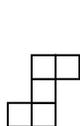
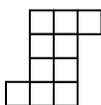


## THE TILE PROBLEM

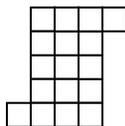
*The first 3 figures in a pattern of tiles are shown below.*



*Figure 1*



*Figure 2*



*Figure 3*

*Find the total number of tiles in the nth figure.*

### Approaches

#### Methods Using the Physical Arrangement of the Tile Sequence

##### Method 1

Fill in Figure 1 so that you have a 3 by 3 arrangement of tiles (shown below). You then have 9 tiles but to find the original number of tiles, you must subtract the tiles (shaded) that you added to fill in the first and last columns. So you have  $9 - 2(2) = 5$  tiles. Fill in Figure 2 so that you have a 4 by 4 arrangement of tiles. You then have 16 tiles but must subtract the  $2(3)$  tiles that you added. So you have  $16 - 2(3) = 10$  tiles. In Figure 3 you then have  $25 - 2(4) = 17$  tiles. Continuing with this pattern, the number of tiles in the 6<sup>th</sup> figure can be obtained by visualizing an 8 by 8 arrangement of tiles in which you've added  $2(7)$  tiles to fill out the first and last columns. So you have a total of  $8 \times 8 - 2(7) = 64 - 14 = 50$  tiles in our original figure.



Figure 1

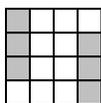


Figure 2

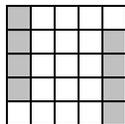


Figure 3

Now you can generalize the pattern. Fill in the nth figure so that you have an  $n+2$  by  $n+2$  arrangement of tiles. So you have  $(n+2)(n+2)$  tiles. To fill out the figure, you've added  $2(n+1)$  tiles ( $n+1$  tiles to the all but the last position in the first column and  $n+1$  tiles to all but the first position in the last column). To get the original number of tiles, you must subtract  $2(n+1)$  from the  $(n+2)(n+2)$  tiles. So you will have  $(n+2)(n+2) - 2(n+1)$  tiles in the nth figure, and these tiles are arranged as a square of dimension  $n+2$

with  $n+1$  tiles removed from the top of the first column and  $n+1$  tiles removed from the bottom of the last column.

Method 2

Figure 1 has 1 center tile and 2 rows of 2 tiles. Figure 2 has a center of  $2 \times 2$  tiles and 2 rows of 3 tiles. Figure 3 has a center of  $3 \times 3$  tiles and 2 rows of 4 tiles. So, the 6<sup>th</sup> figure would have a center of  $6 \times 6$  tiles and 2 rows of 7 tiles for a total of  $36+14 = 50$  tiles.

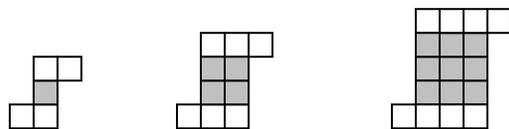


Figure 1      Figure 2      Figure 3

According to the pattern described above, the  $n$ th figure would have a center of  $n \times n$  tiles and 2 rows of  $n+1$  tiles for a total of  $n^2 + 2(n+1)$  tiles.

Method 3

Figure 1 has a  $1 \times 3$  rectangle of tiles with 1 tile added to the bottom row and 1 tile added to the top row. Figure 2 has a  $2 \times 4$  rectangle of tiles with 1 tile added to the bottom row and 1 tile added to the top row. Figure 3 has a  $3 \times 5$  rectangle with 1 tile added to the bottom row and 1 tile added to the top row. Figure 6 would then have a  $6 \times 8$  rectangle with 1 tile added to the bottom row and 1 tile added to the top row, for a total of  $48 + 2 = 50$  tiles.

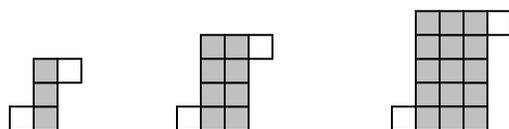


Figure 1      Figure 2      Figure 3

Following the pattern described above, the  $n$ th figure would have a  $n \times (n+2)$  rectangle with 1 tile added to the bottom row and 1 tile added to the top row, for a total of  $n(n+2) + 2$  tiles.

Method 4

In Figure 1, move the top row to fit under the bottom row to form a  $2 \times 2$  rectangle of tiles plus 1 tile. In Figure 2, move the top row to fit under the bottom row to form a  $2 \times 3$  rectangle and a square of  $2 \times 2$  tiles. In Figure 3, move the top row to fit under the bottom row to form a  $2 \times 4$  rectangle and a  $3 \times 3$  square of tiles. In the 6<sup>th</sup> figure I would have a  $2 \times 7$  rectangle and a  $6 \times 6$  square of tiles for a total of 50 tiles.

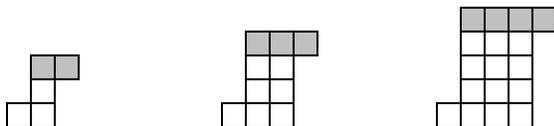


Figure 1

Figure 2

Figure 3

In the  $n$ th figure, I would have a 2 by  $n+1$  rectangle and a  $n$  by  $n$  square of tiles for a total of  $2(n+1) + n^2$ .

Method 5

To go from Figure 1 to Figure 2, fill in the last column with 2 tiles and add a top row of 3 tiles, for a total of 5 tiles. To go from figure 2 to figure 3, fill in the last column with 3 tiles and add a top row of 4 tiles, for a total of 7 tiles. To go from figure 3 to figure 4, fill in the last column with 4 tiles and add a top row of 5 tiles.

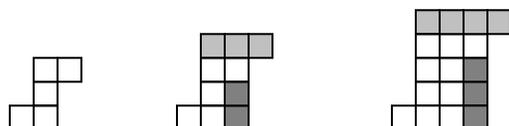


Figure 1

Figure 2

Figure 3

So, to go from figure  $n$  to figure  $n+1$ , fill in the last column with  $n+1$  tiles and add a top row of  $n+2$  tiles. Each time I am adding  $n+1+n+2 = 2n+3$  tiles in total. So I could say that I get the next number of tiles from adding  $2n+3$  to the  $n$ th number of tiles. To put the process in more formal mathematical terms,

$$F_{n+1} = F_n + (n+1) + (n+2) = F_n + 2n + 3.$$

So, to get the number of tiles in the  $n+1$ <sup>st</sup> figure, I add  $2n+3$  tiles to the  $n$ th figure, beginning with the fact that

$$F_1 = 5.$$

This is a recursively defined function or sequence. A recursively defined sequence is a sequence for which the first term or the first few terms are given and each successive term in the sequence can be computed using a formula (called a recurrence relation)

involving one or more of the preceding terms. For example, using our sequence as an example, the first term is given ( $F_1 = 5$ ) and the recurrence relation is  $F_{n+1} = F_n + 2n + 3$ .

### Summary and Discussion

In each of the methods described above, the student has used the physical arrangement of tiles to obtain the number of tiles in Figure 6 and then used the physical arrangement of tiles to generalize the solution process for the number of tiles in the  $n$ th figure. Pedagogically such a problem provides the opportunity for students to see visually that different-looking algebraic expressions are equivalent. Consider the formulas produced by each of the methods:

Method 1:  $f(n) = (n+2)(n+2) - 2(n+1)$

Method 2:  $f(n) = n^2 + 2(n+1)$

Method 3:  $f(n) = n(n+2) + 2$

Method 4:  $f(n) = 2(n+1) + n^2$

Method 5:  $F_1 = 5$

$$F_{n+1} = F_n + (n+1) + (n+2)$$

Each of the formulas from methods 1 through 4 can be converted algebraically to  $n^2 + 2n + 2$ . Note also that the relationship between the figure number and the number of tiles is a function, a function with a discrete domain; that is, the domain is  $\{1, 2, 3, 4, \dots\}$  and the range is the sequence 5, 10, 17, 26, ...

Note that the formula produced in method 5 differs significantly from the formulas produced in methods 1 through 4. Methods 1 through 4 produced a “closed” or “explicit” formula; in such a formula, when you input the figure number  $n$ , the formula outputs the number of tiles in the  $n$ th figure. In contrast, method 5 produced a “recursive” or “implicit” formula for the number of tiles in the  $n$ th figure. A recursive formula has two parts: an initial condition (the number of tiles in the first figure) and a recurrence relation (a formula that uses the number of tiles in the previous figures to obtain the number of tiles in the next figure). For example, let’s say you wanted to know the number of tiles in the sixth figure? All you need to do with the explicit formula is replace  $n$  with 6. The metaphor of a box with an input and output is often used for an explicit formula. Using the recursive formula, however, you must start with the number of tiles in the first figure, add the appropriate number of tiles to get the number of tiles in the second figure, add the appropriate number of tiles to the second figure to get the number of tiles in the third figure ... To use the recursive formula, you need to go through the whole sequence of numbers.

### **A Simple Method Using the Sequence of Numbers**

#### Method 6

Figure 1 has 5 tiles, figure 2 has 10 tiles, figure 3 has 17 tiles. The next figure would have 26 tiles (by drawing it), and the next figure would have 37 tiles, and the next would have 50 tiles. The figures picture the sequence of numbers 5, 10, 17, 26, 37, 50 ... It appears that each number in the sequence is one more than a perfect square; that is,  $4+1$ ,  $9+1$ ,  $16+1$ ,  $26+1$ , ... or  $2^2+1$ ,  $3^2+1$ , ... So the  $n$ th figure would have  $(n+1)^2 + 1$  tiles.

This solution is based purely on the observation that each number in the sequence is one more than a perfect square, but can I verify visually, using our figures, that each figure does indeed consist of a perfect square plus one tile? Yes! Rearrange the tiles as shown below:

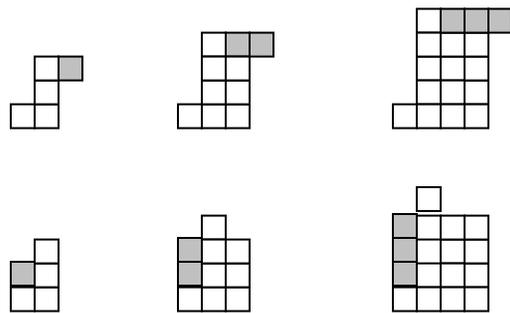


Figure 1

Figure 2

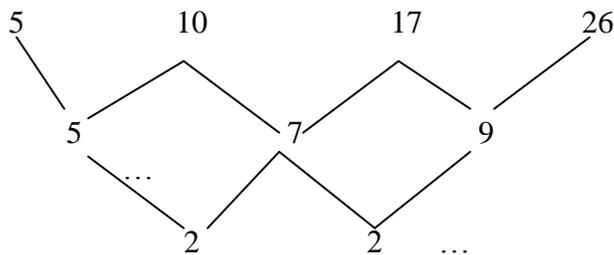
Figure 3

**More Complex Methods  
Using the Sequence of Numbers**

Figure 1 has 5 tiles, Figure 2 has 10 tiles, Figure 3 has 17 tiles. The next figure would have 26 tiles (by drawing it), and the next figure would have 37 tiles. The figures picture the sequence of numbers 5, 10, 17, 26, 37 ... The next three methods describe ways of finding the  $n$ th term in that sequence of numbers.

Method 7: The Method of Finite Differences

Consider the difference between successive differences in the terms of the sequence as shown below:

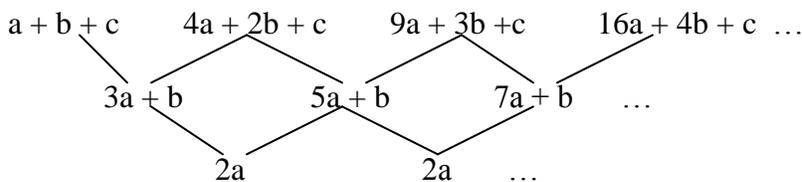


The “first difference” for this sequence is 5, 7, 9, 11, ... (that is, the differences in terms of the original sequence). The “second difference” is a constant 2. Since the second difference is a constant, I know that the function that describes this sequence is quadratic.

So, I know that the number of tiles in the nth figure will be of the form  $an^2 + bn + c$ . In other words, the number of tiles is a quadratic function of the form

$$f(n) = an^2 + bn + c.$$

So figure 1 has  $f(1) = a + b + c$  tiles, and figure 2 has  $f(2) = 4a + 2b + c$  tiles, and figure 3 has  $f(3) = 9a + 3b + c$ , ... In other words, our sequence of numbers looks like the following:



We also know that  $f(1) = 5$ ,  $f(2) = 10$ ,  $f(3) = 17$ ,  $f(4) = 26$ , ... and that the first difference is 5, 7, 9, ... and that the second difference is a constant 2. Therefore,  $2a=2$ , so  $a = 1$ . In addition,  $3a + b = 5$  and since  $a=1$ ,  $3(1) + b = 5$  so  $b = 2$ . Since  $a+b+c = 5$  and  $a=1$  and  $b=2$ ,  $c=2$ . Therefore, the quadratic function I seek is  $f(n) = n^2 + 2n + 2$ . Note that  $n^2 + 2n + 2$  is equivalent to the expressions obtained in previous methods.

### Method 8: System of Linear Equations

When  $n=1$  I want  $f(1) = 5$ , and when  $n=2$  I want  $f(2) = 10$ , and when  $n=3$  I want  $f(3) = 17$ . I can use this information to obtain the following system of three linear equations in three unknowns:

$$\begin{aligned} a + b + c &= 5 \\ 4a + 2b + c &= 10 \\ 9a + 3b + c &= 17 \end{aligned}$$

Now I have a variety of methods to use to solve this system of linear equations.

I can also use some higher level mathematics to solve this system of equations. The system can be written in matrix form as shown below:

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 17 \end{bmatrix}$$

I am thus solving an equation of matrices of the form  $Ax = B$ . If  $A$  and  $B$  were simply numbers, I would multiply both sides by the multiplicative inverse of  $A$  and solve  $x$  in the form  $x = A^{-1}B$ . This can be done in TI-Interactive by using the “Math Box” and its associated palette of functions.

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 5 \\ 10 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Thus, our answer is  $a = 1$ ,  $b = 2$ , and  $c = 2$ .

The same approach could be used with a TI-83, 84, or 89. Furthermore, students who have had a course in Linear Algebra could solve the system without the use of technology.

#### Method 9: Quadratic Regression

You could use TI-Interactive to calculate instantly the quadratic function that fits the points (1,5), (2,10), and (3,17). Open the list tool in TI-Interactive (the spreadsheet-type tool that has columns labeled L1 and L2). Type in 1, 2, 3 under column L1 and 5, 10, 17 in column L2. Using the “Stat Calculation Tool” request quadratic regression.

| L1 | L2 |
|----|----|
| 1  | 5  |
| 2  | 10 |
| 3  | 17 |
|    |    |

Quadratic Regression  
 $\text{regEQ}(x) = 1.x^2 + 2.x + 2.$

The same quadratic regression can be accomplished with a graphing calculator.

### **A Method Using Sums and Gauss’ Trick**

#### Method 10: Gauss

Consider the table shown below

| Number of term | Term                        |
|----------------|-----------------------------|
| 1              | 5                           |
| 2              | 5 + 5                       |
| 3              | 5 + 5 + 7                   |
| 4              | 5 + 5 + 7 + 9               |
| 5              | 5 + 5 + 7 + 9 + 11          |
|                | ...                         |
| n              | $5 + \sum_{i=2}^n (2i + 1)$ |

Our sequence can actually be defined in terms of sums: 5, 5+5, 5+5+7, 5+5+7+9, ... These are partial sums of an infinite series, yet another way to represent the terms of the sequence. There is a technique for finding a closed formula for the terms in such a sum; it is names Gauss’ method after the great mathematician. To calculate the sum of 5 + 7 + 9 + 11 + 13 +15, for example, I could apply the procedure shown below:

$$\begin{array}{rcccccc}
 5 & 7 & 9 & 11 & 13 & 15 \\
 15 & 13 & 11 & 9 & 7 & 5 \\
 \hline
 20 & 20 & 20 & 20 & 20 & 20
 \end{array}$$

So, to find the 7<sup>th</sup> term:

$$5 + 6(20)/2 = 65$$

In other words,

$$\begin{array}{rcccccc}
 5 & 7 & 9 & 11 & \dots & 2n+1 \\
 2n+1 & 2(n-1) + 1 & 2(n-2) + 1 & 2(n-1) + 1 & \dots & 5 \\
 \hline
 2n+6 & 2n + 6 & 2n + 6 & 2n + 6 & \dots & 2n + 6
 \end{array}$$

So to find the nth term:

$$5 + (n-1)(2n+6)/2$$

Which is equal to  $n^2 + 2n + 2$ .

**Summary Impressions**

Middle grades and secondary mathematics problems often consist of a situation that yields a specific number; this problem is different in that the answer wasn’t a specific number but an expression. There were really two general ways of approaching this problem: you could use the picture to reason directly to the desired expression (methods 1

through 5) or you could use the sequence implied by the picture (methods 6 through 10). Through methods 6 through 10, the problem is connected to matrices, systems of linear equations, and regression techniques. The desired formula can be represented in closed form or recursive form in a number of different ways; indeed, all the possibilities for the closed form and the recursive form were not presented but since the approach would be essentially the same, all possibilities were not pursued.

## Generalization and Analysis

In our generalization and analysis, I want to replace the quantities that are given (the inputs) with variables. For the Tile Problem, that means I want to replace the terms of the sequence 5, 10, 17, 26, ... with a variable sequence such as  $y_1, y_2, y_3, y_4, \dots$  that has a constant second difference. I then perform the same procedures on those variables as I did on the original quantities in the original problem. The result is a formula for the solution to this problem regardless of the inputs.

In the analysis of my generalization, I want to consider how changes in the given quantities influence the answer to the problem; in other words, I want to analyze the solution as a function of each of the inputs, keeping other inputs constant. I also examine the relationship between other pairs of quantities in the solution. To examine the solution as a function of each of the inputs and the relationship between other pairs of quantities, I use all I know about characteristics of functions – domain, range, rate of change, maxima and minima, increasing and decreasing, symmetry, end behavior, intercepts, asymptotes, inverses, etc. I will relate these properties of functions to the specific context of the problem.

### Generalization for the Closed Form

My goal then is to generalize the Method of Finite Differences for a sequence of numbers that has a constant second difference. I must take the procedure I used for specific sequences, replace the terms of those sequences by variables, and then repeat the same procedures to form a general solution. So, given a sequence of numbers  $y_1, y_2, y_3, y_4, \dots$  that has a constant second difference, a quadratic function of the form  $an^2 + bn + c$  can be obtained to describe the  $n$ th term in the sequence.

Using the Method of Finite Differences,  $y_1 = a + b + c$  tiles,  $y_2 = 4a + 2b + c$  tiles, and  $y_3 = 9a + 3b + c$ , ...

$$\begin{array}{cccc}
 y_1 = & y_2 = & y_3 = & y_4 = \\
 a + b + c & 4a + 2b + c & 9a + 3b + c & 16a + 4b + c \dots \\
 & \swarrow & \swarrow & \swarrow \\
 & y_2 - y_1 = & y_3 - y_2 = & y_4 - y_3 = \\
 & 3a + b & 5a + b & 7a + b \dots \\
 & \swarrow & \swarrow & \swarrow \\
 & y_3 - 2y_2 + y_1 = & y_4 - 2y_3 + y_2 = & \dots \\
 & 2a & 2a & \dots
 \end{array}$$

Therefore,  $2a = y_3 - 2y_2 + y_1$ , and solving for  $a$  gives

$$a = (y_3 - 2y_2 + y_1)/2 \text{ or}$$

$$a = .5y_1 - y_2 + .5y_3.$$

Since  $3a + b = y_2 - y_1$ , substituting for  $a$  and solving for  $b$  yields

$$3(.5y_1 - y_2 + .5y_3) + b = y_2 - y_1 \text{ and then}$$

$$b = -2.5y_1 + 4y_2 + -1.5y_3$$

Also, since  $a + b + c = y_1$ , substituting for  $a$  and  $b$  and solving for  $c$  yields

$$c = 3y_1 + -3y_2 + y_3.$$

So, given a sequence of numbers  $y_1, y_2, y_3, y_4, \dots$  that must have a constant second difference, a quadratic function of the form  $an^2 + bn + c$  can be obtained to describe the  $n$ th term in the sequence such that

$$a = .5y_1 - y_2 + .5y_3$$

$$b = -2.5y_1 + 4y_2 + -1.5y_3, \text{ and}$$

$$c = 3y_1 + -3y_2 + y_3.$$

Thus, I have derived a general formula for the closed form of the quadratic function that describes the sequence. Note that the result is not as important as the process described to arrive at the result. Note also that you could test this formula on our original sequence 5, 10, 17, 26, ... as a quick check to make sure we've made no simple arithmetic mistakes. If I substitute 5, 10, 17 into  $y_1, y_2,$  and  $y_3$ , I get  $a = 1, b = 2, c = 2$ .

What are the domains of each of the variables in the solution? The given sequence  $y_1, y_2, y_3, y_4, \dots$  can be any real numbers as long as they have a constant second difference. Note that the fact they the sequence must have a constant second difference means that I could randomly pick three terms in the sequence but the remaining terms would then be determined by those three terms. Our solution is written in terms of the first three terms of the sequence. The solution  $a, b, c$  can also be any real numbers.

### Generalization for the Recursive Form

I have an adequate generalization given in the previous section and that is the generalization I'll analyze, but the question comes to mind: "What about a recursively defined form of the function?" Well, the initial term,  $F_1$ , would simply correspond to the first term of the sequence,  $y_1$ . How could I obtain the recurrence relation? Consider again the original sequence and the first and second differences:



$$\begin{array}{ccccc}
 y_2 - y_1 & & y_3 - y_2 & & y_4 - y_3 \\
 & \diagdown & / & \diagdown & / \\
 & y_3 - 2y_2 + y_1 & & y_4 - 2y_3 + y_2 & 
 \end{array}$$

The second difference is a constant  $D = y_3 - 2y_2 + y_1$  so

or

$$y_{n+1} = 2y_n - y_{n-1} + D$$

and thus the recurrence relation could be formed in terms of two previous terms. Let's test this on my original sequence 5, 10, 17, 26, ... to make sure I've made no simple arithmetic mistakes. I would have

$$\begin{aligned}
 y_1 &= 5 \text{ and } y_2 = 10, \text{ then} \\
 y_3 &= 2(10) - 5 + 2 = 17 \\
 y_4 &= 2(17) - 10 + 2 = 34 - 10 + 2 = 26 \\
 y_5 &= 2(26) - 17 + 2 = 52 - 17 + 2 = 37 \\
 &\text{etc.}
 \end{aligned}$$

Alternately, since the second difference is constant, the  $n+1^{\text{st}}$  term in the sequence can be obtained by taking  $y_2 - y_1$  and adding the second difference  $n-1$  times. Thus, our recurrence relation is

$$y_{n+1} = y_n + (y_3 - 2y_2 + y_1)(n-1) + (y_2 - y_1).$$

Let's test this on my original sequence 5, 10, 17, 26, ... as a check that we've made no simple arithmetic mistakes. I would have

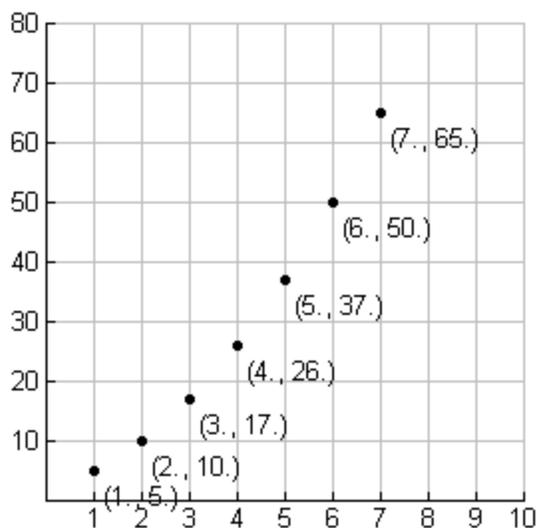
$$\begin{aligned}
 y_1 &= 5. \\
 y_2 &= 5 + (17-20+5)(0) + (10-5) = 5+5 = 10 \\
 y_3 &= 10 + 2(1) + 5 = 17 \\
 &\text{etc.}
 \end{aligned}$$

### Analysis of Generalization

Now I'm ready to tackle an analysis of the generalization. Recall that the solution to the tile problem was the function:

$$f(n) = n^2 + 2n + 2$$

Since this function is being used to produce a particular sequence of numbers  $\{5, 10, 17, 26, \dots\}$ , the domain is  $\{1, 2, 3, 4, \dots\}$  or  $\mathbb{N}$ , the natural numbers. A graph and table of values can be produced using TI-Interactive.



Note that the graph of the function is a portion of a parabola, since I know the function is a quadratic function. Note that the dots representing the points on the graph of the function get farther apart as  $n$  increases; this is because the rate of change of the function is increasing – the first difference gets larger and larger as  $n$  increases.

Note that I still don't have an idea of how the solution changes with changes in the input. My input is a sequence  $y_1, y_2, y_3, y_4, \dots$  that has a constant second difference. As already stated, the  $n$ th term of the sequence can be obtained using the function

$$f(n) = an^2 + bn + c,$$

but note that  $a, b, c$  depend on the inputs as follows:

$$a = .5y_1 - y_2 + .5y_3$$

$$b = -2.5y_1 + 4y_2 - 1.5y_3, \text{ and}$$

$$c = 3y_1 - 3y_2 + y_3.$$

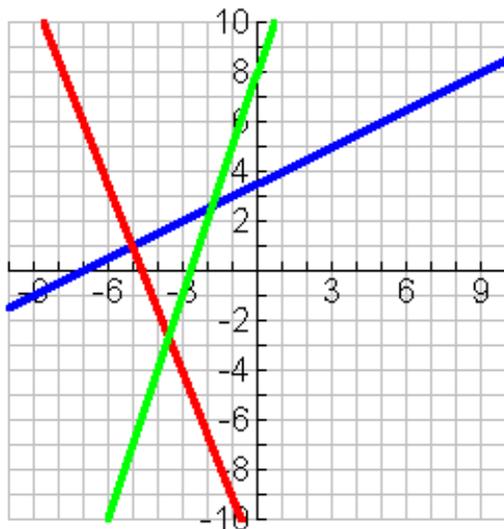
I ultimately want to see how the function  $f(n) = an^2 + bn + c$  depends on the original inputs, the first three terms of the sequence. I've characterized how  $f(n)$  depends on  $a, b, c$ . Now I could characterize separately how  $a$  depends on the first three terms, how  $b$  depends on the first three terms and how  $c$  depends on the first three terms. For example, suppose that I make the first term of the sequence a variable, while I hold the second term at -1 and the third term at 5. Usually in this problem analysis, I would use our original inputs and hold the second term at a value of 10 and third term at a value of 17, but it really doesn't matter what those original terms are... so I picked some smaller values that are easier to work with. Then

$$a = 3.5 + .5y_1 \text{ (blue)}$$

$$b = -11.5 - 2.5y_1 \text{ (red)}$$

$$c = 3y_1 + 8 \text{ (green)}$$

Thus, a, b, and c are all linear functions of the first term in the sequence. Using the software *TI Interactive* I can examine a graph of each of the functions.



| x  | a    | b     | c  |
|----|------|-------|----|
| 1  | 4    | -14   | 11 |
| 2  | 4.5  | -16.5 | 14 |
| 3  | 5    | -19   | 17 |
| 4  | 5.5  | -21.5 | 20 |
| 5  | 6    | -24   | 23 |
| 6  | 6.5  | -26.5 | 26 |
| 7  | 7    | -29   | 29 |
| 8  | 7.5  | -31.5 | 32 |
| 9  | 8    | -34   | 35 |
| 10 | 8.5  | -36.5 | 38 |
| 11 | 9    | -39   | 41 |
| 12 | 9.5  | -41.5 | 44 |
| 13 | 10   | -44   | 47 |
| 14 | 10.5 | -46.5 | 50 |

How does a depend on the first term? The x intercept is  $(-7, 0)$  and the y intercept is  $(0, 3.5)$ ; in the context of the problem, these intercepts indicate that when the first term is less than  $-7$ , a will be negative, and when the first term is larger than  $-7$ , a will be positive. If the first term is  $0$ , then  $a = 3.5$ . Note that the values of  $-7$  and  $3.5$  will change depending on the values of the second and third terms in the sequence; in other words, the second and third terms will cause a vertical shift in the line, moving it up or down. Since the slope is  $.5$ , a increases with increasing values of  $y_1$ ; to be more specific, for every increase of  $1$  in  $y_1$ , there must be an increase of  $\frac{1}{2}$  in a. Note that this is true regardless of the value of  $y_2$  and  $y_3$ , since  $.5$  is always the coefficient of  $y_1$ .

How does b depend on the first term? The x intercept is  $(-4.6, 0)$  and the y intercept is  $(0, -11.5)$ ; in the context of the problem, these intercepts indicate that when the first term is less than  $-4.6$ , b will be positive, and when the first term is larger than  $-4.6$ , b will be negative. If the first term is  $0$ , b will be  $-11.5$ . Since the slope is  $-2.5$ , for every increase of  $1$  in  $y_1$  there must be a decrease of  $2.5$  in b.

How does c depend on the first term? The x intercept is  $(-8/3, 0)$  and the y intercept is  $(0, 8)$ ; in the context of the problem, these intercepts indicate that when the first term is less than  $-8/3$ , c will be negative, and when the first term is larger than  $-8/3$ , c will be positive. If the first term is  $0$ , b will be  $8$ . Since the slope is  $3$ , for every increase of  $1$  in  $y_1$  there must be a decrease of  $3$  in c.

Note that regardless of what the second and third terms are, the slopes of all three functions will remain the same, and the only difference will be in the location of the vertical axis intercept and thus the placement of the line. Also note that the value that

changes the most on changes in the first term is  $c$ , and the value that changes the least is  $a$ .

In examining my general solution,

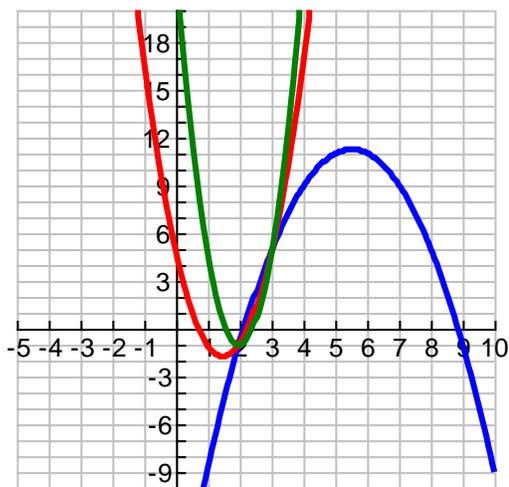
$$a = .5y_1 - y_2 + .5y_3$$

$$b = -2.5y_1 + 4y_2 + -1.5y_3, \text{ and}$$

$$c = 3y_1 + -3y_2 + y_3.$$

I see that an analysis of  $a$ ,  $b$ , and  $c$  as functions of the second term or as functions of the third term would be similar to the analysis of  $a$ ,  $b$ , and  $c$  as functions of the first term. In general, the value of  $a$  would decrease by 1 for every unit increase in the second term and increase by .5 for every unit increase in the third term. Similarly,  $b$  would increase by 4 for every unit increase in the second term and decrease by 1.5 for every unit increase in the third term; and  $c$  would decrease by 3 for every unit increase in the second term and increase by 1 for every unit increase in the third term. Note also, if I'm examining  $a$ ,  $b$ , and  $c$  as functions of the first term, if  $y_2$  is increased, the lines describing  $a$  and  $c$  will be shifted downward, while the line describing  $b$  will be shifted upward. If  $y_3$  is increased, the lines describing  $a$  and  $c$  will be shifted upward, while the line describing  $b$  will be shifted downward.

To get another view of the influence of  $y_1$  on  $a$ ,  $b$ , and  $c$ , shown below are the three functions obtained if  $y_2 = -9$ ,  $y_3 = -1$  and  $y_1 = 5$ , but  $y_2$  and  $y_3$  are held constant at  $-1$  and  $5$  respectively.



$$\begin{aligned} f(x) &:= -x^2 + 11x - 19 \text{ (blue)} \\ g(x) &:= 3x^2 - 9x + 5 \text{ (red)} \\ h(x) &:= 6x^2 - 24x + 23 \text{ (green)} \end{aligned}$$

As expected, with the first term being less than  $-7$ ,  $a$  and  $c$  were negative and  $b$  was positive and the parabola was opening downward. Note also that the sequence produced by this parabola would be  $-9, -1, 5, 9, 11, 11, 9, 5, \dots$ . So the sequence is increasing, has two consecutive terms of the same value and then is decreasing. The sequence produced with the first term being  $5$ , however, decreases and then increases sharply ( $5, -1, 5, 23, 53, 95, \dots$ ).

The second part of this analysis would be to explore the relationship between the various inputs. In the context of this problem, I would want to explore the relationships between the terms in the sequence. The relationship between the terms, however, is very specific in this case – the terms must have a constant second difference. So, the relationship between the terms is actually provided by the second generalization:

$$y_{n+1} = 2y_n - y_{n-1} + D$$

or keeping to the inputs as being the first three terms in the sequence,

$$y_3 = 2y_2 - y_1 + D$$

and keeping  $D = 2$ ,

$$y_3 = 2y_2 - y_1 + 2$$

I could then analyze  $y_3$  as a function of  $y_2$ , or  $y_3$  as a function of  $y_1$ , or  $y_2$  and a function of  $y_3$ , or the inverses of each of those functions, but I can see that all functions are linear. I can also see that  $y_3$  will increase by 2 units for every unit increase in  $y_2$  and will decrease by 1 for every unit increase in  $y_1$ .

### Summary Impressions

Although the function produced by the Tile Problem is a quadratic function, the relationships among inputs and the coefficients of the resulting quadratic function are uniformly linear.

## Concepts

In this section, I begin with an overview of the mathematical concepts that underlie the Tile Problem, and then choose one particular concept to explore deeply. To complete this section and the next section, I had to consult a variety of texts and articles.

To be successful with the problem, a student needs to be able to recognize patterns, to perform the usual arithmetic operations, and to use a variable. Recognition of patterns and performance of the arithmetic operations are stressed in elementary school. Use of variables is also introduced in elementary school, but students do not work with variables regularly until middle school, and often have a great deal of trouble with the concept. By solving the Tile Problem, the student creates a quadratic function, which is introduced in middle school and then explored very deeply throughout high school and college. That function could be represented verbally, graphically as a parabola, algebraically in a variety of forms, and in a table. Depending on the method used to solve the problem, the student could use a variety of additional concepts: recursive forms of a function, explicit forms of a function, series, regression, systems of linear equations, the method of finite differences. One problem can lead to so many mathematical concepts! The concept I'll explore in the rest of this section is the concept of a parabola.

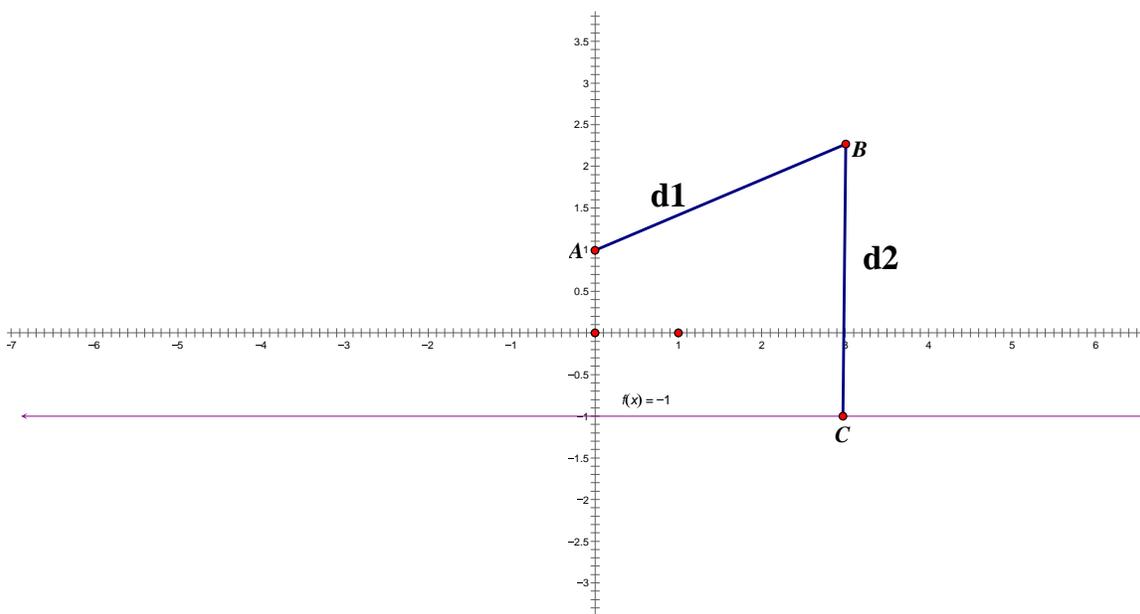
### Definitions of Parabola

What is a parabola? To answer that question, consider the following quotes from the internet and a variety of textbooks at different levels.

1. "In mathematics, the parabola is a conic section, the intersection of a right circular conical surface and a plane parallel to a generating straight line of that surface." From <http://en.wikipedia.org/wiki/Parabola>.
2. "A parabola is the set of all points in the plane equidistant from a given line (the directrix) and a given point not on the line (the focus)." From <http://mathworld.wolfram.com/Parabola.html>.
3. "The graph of a quadratic function is known as a parabola and has a distinctive shape that occurs often in nature." From *Functions and Change: A Modeling Approach to College Algebra* by Crauder, Evans, Noell, 2007. This is a text used at Kennesaw State University in a college algebra course.
4. "Quadratic relationships are characterized by their U-shaped graphs which are called parabolas." From *Frogs, Fleas, and Painted Cubes. Quadratic Relationships. Connected Math 2.* by Lappan, Fey, Fitzgerald, Friel, Phillips, 2006. This is a middle school text.
5. "A parabola is the graph of an equation where one of the variables is squared, or in other words, raised to the power of 2 (such as  $y = x^2$ )." From *The Essentials of High School Math. Covering the Standards Set by the National Council of Teachers of Mathematics* by Davis, 2009. This is a book for high school students.
6. "Quadratic functions are those that are of the form  $y=f(x)=ax^2+bx+c$  where  $a \neq 0$ . The graphs of these functions are parabolas." From *The Mathematics that every*

*Secondary School Math Teacher Needs to Know* by Sultan and Artzt, 2011. This is a book for college students preparing to be secondary math teachers.

Are these definitions equivalent? Definition #1 defines a parabola as a conic section; definition #2 defines it as a locus of points in the plane; and definitions 3, 4, 5, and 6 each associate a certain kind of function or equation with the parabola. Definitions 1 and 2 are very geometric, whereas the remaining definitions are very algebraic. Are they equivalent? To test equivalence, I need to be able to take the description in definition #2 and turn it into an equation. Consider the picture below, showing a line  $f(x) = -c$  (a directrix) strategically placed, and a point  $(0,c)$  (a focus) also strategically placed.



We want  $d1$  to be equal to  $d2$ . Let the coordinates of  $B$  be  $(x,y)$ . I know the coordinates of  $A$  are  $(0,c)$  and the coordinates of  $C$  are  $(x,-c)$ . Therefore,

$$d1 = \sqrt{(x - 0)^2 + (y - c)^2}$$

$$d2 = \sqrt{(x - x)^2 + (y + c)^2}$$

Since  $d1$  is equal to  $d2$ , I can set the two right-hand sides equal and square both to get

$$x^2 + (y-c)^2 = (y+c)^2$$

which is equivalent to

$$x^2 + y^2 - 2cy + c^2 = y^2 + 2cy + c^2$$

which is equivalent to

$$x^2 - 2cy = 2cy$$

or

$$y = \frac{1}{4c}x^2$$

This is a quadratic function with vertex at (0,0). The above sequence of steps is reversible, so if I begin with the equation I can show that it must represent a set of points equidistant from the directrix and focus. Note that I could generalize this to any quadratic function by a vertical and horizontal shift. Also, I could generalize to any directrix and any point by rotations of the axes. Note that this analysis appeared in an article from the *Mathematics Teacher* entitled “On Enlarging the Focal Point of a Parabola” by Dence, Volume 98, Number 9, 2005. In that article, the author gives definitions 1 and 2 as the definitions of a parabola.

Thus, the geometric definitions given in 1 and 2 are consistent with the algebraic definitions given in the remaining definitions. Note that in definition 6 it is stated “the graphs of these functions are parabolas” but the converse is not true – parabolas are the graphs of these functions. If x and y are reversed in the above equation, the parabola would lie along the x axis and y would no longer be a function of x! Also in definition 4, it does not say that ALL u-shaped curves are parabolas; it does say that quadratic relationships have u-shaped curves that are parabolas. Students often jump to the conclusion that anything u-shaped is a parabola; this is not true!

### Representations and Properties of Parabolas

Parabolas are a locus of points; they are points in the coordinate plane that represent a certain algebraic relationship between x and y. The parabolas could lie along the y-axis, along the x-axis, or along any line in the plane. The parabolas can be represented algebraically, graphically, and in tabular form.

In tabular form, a parabola could be identified by the constant second difference. In other words, if asked which of the following variables represents a parabola, we would have to answer z because that is the only one with a constant second difference. Another way of saying the same thing would be that the rate of change of a parabola is linear.

| x | y  | g  | z  |
|---|----|----|----|
| 1 | -2 | -2 | -2 |
| 2 | 0  | -2 | -2 |
| 3 | 2  | 9  | 0  |
| 4 | 4  | 18 | 4  |
| 5 | 6  | 35 | 10 |
| 6 | 8  | 56 | 18 |
|   |    |    |    |

Graphically, parabolas are u-shaped. Let's consider only parabolas that are quadratic functions; that is, parabolas that have an axis of symmetry parallel to the y axis. They are typically the first non-linear functions that students study in detail. One important property of the graph of the parabola is the vertex, typically called  $(h,k)$ , the point where the parabola has a maximum or minimum. Another important property is that the rate  $y$  changes with respect to  $x$  is not constant; that is, the graph is most shallow around the vertex and most steep at the ends of the parabola, indicating that the rate  $y$  is changing with respect to  $x$  is close to 0 near the vertex and far from 0 far from the vertex. If the vertex is a maximum, then the function is increasing for  $x < h$  and is decreasing for  $x > h$ . If the vertex is a minimum, then the function is decreasing and then increasing.

Intercepts are also an important feature of graphs of quadratic functions, particularly the  $x$  intercepts. There could be one  $x$  intercept (so that the parabola touches the  $x$  axis at its vertex), two  $x$  intercepts, or no  $x$ -intercepts. These graphical features as well as others are intimately connected with the algebraic representation of quadratic functions and will be discussed after presentation of the algebraic forms of the parabola.

Algebraically, parabolas can be represented as follows:

- As quadratic functions in standard form

$$f(x) = ax^2 + bx + c \quad \text{where } a, b, \text{ and } c \text{ are constants and } a \text{ is not } 0.$$

- As quadratic functions in vertex form

$$f(x) = a(x-h)^2 + k \quad \text{where } a, h, k \text{ are constants and } a \text{ is not } 0, \text{ and } (h,k) \text{ is the vertex.}$$

- As quadratic functions in conic section form

$$y - k = (1/p)(x-h)^2 \quad \text{where } p, h, k \text{ are constants and } p \text{ is not } 0, \text{ and } (h,k) \text{ is the vertex, } p \text{ is the distance from the vertex to the focus (or the directrix). In this form, we could interchange } x \text{ and } y \text{ and find the equation for a parabola that lies along a horizontal axis of symmetry and is not a function.}$$

- As quadratic functions in factored form

$$y = a(x-d)(x-e) \quad \text{where } a, d, e \text{ are constants, and } d \text{ and } e \text{ represent the } x\text{-intercepts of the graph of the function or are complex numbers and there are no } x \text{ intercepts.}$$

- As a quadratic relation in polar form

$$r = \frac{d}{1 \pm \sin(\theta)} \text{ or } r = \frac{d}{1 \pm \cos(\theta)}$$

where the sin is used for a vertical axis and the cos is used for a horizontal axis and the focus is at the origin.

All of the equations of a parabola given above assume that the axis of symmetry is parallel to the y axis (if the parabola represents a function) or parallel to the x axis (in which case the parabola does not represent a function). The axis of symmetry could be rotated so that it is parallel to a line different from  $x=0$  or  $y=0$ .

The relationships between the various algebraic forms are important. As shown in the section on definitions,  $(h,k)$  is the same in the vertex form as in the conic section form and the  $a$  of the vertex form is  $1/p$  in the conic section form. It is most useful to note the connection between the vertex form and the standard form. To convert from standard form to vertex form, we can use the method of completing the square much as we do in solving the quadratic formula:

Given  $f(x) = ax^2 + bx + c$

Factor out a from the first two terms:

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

Complete the square:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

(Note that you're adding and subtracting a  $x$  ( $b^2/(4a)$ )).

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

So we have that  $h = -b/(2a)$  and  $k = c - (b^2/(4a))$ .

If we find the x intercepts by setting  $f(x)$  to 0, we would then derive the quadratic formula:

$$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\frac{\pm\sqrt{b^2 - 4ac}}{2a} = x + \frac{b}{2a}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x$$

Now I can connect the graphical representation with the algebraic representation by noting that if the discriminant ( $b^2 - 4ac$ ) is zero, then there will be one x-intercept and the parabola will touch the x-axis at its vertex. If the discriminant is less than zero, then there will be no x-intercepts because the only solutions are complex numbers, and if the discriminant is greater than zero, then there will be two x-intercepts.

To relate the factored form of the quadratic function to the standard form, note that starting with

$$y = a(x - d)(x - e)$$

We multiply out the binomials to get

$$y = ax^2 - a(d + e)x + ade$$

Thus,  $b = -a(d+e)$  or  $-b/a$  is the sum of the roots, and  $c = ade$  or  $c/a$  is the product of the roots.

## Applications

There are a wide range of applications of parabolas. The same applications seem to appear in textbook after textbook. These include area problems in which gives the student each side of a rectangle in terms of a variable  $x$  and also gives the student the area of the shape. Optimization problems are typical since the quadratic function is the first function that has a maximum and a minimum. In chapters involving conic sections, there is always a problem about the shape of a satellite dish or telescopic mirror. The most common application of parabolas comes in modeling the flight of a projectile or a falling object.

A typical problem involving projectiles is the following, taken from Algebra 2 by Larson, Kanold, and Stiff, published in 1993.

“A football player kicked a 41-yard punt. The path of the ball was modeled by  $y = -0.035x^2 + 1.4x + 1$ , where  $x$  and  $y$  are measured in yards. What was the maximum height of the ball? The player kicked the football toward midfield from

the 18 yard line. Over which yard line was the ball when it was at its maximum height?”

This problem requires that the student find the y coordinate of the vertex (15 yards) and realize that the x coordinate of the vertex which is 20 must be shifted by 18 units since the football started at the 18 yard line, to get the 38 yard line.

In fact, the general quadratic function that gives height as a function of time for a projectile is

$$h(t) = (-1/2)gt^2 + v_0t + h_0$$

where  $g$  is acceleration due to gravity ( $9.8 \text{ m/sec}^2$ ),  $v_0$  is the initial velocity of the projectile, and  $h_0$  is the initial height of the projectile. The formula can be justified using calculus.

In an activity for first-year algebra students, a teacher describes a math field trip in which students took pictures of a fountain and then returned to class to place the streams of water onto a coordinate system and find equations for the parabolas. (From “Parabolas Under Pressure” by Madden in *Mathematics Teacher*, Volume 102, No. 9, 2009.) This sounds a lot more fun than solving the problem out of the text described above; and such an activity would convince students that the path of a projectile is indeed a parabola.

### Summary Impressions

The study of parabolas seems to be provide students with many opportunities to make connections between algebraic and graphical representations, as well as between different algebraic representations. It is important that students realize that the “parabola” they encounter in one section of their text in vertex form is the same “parabola” that they encounter in the conic section chapter of their text.

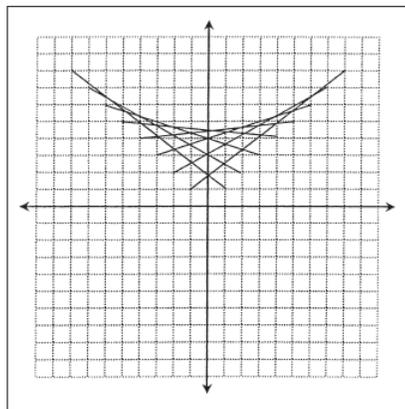
## Connections

In this section, the focus is on connections between the concept of parabola and other concepts in the curriculum from kindergarten to college mathematics courses. Parabolas, in the form of quadratic functions, are first introduced explicitly in the middle grades. They are studied in depth in high school algebra courses. Understanding of parabolas is inevitably connected to a student's understanding of variables, squaring, adding unlike and like terms in a polynomial, patterns, rates of change, and the Cartesian coordinate system. In most textbooks, it appears that the quadratic function is the first non-linear function that the student encounters. The chapter on quadratic functions always seems to come after the chapter on linear functions. Thus, it is the first time that the student encounters a non-constant rate of change, a graph with a maximum and minimum and thus a restricted range, a graph that has increasing and decreasing parts, a function that is NOT one-to-one and thus whose domain has to be restricted to form an inverse. Understanding of maxima, minima, increasing, decreasing, one-to-one, restricted domains and ranges, inverses, are all concepts that figure prominently in the remainder of the student's mathematical career. Quadratic functions are also usually the first time that the student encounters horizontal and vertical shifts and stretches! An understanding of such transformations is critical for the understanding of the graphs of every function the student will study! Quadratic functions also lead to an introduction to complex numbers and the idea that real numbers are not the only kind of numbers in the mathematical universe.

In this section, I chose to concentrate on the connection between the graphs of linear and quadratic functions, a connection that is not always so obvious to students but that is exploited by several writers in articles from *Mathematics Teacher*. I also include a connection between parabolas and complex numbers also from the *Mathematics Teacher*. I chose these particular connections because they focus on graphical representations, and after all, the parabola is really the graphical representation of a quadratic function. One last connection that I couldn't resist is the connection to triangular numbers and in turn to Pascal's Triangle and combinatorics.

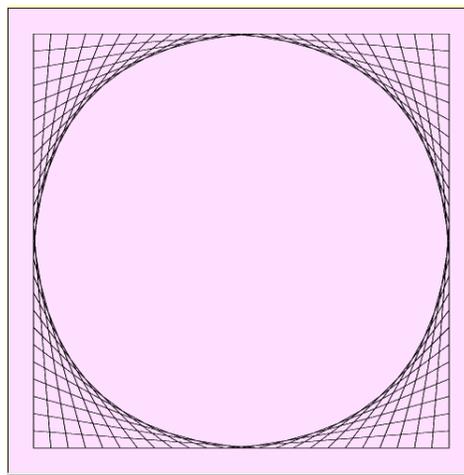
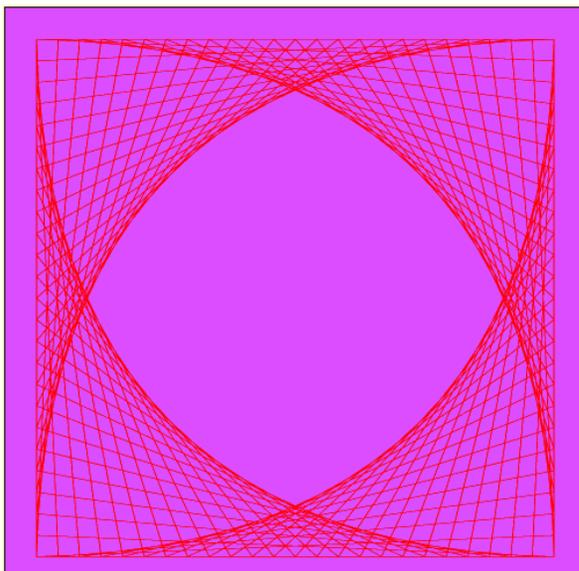
### Linear Functions and Parabolas

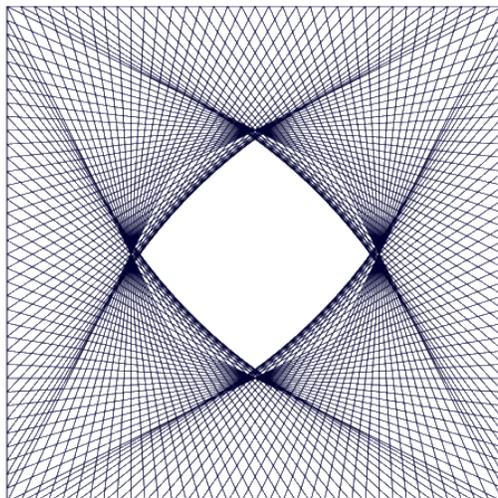
In the article from the *Mathematics Teacher* by Trisha Bergthold entitled "Curve Stitching: Linking Linear and Quadratic Functions" (Volume 98, No. 5, 2005), the author not only provides a connection between lines and parabolas, but also between mathematics and art. Each student is provided a coordinate grid printed on card stock and a needle and thread. They create eight stitches as shown below starting at points along the



line  $y=x$  and ending at points on the line  $y = -x$ . The author recommends that the teacher ask “What curve are we really looking at? What is its equation? Can we identify some ordered pairs that lie on this curve?” The students can then find an equation for the parabola that fits the curve using many of the methods I used in the Approaches section – Method of Finite Differences, matrices, system of linear equations. There are several possible parabolas that are created by the intersections of the stitches; however, if we look at the most obvious topmost parabola, it is clear that the lines joining successive intersections are actually tangents to the parabola.

It turns out that “curve stitching” is quite popular among mathematicians. The web site [http://www.deimel.org/rec\\_math/curve\\_stitch.htm](http://www.deimel.org/rec_math/curve_stitch.htm) has a description of how one mathematician was drawn to curve stitching and provides links to a variety of examples of curve stitching including the designs shown below.

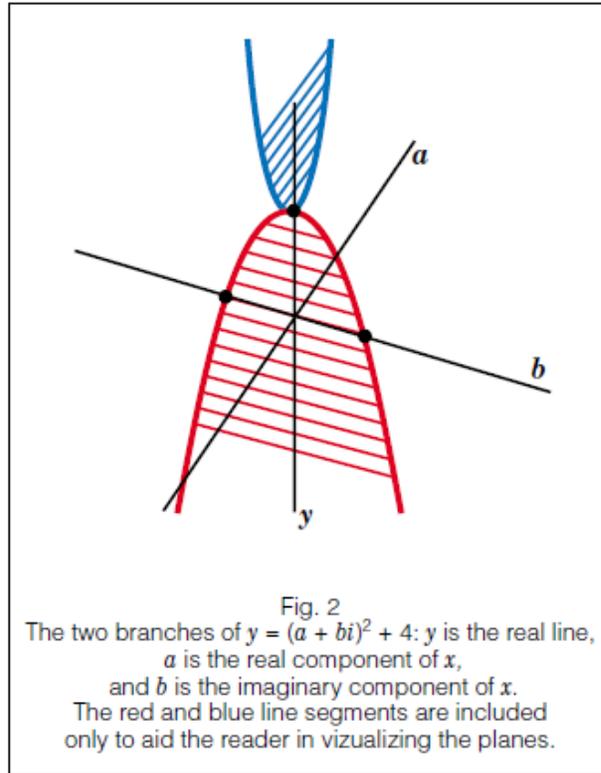




In another article from *Mathematics Teacher* by Moore-Russo and Golzy entitled “Helping Students Connect Functions and Their Representations” (Volume 99, No. 3, 2005), the authors suggest that we should start with the graphs of linear functions such as  $y_1 = 2$  and  $y_2 = .5x$  and then ask students to describe, without resorting to the algebra, but reasoning purely from the graphs, the sum and the product of the two functions. Thus, they suggest that we should introduce quadratic functions as the product of two linear functions, which in fact they are the product of two linear functions. As the authors state “It also sets the stage for a more in-depth look at the role of linear factors in the graphs of quadratic functions and other polynomials.”

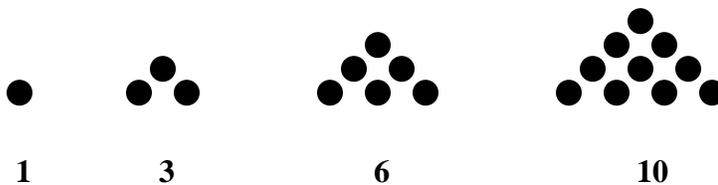
### Complex Numbers and Parabolas

As pointed out in Concepts, if the discriminant of a quadratic equation of the form  $ax^2 + bx + c = 0$  is less than zero, then there are no real solutions to this equation. Also, the function  $f(x) = ax^2 + bx + c$  then has no x-intercepts. I had never thought about how to visualize these complex roots. In “Visualizing the Complex Roots of Quadratic and Cubic Equations” by Lipp in *Mathematics Teacher* (Volume 94, no. , 2001), the author shows that if we take a parabola such as  $y = x^2 + 4$  which has no x intercepts, and allow the domain to include complex numbers of the form  $a+bi$ , then by substituting  $a+bi$  in for  $x$ , we get  $y = a^2 - b^2 + 2abi + 4$ . Since  $y$  is real, the imaginary term must vanish, according to the authors, so we get  $y = a^2 - b^2 + 4$  and this equation has two branches:  $a^2+4$  when  $b$  is 0 and  $-b^2 + 4$  when  $a$  is 0. Also note that  $b$  represents the imaginary part of the complex number while  $a$  represents the real part. So we get the picture shown below.

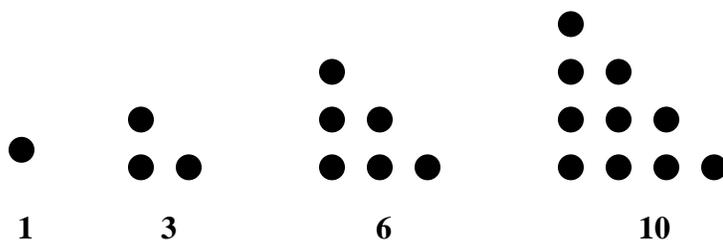


### Triangular Numbers

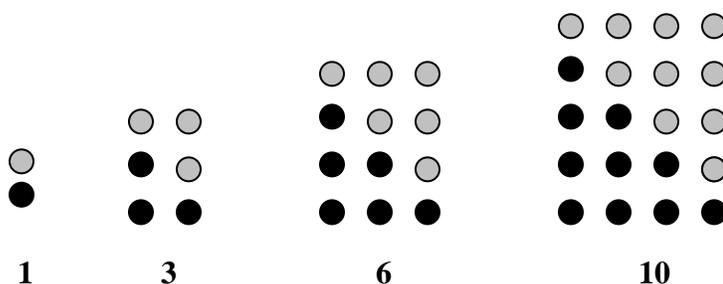
Some of the most famous sequences of numbers are sequences that have a constant second difference and thus have a quadratic closed form. Figurate numbers are sequences of numbers associated with geometrical arrangements of dots. Interest in these numbers dates back to the sixth century B.C. and the followers of Pythagoras. Triangular numbers are a sequence of numbers that can be represented by triangular arrangement of dots as shown below.



There are also square numbers, rectangular numbers, pentagonal numbers, etc. One way of proving the formula for the  $n$ th triangular number is by rearranging the dots as shown below.



By filling out the arrangement of dots as shown below, so that each arrangement consists of an  $n$  by  $n+1$  rectangular array of dots, we can see that the number of dots in such an  $n$  by  $n+1$  array is  $n(n+1)$ . In the triangular arrangement we have exactly half of the  $n(n+1)$  dots. So, the formula for the  $n$ th triangular number is  $n(n+1)/2$ .



Another interesting property of the triangular numbers can be seen in the following table:

| n | nth triangular number |     |
|---|-----------------------|-----|
| 1 | 1                     |     |
| 2 | 1+2                   | =3  |
| 3 | 1+2+3                 | =6  |
| 4 | 1+2+3+4               | =10 |
| 5 | 1+2+3+4+5             | =15 |
| 6 | 1+2+3+4+5+6           | =21 |
| . | .                     |     |
| . | .                     |     |
| . | .                     |     |

So, the  $n$ th triangular number is also  $\sum_{i=1}^n i$ ; that is, the sum of the first  $n$  numbers.

There is another interesting way to arrive at the formula for the  $n$ th triangular number, a method attributed to the mathematician Carl Friedrich Gauss (1777-1855) who was known as a child prodigy in mathematics. The story is that Gauss' teacher gave his class some busy work -- the task of summing the first 100 numbers. Gauss came up with the answer almost immediately by recognizing the following pattern:

$$\begin{array}{r}
 1 + 2 + 3 + 4 + 5 + \dots + 95 + 96 + 97 + 98 + 99 + 100 \\
 \hline
 100 + 99 + 98 + 97 + 96 + \dots + 6 + 5 + 4 + 3 + 2 + 1 \\
 \hline
 101 + 101 + 101 + 101 + 101 + \dots + 101 + 101 + 101 + 101 + 101 + 101 \\
 = 100 ( 101)
 \end{array}$$

Since we have summed 1 through 100 twice, the sum of the first 100 numbers would be  $100(101)/2$ . By generalization, the sum of the first  $n$  numbers would be  $n(n+1)/2$ .

Note that the formula  $\sum_{i=1}^n i = n(n+1)/2$  is usually proven using the method of mathematical induction. In this technique, the first step is to show that  $\sum_{i=1}^n i = n(n+1)/2$  is true for  $n=1$ . This is true because for  $n=1$ ,  $\sum_{i=1}^n i = 1$  and  $n(n+1)/2 = 1$ . The next and final step is to show that if  $\sum_{i=1}^n i = n(n+1)/2$  is true for all  $n \leq k$  then it is true for  $n=k+1$ . This is true because  $\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + k + 1$ . By assumption,  $\sum_{i=1}^k i + k + 1 = k(k+1)/2 + k + 1$ . To complete the proof, we just need to show that  $k(k+1)/2 + k + 1$  is the same as  $(k+1)(k+2)/2$ .

The triangular numbers thus appear as interesting figurate numbers, as the sum of the first  $n$  natural numbers, and they appear in solving other problems. They also appear in Pascal's Triangle. Seeing the structural connections between different problem situations is important. For example, in an article from *Mathematics Teacher* entitled "Choices and Challenges," the author states that

"We need to change our practice so that students see and understand mathematical connections. How many double-dip ice-cream cones can you make with ten flavors of ice cream? If ten people are at a gathering and each person shakes hands with every other person, how many handshakes will occur? Among ten cities, how many routes are possible if each city is connected to every other city? What are the triangular numbers? UNDER CERTAIN ASSUMPTIONS, these problems are basically the same – and students should come to recognize why."

### Summary Impressions

The connections between parabolas and other concepts are overwhelming. I've explored in some depth how they connect to lines, complex numbers, sequences, and series. However, there are so many interesting connections that have not been explored in

this section but which I feel should be mentioned. For example, we could connect our two dimensional parabola with three dimensional representations including hyperbolic paraboloids which have quite an interesting modern application in the shape of Pringles potato chips ([http://www.math.hmc.edu/~gu/curves\\_and\\_surfaces/surfaces/hyp-par.html](http://www.math.hmc.edu/~gu/curves_and_surfaces/surfaces/hyp-par.html)). We could connect our parabolas to children's literature including a series about bats by Appelt called *Bats on Parade* and *Bat Jamboree*, with activities described by Roy and Beckmann in an article in *Mathematics Teaching in the Middle School* (volume 13, no. 1, 2007).