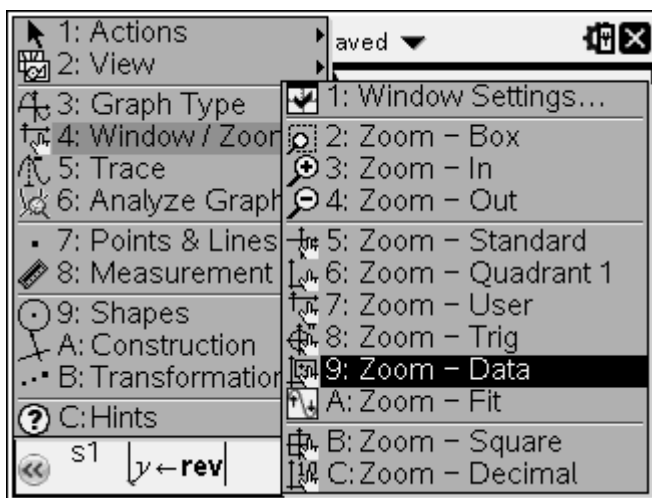


You will notice that the function display at the bottom of your screen has been changed to enter graphing information for a scatter plot rather than functions to be graphed. With the cursor to the right of *x*, press **var** to access a menu containing the possible variables that can be plotted as *x*-values. Select **incr**. Now press **TAB** (so that the cursor moves to the right of *y*), press **TAB** again, and select **rev**. Finally, press **ENTER**.





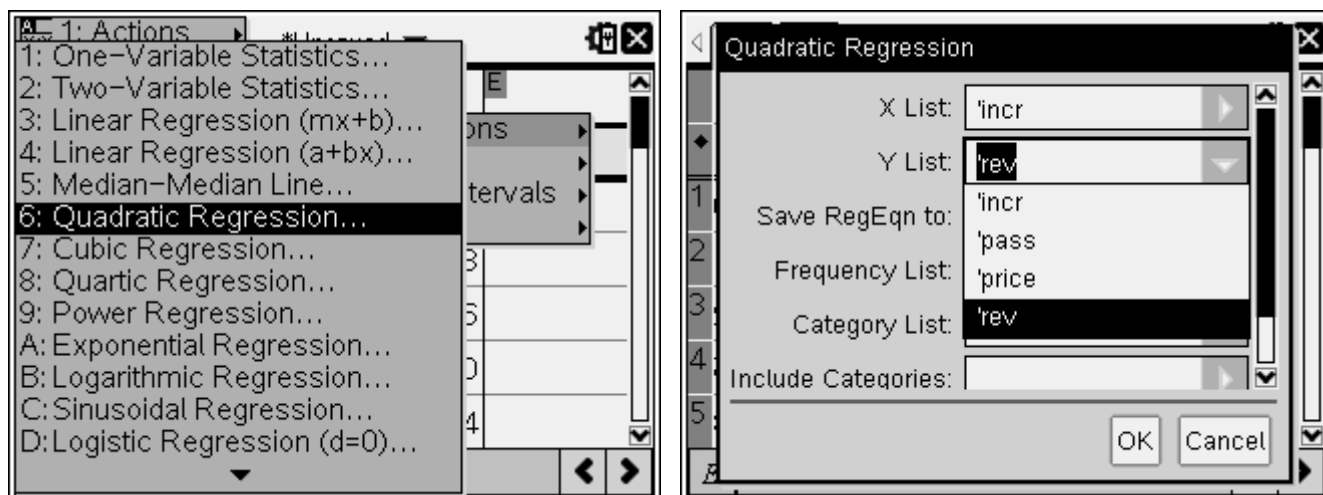
Next, press **menu**, then select **4:Window/Zoom**, followed by **6:Zoom Data**. Voila, you have a scatter plot of the data.



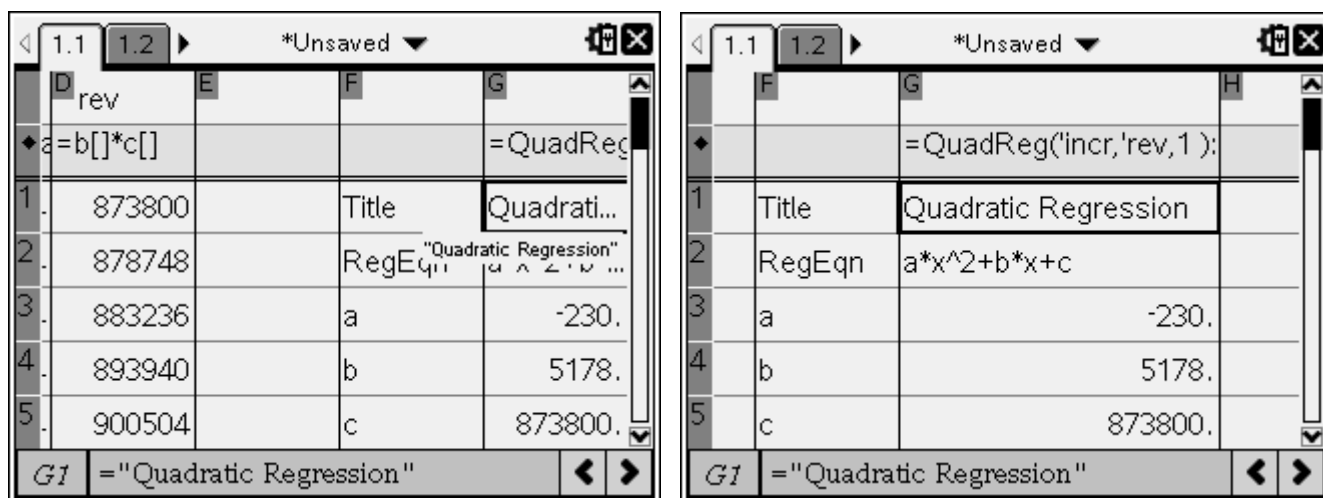
The graph of the data suggests that the relationship between the number of \$10 price increases and the ticket sales revenue might just be quadratic. Performing a quadratic regression on the data can test this conjecture. To do this, press **ctrl** and then left arrow on the *NavPad* to switch back to **Lists & Spreadsheets** application. Use the right arrow on the *NavPad* to scroll over to an open column.

	B	C	D	E
	pass	price	rev	
	$=1700-23^*$ $=514+10*a=b[*]c[*]$			
1	1700	514	873800	
2	1677	524	878748	
3	1654	534	883236	
4	1585	564	893940	
5	1516	594	900504	

Press **menu**, then select **4:Statistics**, followed by **1:Stat Calculations**, followed finally by **6:Quadratic Regression**. A dialog box will appear. Press right arrow on the *NavPad* until the submenu for the **X List** appears, and then select **inc**. Next, press **TAB** followed by right arrow on the *NavPad* to access the submenu for the **Y List** and select **rev**.

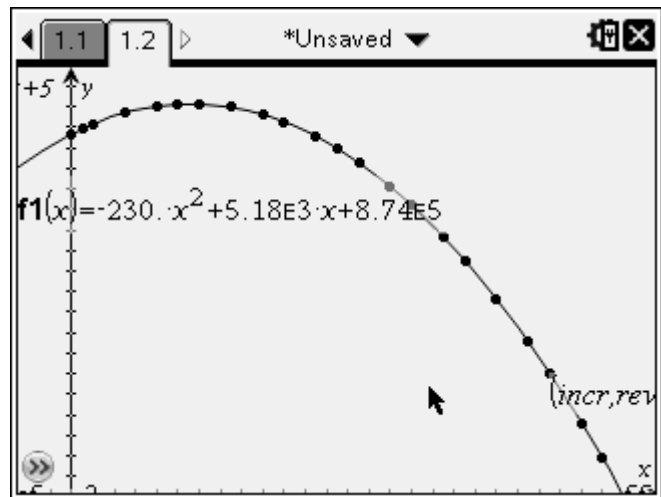
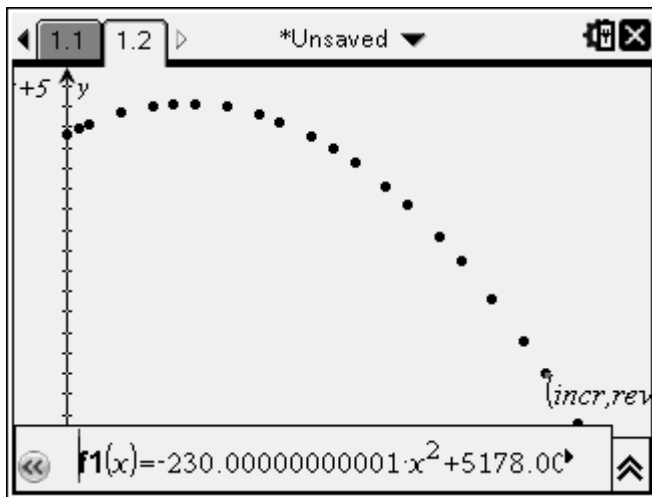


Now press **ENTER**. The information from the quadratic regression will appear in the two columns to the right of the column you scrolled to early. To see this information, you'll need to resize the column following the previous direction.

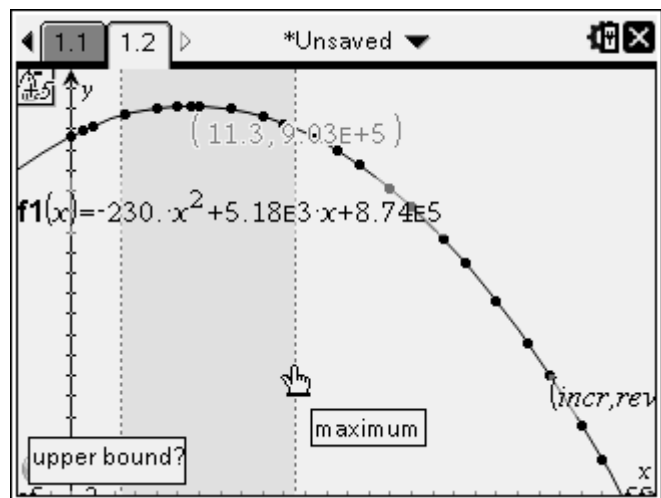
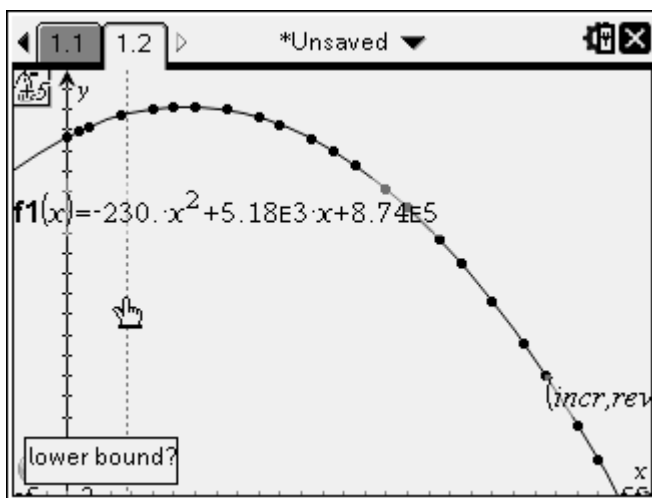


To plot the quadratic regression equation, you'll need to switch pages back to the **Graphs** application by pressing **ctrl** and then the right arrow on the *NavPad*. Press **menu**, and select **3:Graph Type**, followed by **1:Function**. Use up and down arrows on the *NavPad* to scroll to your regression equation and press **ENTER** to graph the equation, which should be (approximately)

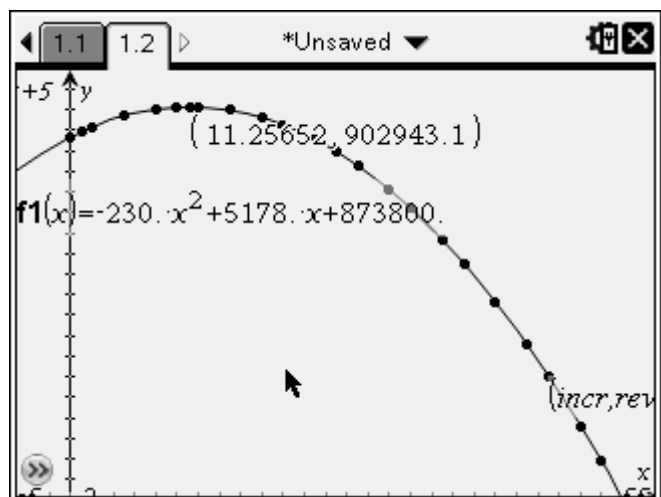
$$y = -230x^2 + 5178x + 873800.$$



We can now use the regression equation graph to estimate the number of \$10 prices increases required to produce the greatest ticket revenue. Press **menu**, then select **6:Analyze Graph**, followed by **3:Maximum**. On the graph screen, the words "lower bound?" will appear. Use the arrows on the NavPad to move the "hand" pointer until it is to the left of the maximum value of the function.

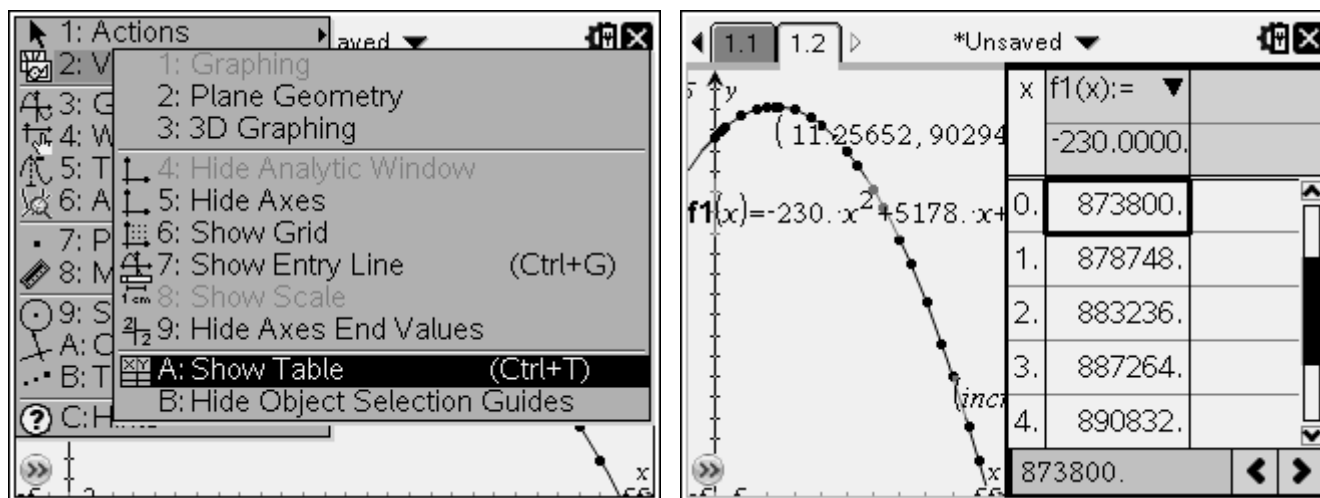


Now press **ENTER**. On the graph screen, the words "upper bound?" will appear (along with the word "maximum." Again, use the arrows on the NavPad to move the "hand" pointer until it is to the right of the maximum value of the function. Then press **ENTER**. A point will appear on the graph at the maximum value of the function along with its coordinates!

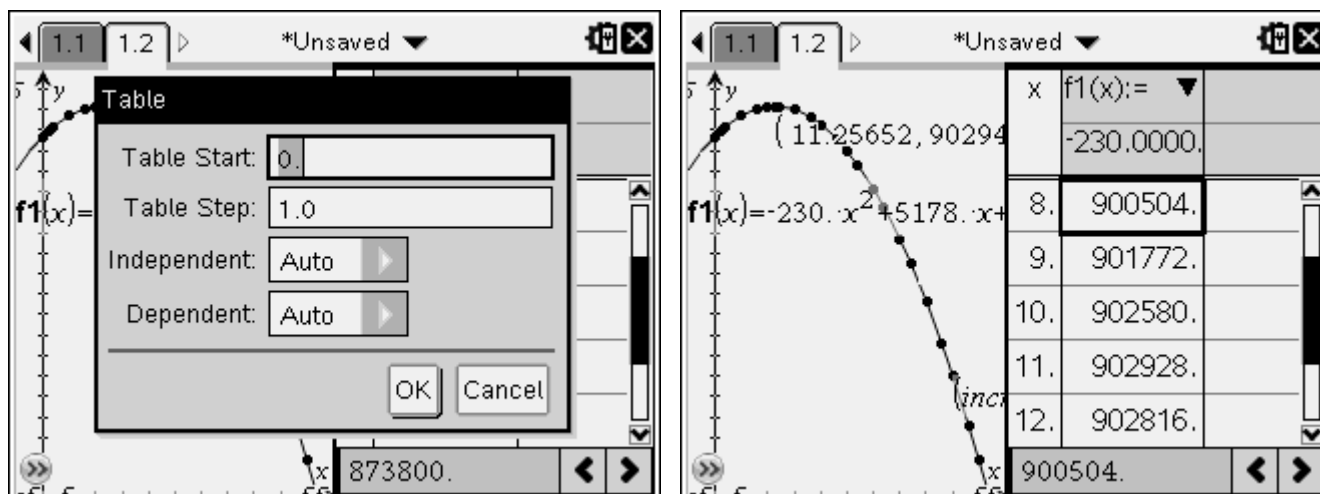


Based on these results it appears that the airline could generate maximal ticket sales of approximately \$902943.10 by increasing the ticket price to \$626.57 ($\$514 + \10×11.2565) allowing for any price change based on information provided. BUT, how can you have 11.2565 price increases of \$10? Doesn't make much sense, does it? . Better might be to look at integral values of the number of \$10 price increases, which can be done in a table.

Press **menu**, then select **2:View**, followed by **A:Show Table**.



A table will appear side by side with the graph. If the values of x in the table are not integral values starting at 0, press **menu**, then select **5:Table**, followed by **5:Edit Table Settings**. Adjust these so that **Table Start** is 0 and **Table Step** is 1. Then press **ENTER**.



Scroll down until the max value of the function appears in the **f1(x)** column. Based on these results, it appears that the airline could generate maximal ticket sales of \$902929 by increasing the ticket price to \$624 ($\$514 + \10×11).

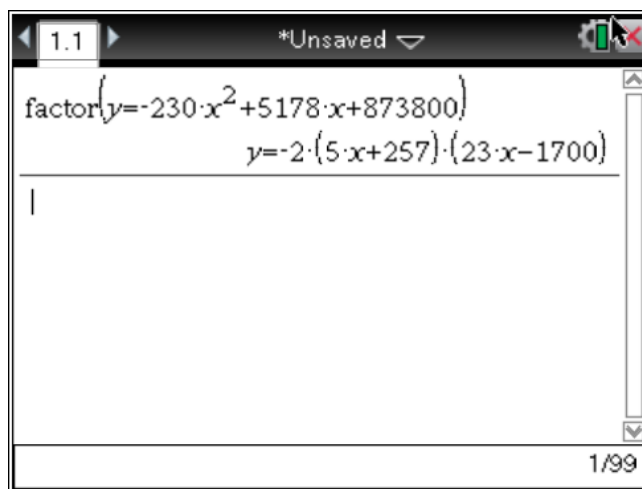
Now the question is, where does CAS come into play with this problem, considered we've solved the problem, without using any CAS capabilities? The answer to this questions lies in using CAS to tie together pieces of this problem in such a way that students will understand how they can solve the problem WITHOUT having to go through all the data collection, graphing, and regression analysis.

Let's return to our regression equation,
 $y = -230x^2 + 5178x + 873800$. Factoring this equation using CAS, we see the equation for this problem situation can be written as:

$$y = -2(5x + 257)(23x - 1700) \quad \text{or}$$

$$y = (10x + 514)(1700 - 23x),$$

which is exactly the equation we would have expected students to derive (or we might have derived for students). The difference here is that the equation should be more meaningful to them since it was derived from actual data and in the context of solving the problem.



Internet Nuggets

There's a library of programs that apparently are designed specifically for the Nspire housed by ticalc.org: <http://www.ticalc.org/pub/nspire/basic/math/>

A good first place to look for TI-Nspire activities is TI-MathNspired, <http://education.ti.com/calculators/timathnspired/>

Lot of TI-Nspire activities can be found at <http://education.ti.com/calculators/downloads/US/Activities/Search/Subject?s=5022&sa=5023&d=1008>

There is a good TI Nspire google group: <http://groups.google.com/group/tinspire>

To CAS or Not To CAS During Assessments

No, I did not just contradict myself. Student should and should not be allowed to use the CAS during an assessment. It all depends on WHAT IT IS YOU ARE TRYING TO ASSESS. That is the key issue. If you trying to assess your students' understanding of a specific symbolic manipulation technique, then you darn well better take CAS out of their hands. If you are assessing your students' ability to turn a problem situation into mathematical symbols and then correctly solve this problem, then perhaps you should let them use CAS so they do not make arithmetic or algebraic errors in the problem solving process that will hamper your ability to assess what you wish to assess.

Using a tool as powerful as the TI-Nspire CAS as a regular part of assessment will require some rethinking about what you assess. Clearly, you would not want your students to use CAS to solve the problem $x^2 - x = 2x + 4$, but you would if you asked them to complete the following:

- Describe three (or more) different ways you could solve the equation $x^2 - x = 2x + 4$. Explain how your solution methods are related to each other. Then choose one of your methods and use it to solve the equation, being sure to explain the steps you are taking to solve the equation.

Such an assessment gets at more than solving the equation. It gets at understanding how different representations of the equation are related and how they might be used to solve the equation.

Use of Multiple Representations

Giving students opportunities to demonstrate their understanding of the connections between multiple representations of mathematical ideas and concepts will help in their development into be flexible thinkers and problem solvers. In *Principles and Standards for School Mathematics* (NCTM, 2000), the National Council of Teachers of Mathematics proposed that "Instructional programs from prekindergarten through grade 12 should enable all students to - ... Select, apply, and translate among mathematical representations to solve problems" (pg. 360). One highly advocated theoretical view on mathematical learning is based on the premise that multiple representations can be utilized to help students develop deeper, more flexible understandings of concepts and processes. If we consider multiple representations of concepts to be important, then that means we will have to look for ways to assess students understandings of these representations, and technology can be invaluable in doing this! What follows are my views on the use of multiple representations in the teaching and learning of mathematics along with a rubric that I have been working on (for many, many years) for assessing work on problems involving use of technology and different representations. You may find this information and rubric useful in your own teaching (and I would appreciate any comments and suggestions you might have concerning the rubric)

WHY USE GRAPHICAL, NUMERICAL, AND SYMBOLIC REPRESENTATIONS IN THE TEACHING AND LEARNING OF MATHEMATICS? BECAUSE THEY:

- are inherent in mathematics;
- provide multiple concretizations of a concept;
- may be used to mitigate difficulties students have in accomplishing certain tasks;
- provide students with more mathematical tools for solving problems (assuming they view the different representations *as* tools);
- may make mathematics more interesting to students;
- can help students develop deeper, more flexible understandings of a concept by having them:
 - generate the different representations of the concept and then view and reflect upon common properties and differences between the different representations,
 - “see” the concept or idea from several points of view (i.e., different representations) and then build a web of connections between the different viewpoints (representations), and
 - utilize the different representations to model and solve a problem.

ASSESSING WORK THAT INVOLVES USE OF TECHNOLOGY AND MULTIPLE REPRESENTATIONS; FOCUS ON STUDENT’S ABILITY:

- To use appropriate solution methods (not just their ability to get the correct answer);
- To recognize and use different representations (symbolic, numerical, graphical) of a given problem situations and of concepts related to that problem situation;
- To “construct” different representations (symbolic, numerical, graphical) using appropriate tools (such as graphing calculators, CAS calculators, and various types of computer software);
- To recognize how the same procedure or concept can be applied to different representations of the problem situation;
- To recognize connections between solution methods that use different types of representations;
- To justify their use of various procedures, concepts, processes, representations, and connections, to solve the problem.

NOVICE LEVEL

- Use (at least attempted) of only one type of representation
- Use of inappropriate procedures, concepts, or methods
- Inaccurate, incomplete, or missing descriptions of solutions
- No justification of processes or representations used in problem solution
- No evidence of recognition of connections between solutions involving different representations
- Serious mathematical errors present

APPRENTICE LEVEL

- Use (at least attempted) of two different types of representations
- Use of some appropriate procedures, concepts, and methods
- Poorly-developed descriptions of solutions
- Vague or unclear justifications of processes and representations used in problem solution
- Evidence of vague, unclear recognition of connections between solutions involving different representations
- Minor mathematical errors present

COMPETENT LEVEL

- Use of two (possible more) different types of representations
- Use of mostly appropriate procedures, concepts, and methods
- Reasonably well-developed descriptions of solutions
- Fairly clear justifications of processes and representations used in problem solution
- Evidence of at least partial recognition of connections between solutions involving different representations
- Few, if any, minor mathematical errors present

PROFICIENT LEVEL

- Use of at least two different types of representations
- Use of appropriate procedures, concepts, and methods
- Well-developed, accurate descriptions of solutions
- Justifications/explanations of processes and representations used in problem solution
- Evidence of recognition of connections between solutions involving different representations
- No mathematical errors present

DISTINGUISHED LEVEL

- Use of three or more different types of representations
- Use of appropriate, possibly unique, procedures, concepts, and methods
- Very well-developed, accurate, concise descriptions of solutions
- Precise, detailed justifications of processes and representations used in problem solution
- Discussion of the non-use of certain methods or representations
- Clear evidence of understanding of connections between solutions involving different representations

DISTINGUISHED LEVEL

Leaps Tall Buildings in a Single Bound

PROFICIENT LEVEL

Leaps Short Buildings in a Single Bound

COMPETENT LEVEL

Leaps Short Buildings with a Running Start and Maybe a Boost

APPRENTICE LEVEL

Bumps into Buildings

NOVICE LEVEL

Does Not Recognize Buildings