Connecting Math and Music

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Friday 9:30 - 10:30 am
NCTM Annual Meeting
Session 374, Room 108
April 19, 2013

Session files will be posted on NCTM by April 26.
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Why Connect Math & Music?

* Music relates to many math concepts at all grade levels
* These relationships exist at the physical, scientific, sensory, affective and symbolic levels
* These relationships engage students
* Involvement in music correlates with academic success

Presentation Goals

* To provide an overview of the connections. (This presentation can only scratch the surface in one hour.)
* To explain the mathematical basis of western music, which necessarily includes some of its history.
* To use an app, a musical instrument, and video to illustrate basic concepts of **rhythm, pitch, and volume.**
* To introduce musical notation and the physics of sound.
* To share references, which include links to ideas for classroom activities, resources for further study, and research on how playing and listening to music helps with academics.

Math Concepts Related to Music

* Elementary School: Divisibility, least common multiple, fractions, powers of 2
* Middle School: Ratio and Proportion
* High School: Arithmetic and Geometric sequences, logarithmic scales, applications of trigonometry (especially trigonometric graphs)
* Beyond High School: Calculus, Differential Equations, Fourier Analysis, Signal transmission and conversion

A Few Questions

Raise your hand high if:

* You teach elementary school
* You teach middle school
* You teach high school
* You are an educator, but you don't teach K-12
* You already know more than a little about the connections between math and music
* You play a musical instrument or sing

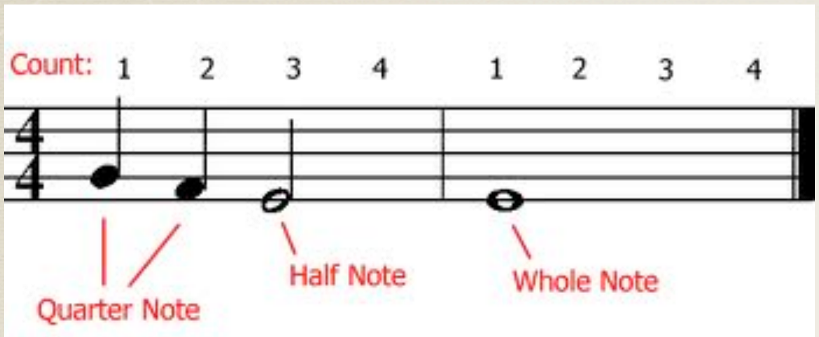
The Rhythm of Math®
Keith Terry and Linda Akiyama

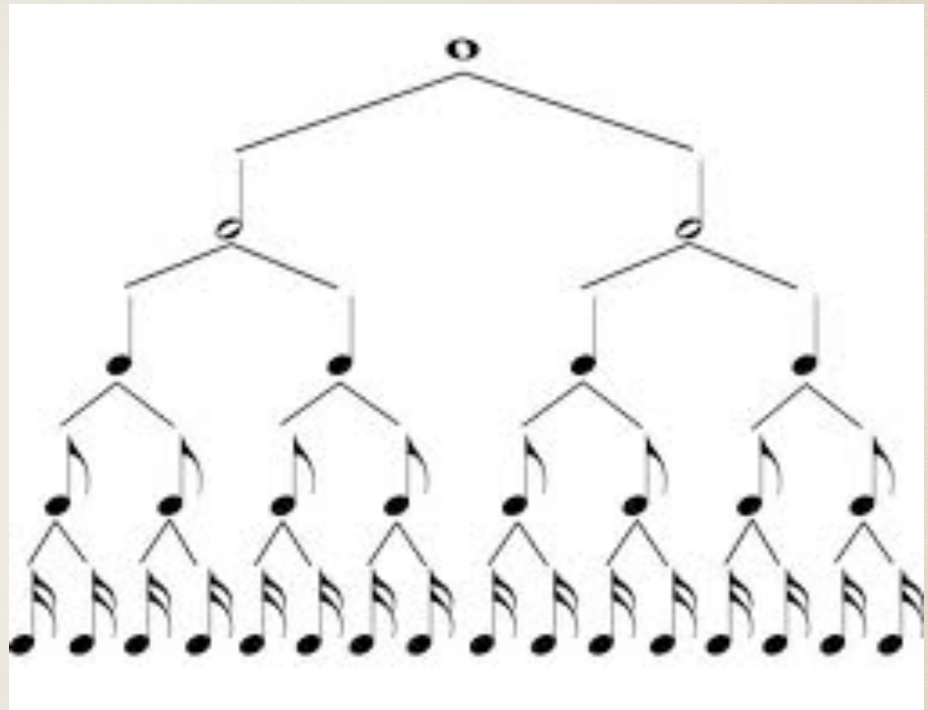
* The Rhythm of Math is a Math Teaching Unit for grades 2-5. Link will be included in the references.
* Rhythm Blocks
* Some simple polyrhythms
* An illustration of the commutative law of addition

1. Musical Rhythm

* **Rhythm**: A repeated pattern of sounds and silences. Rhythm recalls the regularity of walking and heartbeat
* **Beat**: The basic unit of time in music
* **Tempo**: The speed at which the music is played. It is measured by a metronome in beats per minute (bpm)

Musical Notation





Note the powers of two in the denominator.
Alternatively, note the powers of one-half.

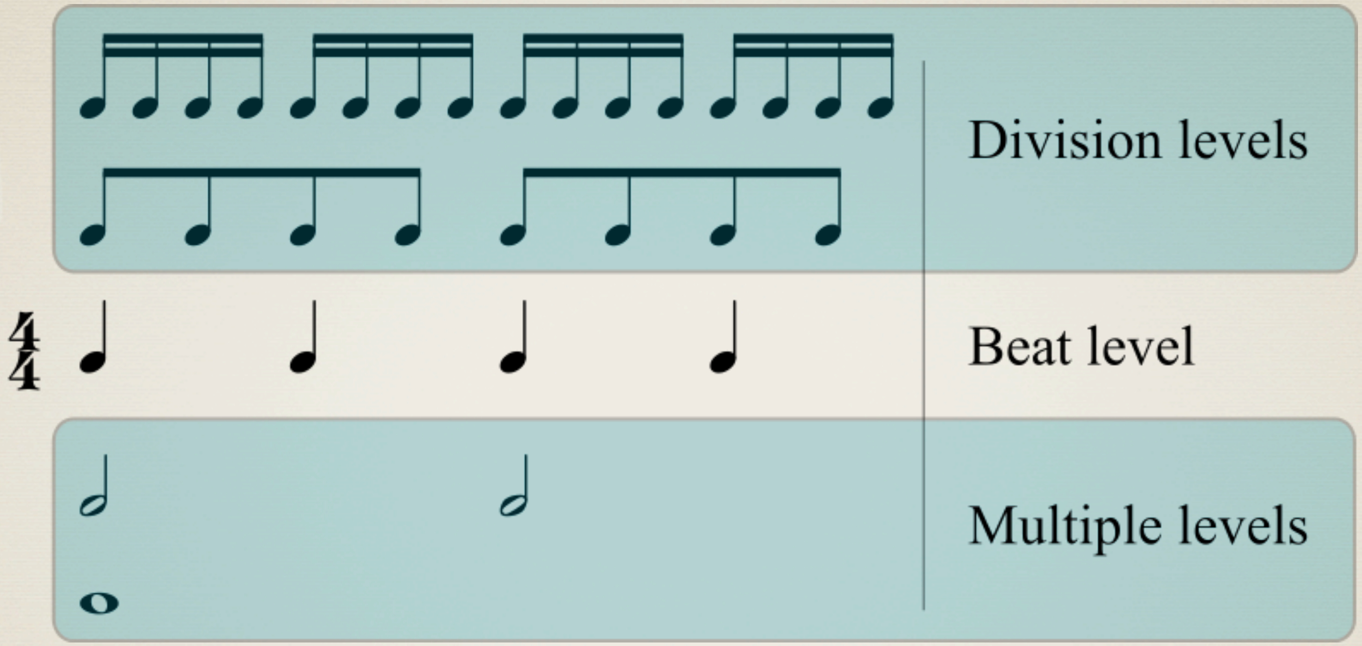
Time Signature (Meter)

* Time signature: A notational convention in Western music consisting of a pair of vertically stacked numerals

* In this case:
  3 = number of beats per measure
  4 indicates the note value of one beat (quarter note in this case).

* Note the similarity to fractions: The whole note is divided into 4 equal quarter notes. Three of these make a measure. Each measure takes 3/4 of a whole note.

Simple Meter



* The beat is a quarter note, the denominator of the time signature

* In simple meter, each beat divides (sometimes repeatedly) in half

Compound Meter

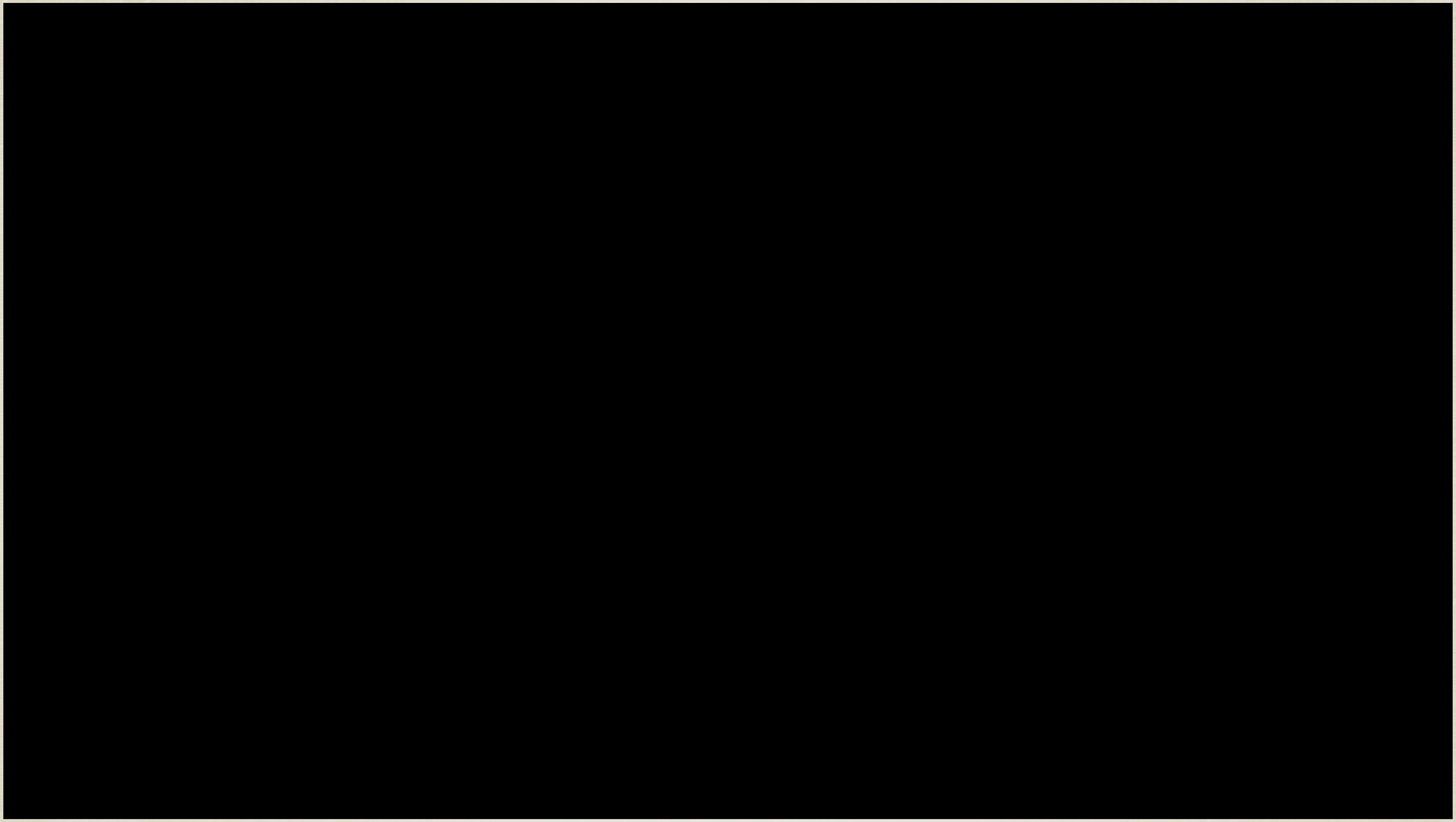
* In Western music, the predominant form of compound meter divides the beat into 3 parts instead of 2
* If each beat of a 3/4 time signature is divided in half, it would seem to be equivalent to 6/8. But by convention, 3/4 would use 3 groups of 2 (if split into eighths), while 6/8 uses 2 groups of 3.
* *America*, from *West Side Story*, uses both:

Irrational Rhythm

* Irrational Rhythm: Any rhythm that involves dividing the beat into a different number of equal subdivisions from those usually permitted by the time-signature. This is indicated by a number, and sometimes a bracket or slur to group notes, indicating the fraction involved. The most common type is the "triplet."

Polyrhythm

* Polyrhythm: The simultaneous sounding of two or more independent rhythms, one of which is typically irrational. Very common in Africa.



Syncopation

* Syncopation is a general term for a disturbance or interruption of the regular flow of rhythm; a placement of rhythmic stresses or accents where they wouldn't normally occur
* Syncopation is used in many musical styles, and is fundamental in African-derived styles such as ragtime, jazz, funk, reggae, and rap
* Syncopation has been an important element of musical composition since at least the Middle Ages

Symmetry in Music

* Musical can exhibit translational symmetry,

Figure 9.2 Opening of Beethoven's *Moonlight Sonata.*

* reflectional symmetry,

Figure 9.4 Bar from Bartók's *Fifth String Quartet*. With kind permission of Bossey & Hawkes.

* rotational symmetry

Figure 9.10 Rotational symmetry.

From pages 313 and 316 of *Music: A Mathematical Offering*

2. Musical Pitch

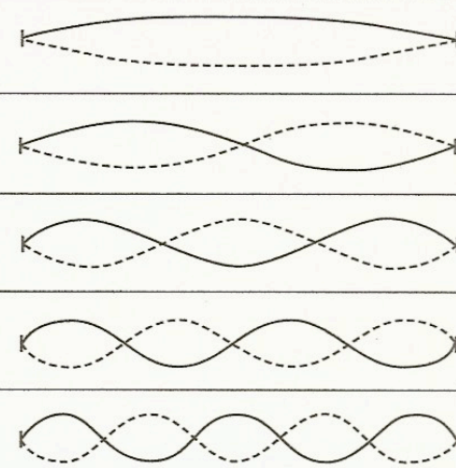
* Origins: Pythagoras
* Chapter 5: The Music of the Spheres
  From *The Ascent of Man*, a thirteen-part documentary television series produced by the BBC and Time-Life Films first transmitted in 1973, written and presented by Jacob Bronowski, a professor of humanities at MIT at the time.

String Length vs Frequency

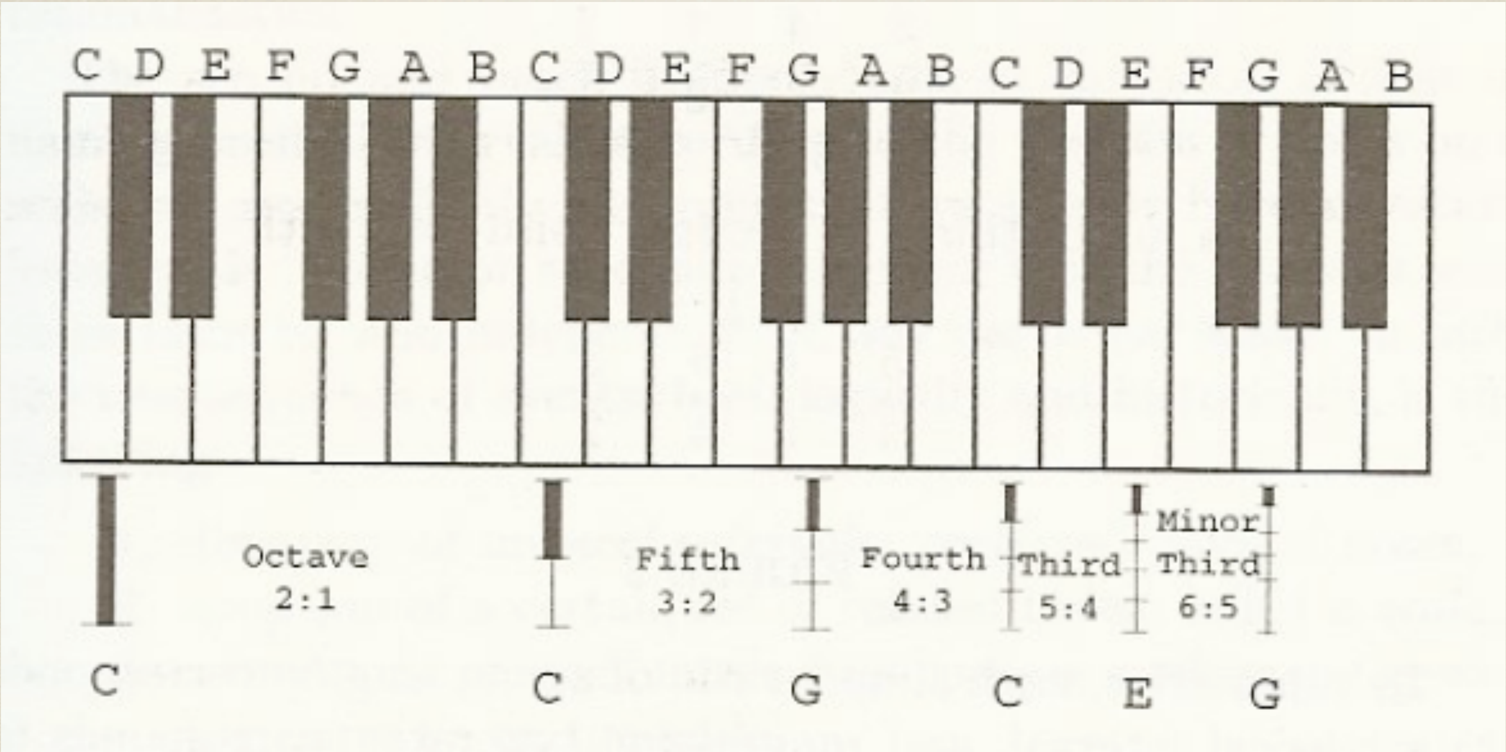
- If the string is cut in half, the frequency doubles
- In general, frequency is <u>inversely proportional</u> to string length
- The frequency of the sound wave determines the pitch we hear

Standing-Wave Vibrations

	Frequency		
	Example	General	Vibrating string pattern
First harmonic	130	*f*	
Second harmonic			
Third harmonic			
Fourth harmonic			
Fifth harmonic			

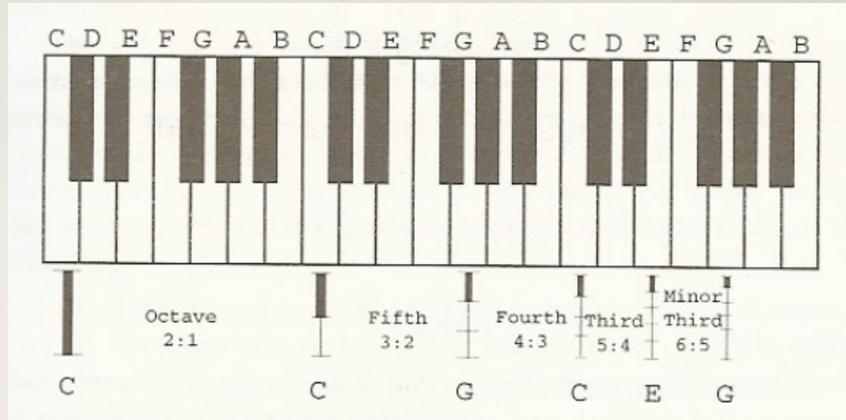


Intervals on a Keyboard



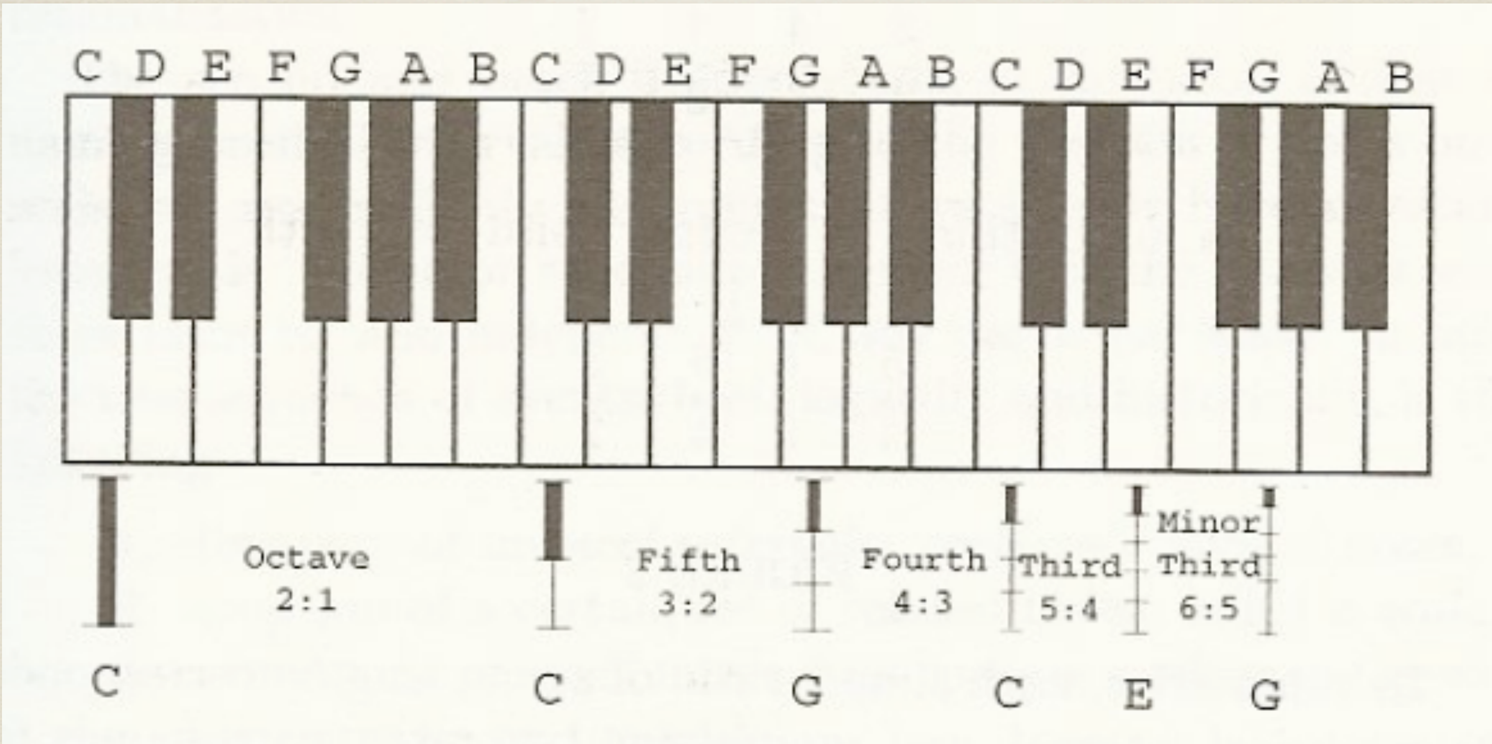
* The vertical segments below the keyboard show relative string lengths of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$.

Interval Naming Confusion



- An interval is named after the position on the scale of the upper note, assuming the lower note is note number 1.
- A fifth plus a fourth should be a ninth, not an octave (eighth).
- If we counted spaces instead, it would work out: 4 + 3 = 7.
- More importantly, the interval names assume that we already have a scale. We can't name intervals without using a scale.
- The true sequence, both logically and historically, is:
  1) discovery of musical intervals, 2) adoption of a set of related notes (a scale), 3) naming the intervals using the adopted scale

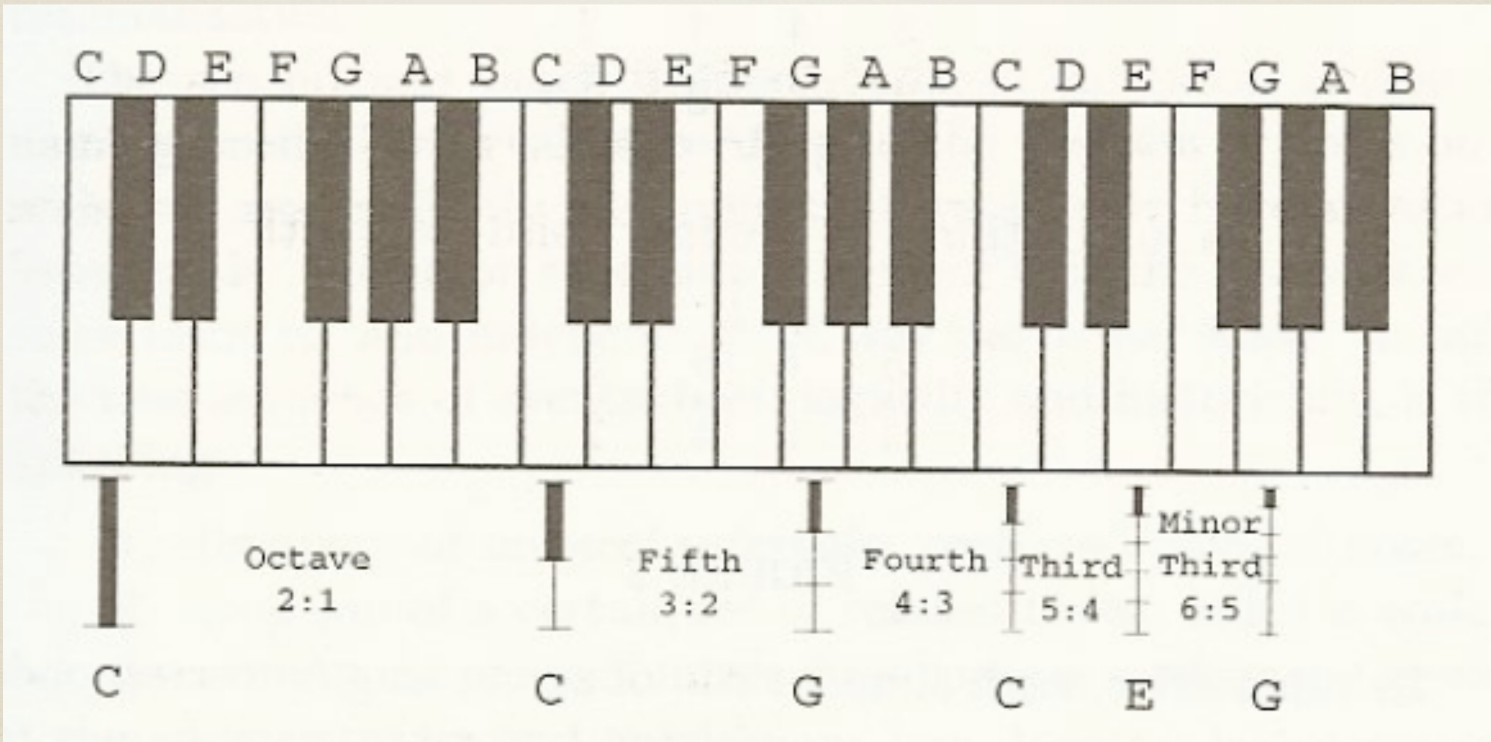
Frequency Ratios



* The frequency ratios of the intervals shown are

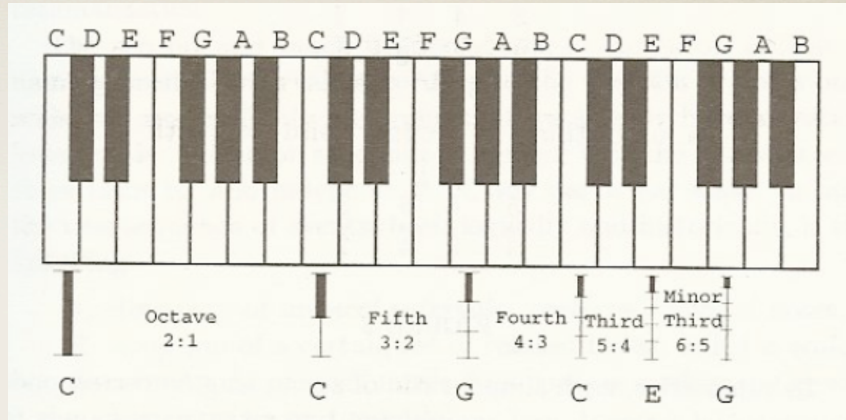
  1:2 2:3 3:4 4:5 5:6.

* The C major scale has just white keys. It has whole and half steps. Half steps to keys 4 and 8. The G major scale has one black key. Using all successive keys provides for all half steps. Major third has 4 half-steps. Minor third has 3.



* This can be illustrated using Piano HD on an iPad.

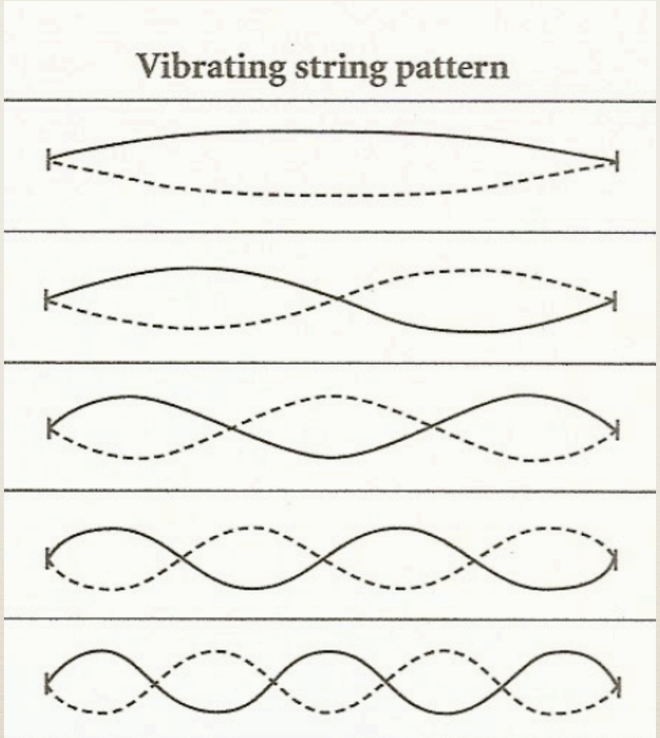
Adding Musical Intervals



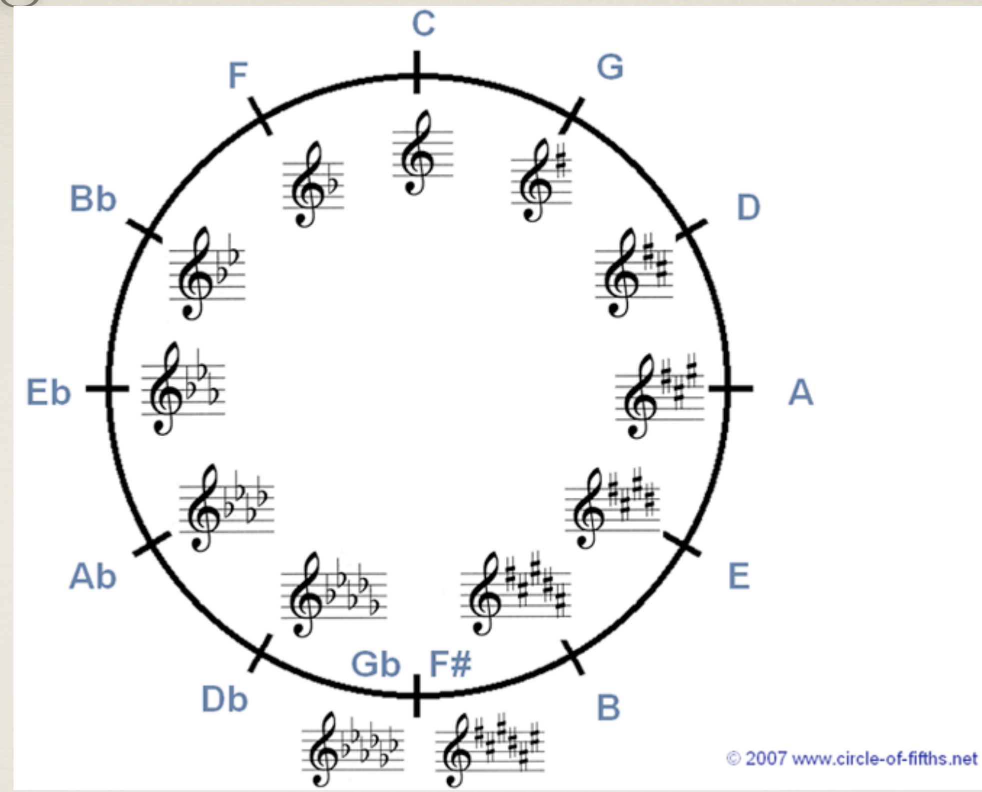
- Consider the C, G, C sequence in the middle
- From C to G, we take two-thirds of the string.
- From G to C, we take three-fourths of the string length.
- From C to C, we take $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$ of the string length.
- Each simple ratio of string lengths is a musical interval.
- *Multiplying* two ratios corresponds to *adding* musical intervals! Logarithms are built into pitch perception!

The Harmonic Series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$



Choosing a Scale: The Circle of Fifths



- Restrict to the most harmonious notes (simplest string length ratios): 2:1 (octave) and 3:2 (fifth)
- Go up by fifths, down by octaves. Start at C.

A Suspicious Equation

* $C \to G \xrightarrow{1} D \to A \xrightarrow{2} E \to B \xrightarrow{3} F\sharp \xrightarrow{4} C\sharp \to$
  $G\sharp(A\flat) \xrightarrow{5} E\flat \to B\flat \xrightarrow{6} F \xrightarrow{7} C$
* If we just go up by fifths without dropping by octaves, we cover 7 octaves
* 12 fifths = 7 octaves. Using frequencies:
* $\left(\frac{3}{2}\right)^{12} = 2^7 \to \frac{3^{12}}{2^{12}} = 2^7 \to 3^{12} = 2^{19}$. Why suspicious?

$$\left(\frac{3}{2}\right)^{12} = 129.746\ldots \quad 2^7 = 128$$

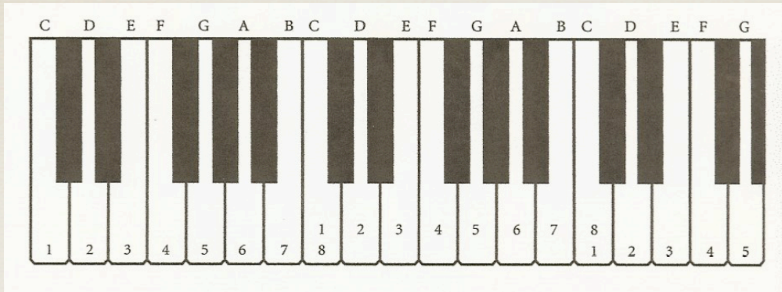
Musical Implications

- 12 fifths is *almost*, but not quite, equal to 7 octaves. What should be done about this?
- Over the centuries, many modifications were proposed.
- Ptolemy's idea: Using the circle of fifths, a major third corresponds to $\frac{81}{64} = \left(\frac{3}{2}\right)^4 \left(\frac{1}{2}\right)^2$.
- A more harmonious ratio would be $\frac{80}{64} = \frac{5}{4}$
- Advantage of this *just intonation*: simpler ratios, so more pleasing sounds
- Disadvantage: Formerly equal intervals now different

Equal Temperment

- *Just intonation*: Intervals determined by simple ratios
- *Equal temperment*: Divide the octave into 12 equal intervals. A bit less harmonious, but intervals equal.
- Details first spelled out by French mathematician Marin Mersenne (Mersenne primes) in 1636
- Championed by Bach:The Well-Tempered Clavichord
- Now accepted nearly universally in the West

The Math of Equal Temperment



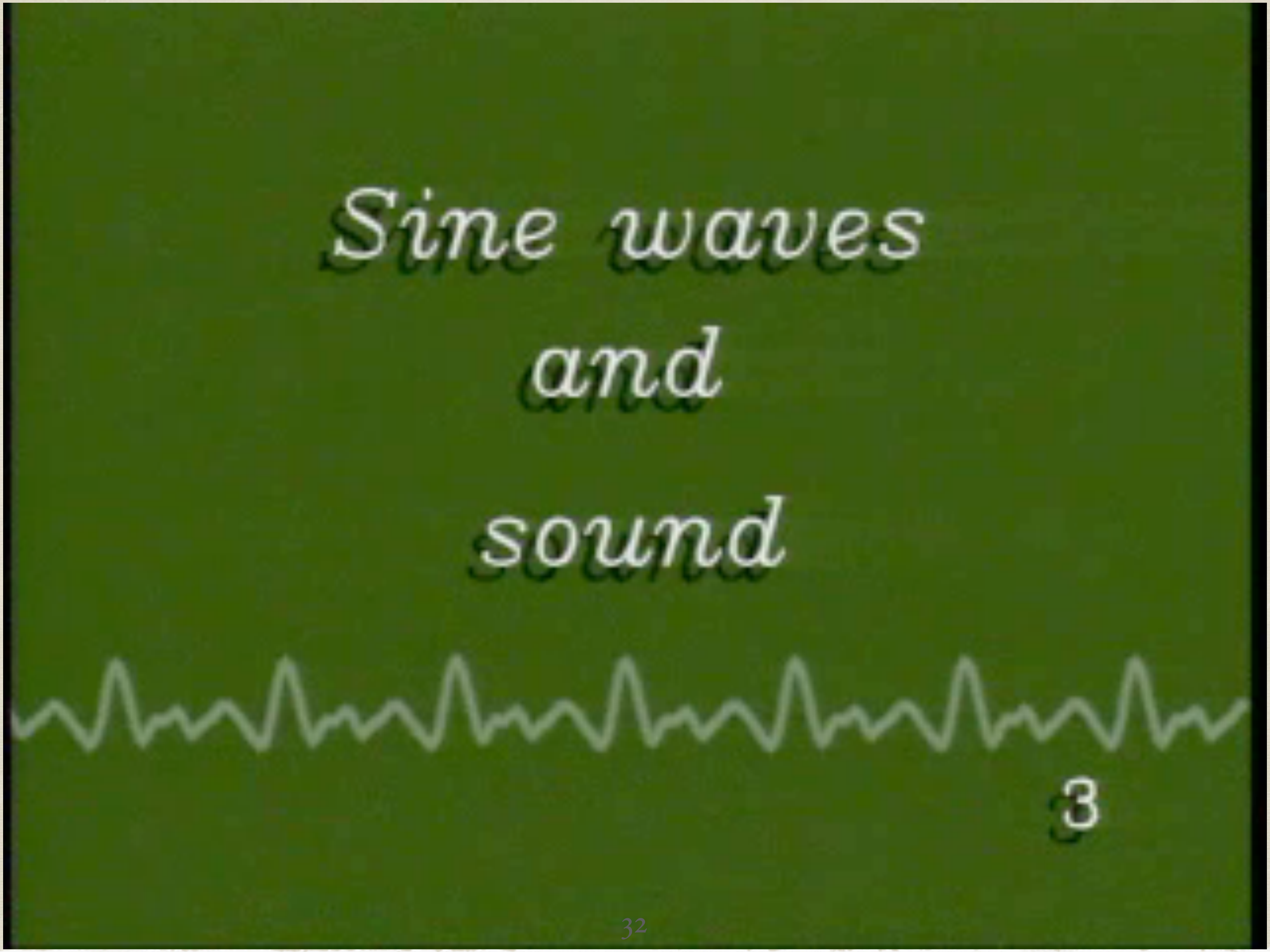
- If r = frequency ratio needed for equal intervals, then $r^{12} = 2 \rightarrow r = \sqrt[12]{2} \approx 1.0595$ (multiplication of string lengths or frequency corresponds to adding intervals).
- A fifth = $r^7 = 1.4983\ldots \approx 1.5$. A fourth = $r^5 = 1.3348\ldots \approx 1.\overline{3}$
- The Chinese were first: Ling Lun anticipated Pythagoras by at least 500 years; Prince Chu Tsai-yu explained the principles of equal temperment in 1596.

Consonance and Dissonance

* The origin of the consonance of the octave (2:1) is the instruments we play! Octaves are so consonant, that we call them "the same note, an octave apart."
* The note we hear is the fundamental frequency. But other frequencies (called *overtones*) occur as well. Differences in overtones is why different instruments have a different "quality of sound" for the same note.
* Stringed and wind instruments naturally produce overtones that consists only of exact integer multiples of the fundamental frequency.
* Percussive instruments don't do this. The western scale is inappropriate, and indeed not used, for gamelan (Indonesian) music.

Taken from the introduction to *Music, A Mathematical Offering*

Sine Waves and Sound



3. Volume (Loudness)

* Understanding our perception of loudness is a bit complicated, necessitating a short detour into the concepts of energy, power, power density, and logarithmic scales. I will post word and pdf versions of the Worksheets 2 and 3 for you to go through at a more leisurely pace later. You may use modified versions with your students if you wish. BUT YOU MUST GIVE ME CREDIT.

Volume (Loudness)

* The human ear is very sensitive. The amount of power (in watts) involved in the production of sound is small.
* A clarinet at its loudest produces 0.05 watts
* A trombone can produce 5 or 6 watts
* Average human speaking voice produces 0.00002 watts
* Since the response of the ear to sounds depends on frequency, a standard of 1000 hertz is used. (Hertz means cycles per second, the unit of frequency)

Decibels

* Sound intensity is measured in decibels (dB). Power *intensity* is measured in watts. Power *density* is measured in watts per square meter (watts/m^2).

* The ear hears logarithmically: **Multiplication** of the power density by a given factor causes an **additive** increase in our perception of sound intensity (loudness)

* The **decibel's logarithmic nature** allows us to represent a very large range of power densities by a convenient number.

* Zero decibels is approximately the weakest sound we can hear, which is $10^{-12} = .000000000001$ watts

* **Multiplying** the power density by 10 corresponds to **adding** one *bel* (= 10 decibels = 10 dB) to the volume of the sound.

Advantages of Decibels

* A very large range of power densities can be represented by small numbers
* Overall decibel gain can be calculated by summing the decibel gains of the individual components of the gain, just as with musical intervals
* The human perception of sound intensity is more nearly proportional to the logarithm of power density than to the power density itself
* The Richter scale for measuring earthquake intensity and the magnitude scale for measuring star brightness are also logarithmic scales

Sound Intensity in Decibels

	Power Density	Sound Intensity
Weakest sound heard	10^{-12} W/m²	0dB
Whisper Quiet Library	10^{-9} W/m²	30dB
Normal conversation (3-5 ft)	10^{-6} -10^{-5} W/m²	60-70dB
City Traffic (inside car)	3.162×10^{-4} W/m²	85dB
Subway train at 200 ft	3.162×10^{-3} W/m²	95dB
Level at which sustained exposure may result in hearing loss	1-3.162×10^{-3} W/m²	90-95dB
Power saw at 3 ft	0.1 W/m²	110dB
Loud Rock Concert	0.3162 W/m²	115dB
Pain begins	3.162 W/m²	125dB
Loudest recommended exposure WITH hearing protection	100 W/m²	140dB

Approximate Audible Intensity of Musical Instruments (≈2 meters)

Instrument	Sound Intensity
Violin	45-95dB
Trumpet	55-95dB
Piano	60-100dB
Trombone	85-115dB
Bass Drum	35-115dB
Cymbal	40-110dB

Music and Math Achievement

* “Most research shows that when children are trained in music at a young age, they tend to improve in their math skills. The surprising thing in this research is not that music as a whole is enhancing math skills. It is certain aspects of music that are affecting mathematics ability in a big way. Studies done mostly in children of young age show that their academic performance increases after a certain period of music education and training. One particular study published in the journal 'Nature' showed that **when groups of first graders were given music instruction that emphasized sequential skill development and musical games involving rhythm and pitch**, after six months, the students scored significantly better in math than students in groups that received traditional music instruction.”

From *The Correlation Between Music and Math: A Neurobiology Perspective*, by Cindy Zhan

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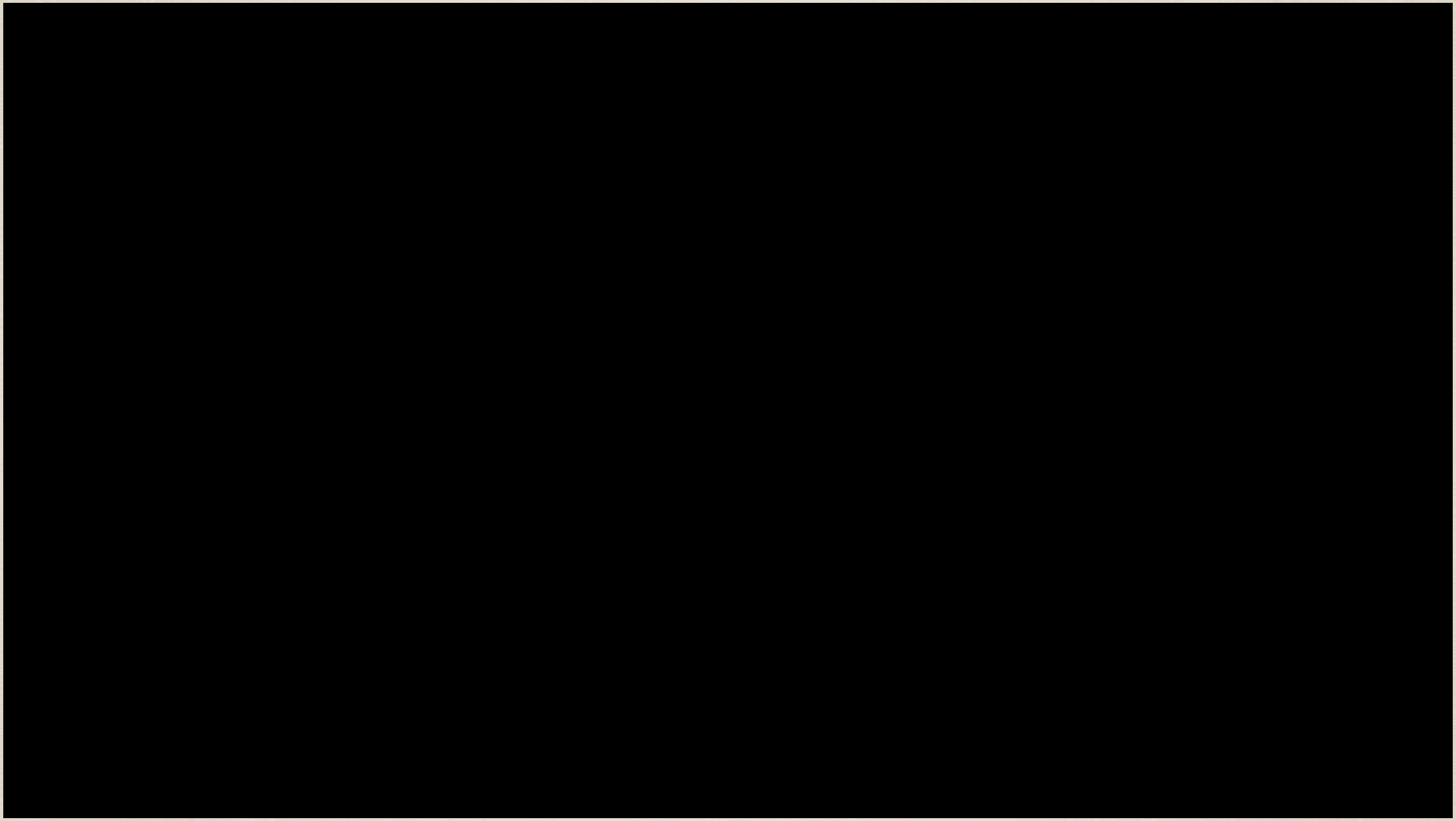
Dessert: A Video

* Stand up and get ready to sing pitches based on what you see Bobby McFerrin do!



Thank You!

I will post the slides, worksheets, a sheet of references, my major source article by the late Robert Osserman, and a sheet illustrating musical intervals using well-known songs.
Rate this session 374 at www.nctm.org/conapp

Send questions, requests or comments to Lew Douglas at lewdouglas@berkeley.edu or Jim Loats at loatsj@msudenver.edu