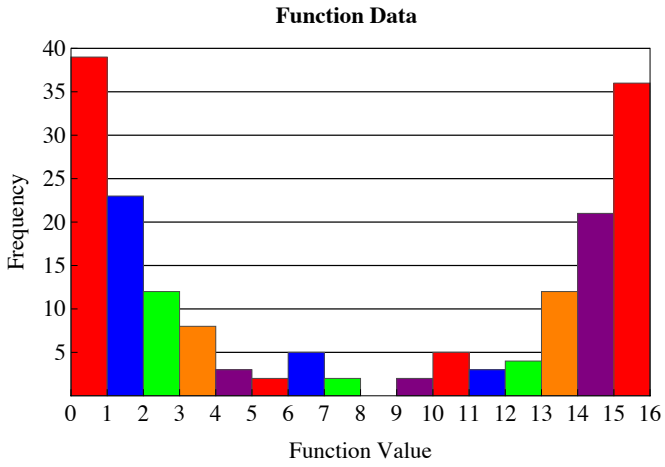

Analyzing Function Behavior Using Graphical Displays – Part 2

LEVEL

Algebra 2 or Math 3 in a unit on analyzing functions

Algebra 2 or Math 3 in a unit on statistics

MODULE/CONNECTIONS TO AP*

Analysis of Functions

Graphical Displays and Distributions

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MODALITY:

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

ABOUT THE LESSON

This lesson is should follow the Algebra 2 activity, "Analyzing Function Behavior Using Graphical Displays – Part 1," where students are first introduced to the relationship between graphical displays and analysis of functions. In the Part 2 lesson, students use average rate of change (the slope of the secant line of a function over a specific interval) to form a connection between the graph of a function and the univariate display of its function values. Through several exercises, students begin to relate the magnitude of the slope over an interval of the function values to the likelihood that a randomly selected function value will fall in a particular bin in a histogram. With practice and experience, students learn to analyze the behavior of functions on a fixed domain by looking at a histogram of the distribution of function values.

OBJECTIVES

Students will

- calculate average rates of change.
- create and interpret boxplots and histograms.
- create and interpret a five-number summary.
- evaluate functions on restricted domains.
- predict distributions of functions values based on a function's graph.
- create function graphs related to specific histograms.

TEACHER PAGES

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COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (⋆) at the end of a specific standard indicates that the high school standard is connected to modeling.

Targeted Standards

S-IC.1: Understand statistics as a process for making inferences about population parameters based on a random sample from that population.⋆
See questions 3b, 4b, 6, 7, 8

F-IF.6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.⋆
See questions 1, 2, 3b, 4b, 6, 7, 8

Reinforced/Applied Standards

S-ID.1: Represent data with plots on the real number line (dot plots, histograms, and box plots).⋆
See questions 3a, 4a, 5c-d

S-ID.3: Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).⋆
See questions 3a, 3d, 5e, 6

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*⋆
See questions 3b, 4b, 6, 7

F-IF.7a: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph linear and quadratic functions and show intercepts, maxima, and minima.⋆
See questions 3 stem, 8c

F-IF.7c: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.⋆
See questions 4c, 8a

F-IF.7e: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.⋆
See question 8b

F-IF.2: Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
See questions 3 stem, 4a, 5a

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COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.3: Construct viable arguments and critique the reasoning of others.
Students justify their predicted histograms based on their knowledge of the behavior of function values.

MP.5: Construct viable arguments and critique the reasoning of others.
Students use the graphing calculator to generate and evaluate random domain values in order to create histograms.

MP.7: Look for and make use of structure.
Students relate the shape of the display of function values to the appearance of the graph of a function on a restricted domain and predict the shape of a histogram of function values.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Create and interpret histograms
- Sketch quadratic, cubic and logarithmic functions

ASSESSMENTS

The following formative assessment is embedded in this lesson:

- Students engage in independent practice.

The following additional assessments are located on the our website:

- Graphical Displays and Distributions – Algebra 2 Free Response Questions
- Graphical Displays and Distributions – Algebra 2 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Graphing calculators
- Mathematica demonstration "Analyzing Functions Using Graphical Displays" explores the relationship between the distribution of randomly selected function values and the behavior of the function and is available on the NMSI website (a free download of Mathematica Player is required).
- Mathematica demonstration "Relating Boxplots to Histograms" explores the relationship between boxplots and histograms and is located on the NMSI website (a free download of Mathematica Player is required).
- NMSI video clip on random domain generation on the TI-84
- NMSI video clip on creating histograms on the TI-84

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TEACHING SUGGESTIONS

This lesson requires students to identify the shape of a distribution. Remind students of the definitions of mound shaped, skewed left, skewed right, and uniform distributions. Emphasize how the relationship between mean and median can be predicted from the shape of the distribution. (See the Teaching Suggestions for "Analyzing Function Behavior Using Graphical Displays – Part 1.")

The students will be asked about outliers. One commonly held definition of an outlier in a set of data is as follows: an outlier is a data point that falls above an upper boundary or below a lower boundary as defined by adding 1.5(IQR) to Q_3 or subtracting 1.5(IQR) from Q_1. $LB = Q_1 - 1.5(IQR)$ and $UB = Q_3 + 1.5(IQR)$

Discuss the meaning of the syntax of the calculator key strokes when generating the random numbers over the limited domains. Review the instructions for the calculator with the students using the last page of the student activity.

In order to ensure that all groups will generate different random numbers, seed the graphing calculators by storing the last four digits of the student's phone number in rand on the graphing calculator.

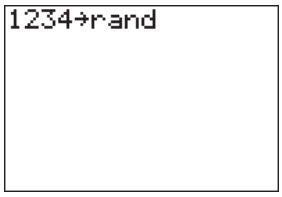
The key sequence is as follows:

Step 1: Students enter the last four digits of their phone numbers

Step 2: Press [STO▶]

Step 3: Press [MATH], arrow over to PRB, press [1]

Step 4: Press [ENTER] and the new number is the seed for the random number generator



The calculator instructions for generating the domain values at random are on the last page of the student activity.

Memory errors may occasionally occur when generating large lists of random numbers since these lists take large amounts of RAM. If memory errors occur, instruct students to clear the lists prior to generating random numbers.

Suggested Modifications for additional scaffolding include the following:

3b, 4b Provide a fill-in the blank format for the answer. For example, "Since the function has a _____minimum/maximum value of _____, the bin with the _____least/greatest frequency will be the bin that contains values closest to zero."

7 Provide complete graphs and justification statements for students to match to the histograms

8 Allow the use of calculators to sketch the histogram.

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NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Create, interpret, and compare dotplots (line plots), stemplots, and bar graphs. (200_06.GD_D.01)	Create, interpret, and compare dotplots (line plots), stemplots, bar graphs, histograms, and boxplots. (200_07.GD_D.01)	Create, interpret, and compare dotplots (line plots), stemplots, bar graphs, histograms, and boxplots. (200_A1.GD_D.01)	Create, interpret, and compare dotplots (line plots), stemplots, bar graphs, histograms, and boxplots. (200_GE.GD_D.01)	Create, interpret, and compare dotplots (line plots), stemplots, bar graphs, histograms, and boxplots. (200_A2.GD_D.01)	Create, interpret, and compare dotplots (line plots), stemplots, bar graphs, histograms, and boxplots. (200_PC.GD_D.01)
Calculate the mean, median, mode, range, and mean absolute deviation from tabular or graphical data or data presented in paragraph form. (200_06.DI_D.01)	Calculate the mean, median, mode, range, and mean absolute deviation from tabular or graphical data or data presented in paragraph form. (200_07.DI_D.01)	Calculate the mean, median, mode, range, and standard deviation from tabular or graphical data or data presented in paragraph form. (200_A1.DI_D.01)	Calculate the mean, median, mode, range, and standard deviation from tabular or graphical data or data presented in paragraph form. (200_GE.DI_D.01)	Calculate the mean, median, mode, range, and standard deviation from tabular or graphical data or data presented in paragraph form. (200_A2.DI_D.01)	Calculate the mean, median, mode, range, and standard deviation from tabular or graphical data or data presented in paragraph form. (200_PC.DI_D.01)

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Analyzing Function Behavior Using Graphical Displays – Part 2

Answers

1.

Bin	Δy	Δy	$\left\lvert\frac{\Delta y}{\Delta x}\right\rvert$	Frequency
$0 \le y\text{-value} < 1$	1	$\sqrt{1}-\sqrt{0}$	$\left\lvert\frac{1}{\sqrt{1}-\sqrt{0}}\right\rvert = 1$	41
$1 \le y\text{-value} < 2$	1	$\sqrt{2}-\sqrt{1}$	$\left\lvert\frac{1}{\sqrt{2}-\sqrt{1}}\right\rvert = 2.414$	21
$2 \le y\text{-value} < 3$	1	$\sqrt{3}-\sqrt{2}$	$\left\lvert\frac{1}{\sqrt{3}-\sqrt{2}}\right\rvert = 3.146$	11
$3 \le y\text{-value} < 4$	1	$\sqrt{4}-\sqrt{3}$	$\left\lvert\frac{1}{\sqrt{4}-\sqrt{3}}\right\rvert = 3.732$	8
$4 \le y\text{-value} < 5$	1	$\sqrt{5}-\sqrt{4}$	$\left\lvert\frac{1}{\sqrt{5}-\sqrt{4}}\right\rvert = 4.236$	5
$5 \le y\text{-value} < 6$	1	$\sqrt{6}-\sqrt{5}$	$\left\lvert\frac{1}{\sqrt{6}-\sqrt{5}}\right\rvert = 4.686$	5
$6 \le y\text{-value} < 7$	1	$\sqrt{7}-\sqrt{6}$	$\left\lvert\frac{1}{\sqrt{7}-\sqrt{6}}\right\rvert = 5.095$	4
$7 \le y\text{-value} < 8$	1	$\sqrt{8}-\sqrt{7}$	$\left\lvert\frac{1}{\sqrt{8}-\sqrt{7}}\right\rvert = 5.474$	3
$8 \le y\text{-value} < 9$	1	$\sqrt{9}-\sqrt{8}$	$\left\lvert\frac{1}{\sqrt{9}-\sqrt{8}}\right\rvert = 5.828$	2

2. a. increases
   b. decreases
   c. increases
   d. decreases
   e. increases
   f. increases

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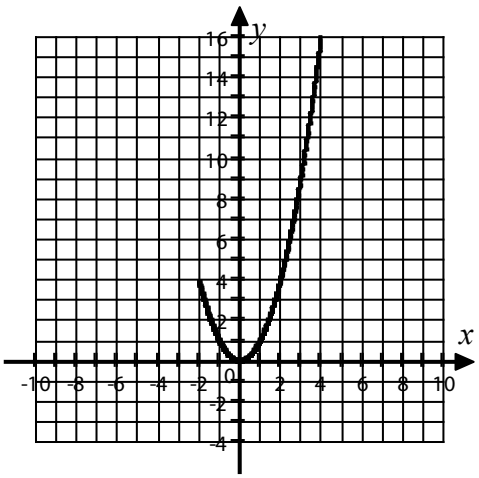
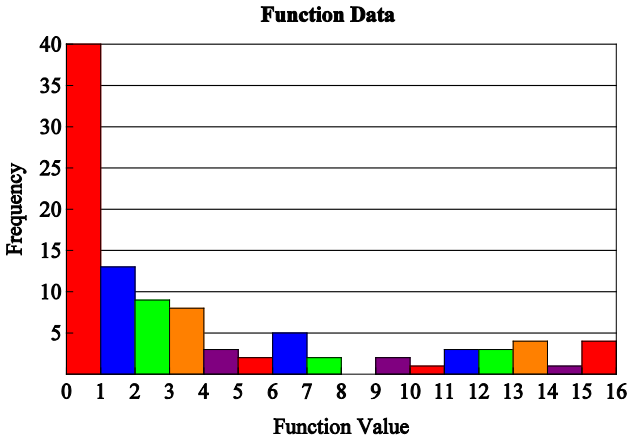
3.





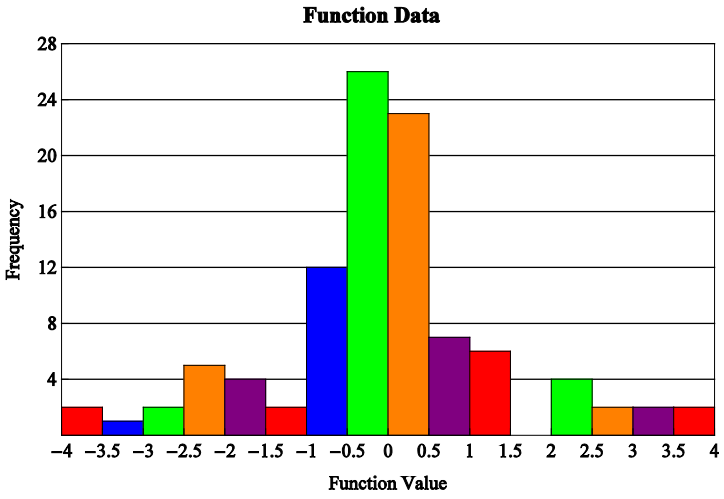
a. The histogram is strongly skewed right.

b. Since the function has a minimum value of 0, the bin with the greatest frequency will be the bin that contains values closest to 0. The frequency is also greater close to 0 since as the absolute value of the slope of the secant line decreases, the frequency in the bin increases. In addition, the frequencies should be greater between 0 and 4 since these values occur twice as often in the given domain.

c. Answers may vary. The median is between the 50th and 51st of the 100 function values. The sample data has a median in the interval $1 \leq f(x) < 2$.

d. The mean should be larger than the median since the distribution is skewed right.

4. a. The distribution of the sample data is mound shaped and approximately symmetric about $f(x) = 0$.



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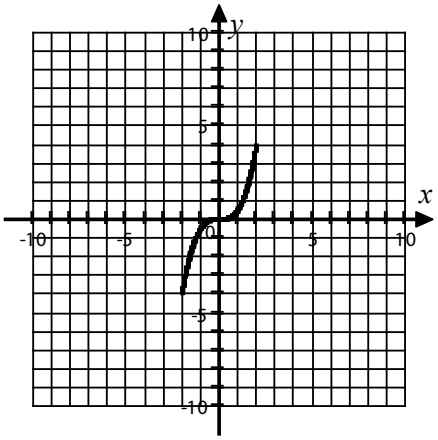
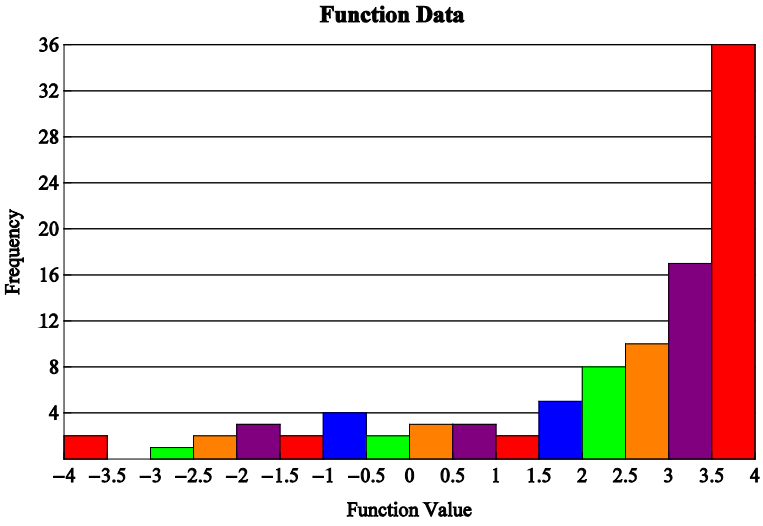
b. Since the frequency is the highest near the center of the data, the absolute value of the slope of the function must be close to zero. As the frequencies decrease, the value of $\left|\frac{\Delta y}{\Delta x}\right|$ will increase.

c. The absolute value of the slope of the secant line is larger near –2 and then begins to decrease until it approaches zero as the function value gets close to zero. As the function values continue to increase away from zero, the values of the slope of the secant line also increase in magnitude.



5. a. $h(x) = -\frac{1}{2}(x+1)^2 + 4$ on the domain [–3, 3].



b. Answers may vary. Sample data are given. Min $= -3.836$; $Q_1 = 1.569$; Med $= 3.036$; $Q_3 = 3.689$; Max $= 3.999$; Range $= 7.835$; $IQR = 2.120$

c. Answers may vary. For the sample data, there are outliers since the minimum value is below the lower boundary, LB.

$LB = 1.569 - 1.5(2.120)$; $LB = -1.611$

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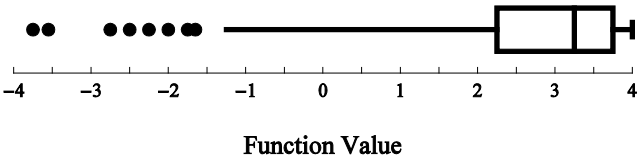
d.



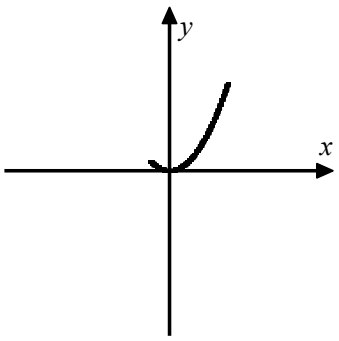
e. Answers may vary. Since the median, Q_3, and max are between 3 and 4, this would indicate that 50% of the function values generated fall in the upper 2 bins, making them much taller than the rest of the bins.

6. a. III; ii

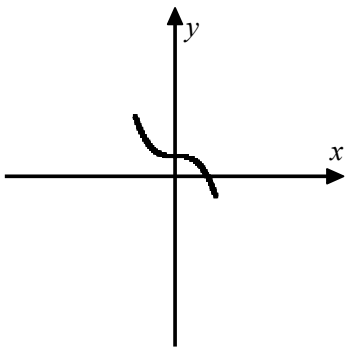
b. I; iii

c. II; i

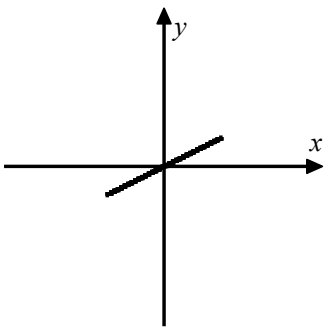
7. Answers may vary. Correct sample graphs and justifications are shown.

a. 

The magnitude of the slope is smallest when the function value is smallest.

b. 

The slope of the function becomes very close to zero near the middle of the function values.

c. 

The slope must stay close to the same value throughout the function.

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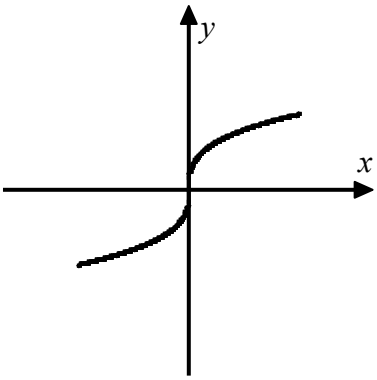
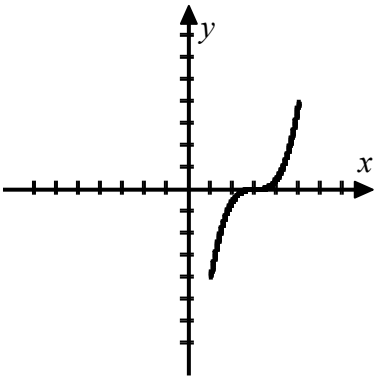
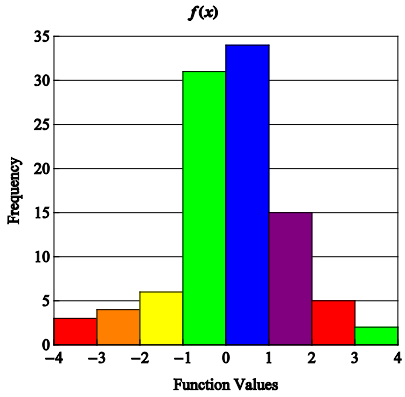
d.



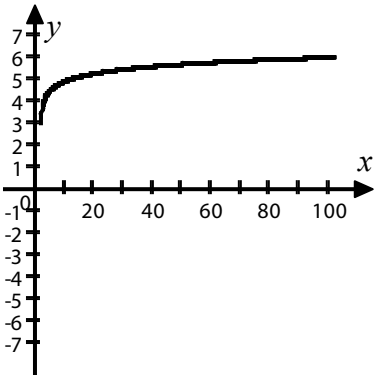
The slope must be close to zero near the minimum and maximum function values and much greater in the middle.

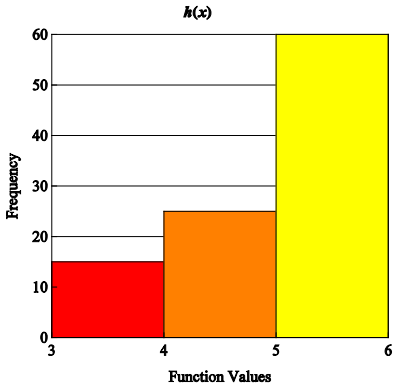
8. a. Range: [–4, 4]



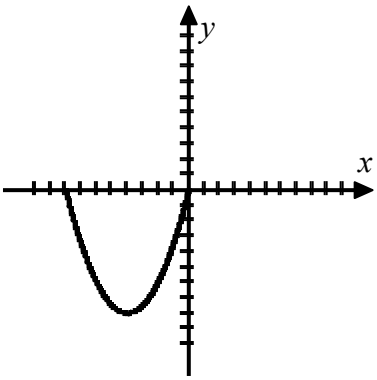


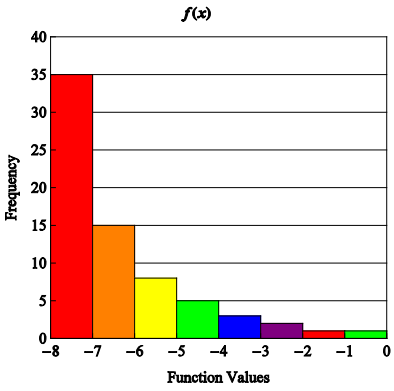
b. Range: [3, 6]





c. Range: [–8, 0]





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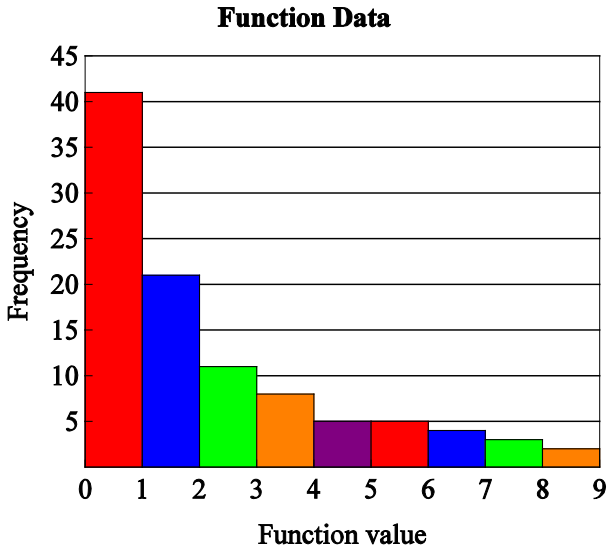
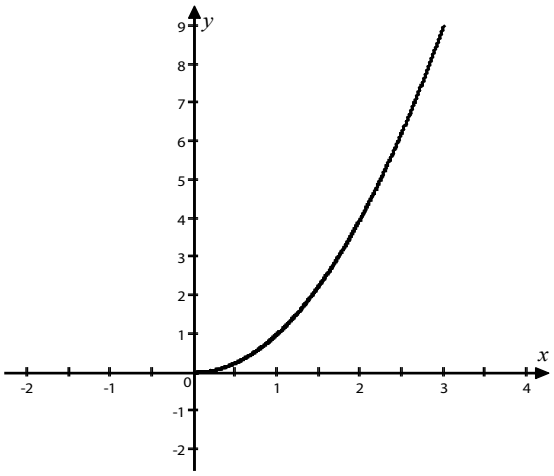
Analyzing Function Behavior Using Graphical Displays – Part 2

1. The data in the histogram was generated by randomly selecting 100 x-values, and then evaluating those values in the function $f(x) = x^2$ on the interval [0, 3].

   Complete the table of values for the magnitude of the average rate of change, $\left|\frac{\Delta y}{\Delta x}\right|$, on the specific y-intervals that correspond to the bins on the histogram.





Bin	Δy	Δy	$\left\|\frac{\Delta y}{\Delta x}\right\|$	Frequency
$0 \le y$-value < 1	1	$\sqrt{1}-\sqrt{0}$	$\left\|\frac{1}{\sqrt{1}-\sqrt{0}}\right\| = 1$	41
$1 \le y$-value < 2	1	$\sqrt{2}-\sqrt{1}$	$\left\|\frac{1}{\sqrt{2}-\sqrt{1}}\right\| = 2.414$	21
$2 \le y$-value < 3	1			11
$3 \le y$-value < 4	1			8
$4 \le y$-value < 5	1			5
$5 \le y$-value < 6	1			5
$6 \le y$-value < 7	1			4
$7 \le y$-value < 8	1			3
$8 \le y$-value < 9	1			2

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2. The tables show the relationship between the average rate of change and the frequency of function values in the bins for functions on restricted domains. The range values for the bins are determined by the given domain. Complete the statements with either "increases" or "decreases."

$f(x) = -\frac{1}{2}x^2 + 4; [0, 4]$

Bin	$\left\|\frac{\Delta y}{\Delta x}\right\|$	Freq
$-4 \le y < -3$	3.871	5
$-3 \le y < -2$	3.603	7
$-2 \le y < -1$	3.313	8
$-1 \le y < 0$	2.995	8
$0 \le y < 1$	2.639	9
$1 \le y < 2$	2.225	14
$2 \le y < 3$	1.707	15
$3 \le y < 4$	0.707	34

$f(x) = x^3 - 2; [0, 2]$

Bin	$\left\|\frac{\Delta y}{\Delta x}\right\|$	Freq
$-2 \le y < -1$	1	47
$-1 \le y < 0$	3.847	16
$0 \le y < 1$	5.485	9
$1 \le y < 2$	6.889	8
$2 \le y < 3$	8.158	7
$3 \le y < 4$	9.333	6
$4 \le y < 5$	10.437	6
$5 \le y < 6$	11.485	1

$f(x) = \log(x - 1) + 1; [1.1, 11]$

Bin	$\left\|\frac{\Delta y}{\Delta x}\right\|$	Freq
$0 \le y < 0.25$	3.312	0
$0.25 \le y < 0.5$	1.806	1
$0.5 \le y < 0.75$	1.016	1
$0.75 \le y < 1$	0.571	2
$1 \le y < 1.25$	0.321	6
$1.25 \le y < 1.5$	0.181	13
$1.5 \le y < 1.75$	0.102	17
$1.75 \le y < 2$	0.057	60

a. As $\left|\frac{\Delta y}{\Delta x}\right|$ decreases, the frequency of the y-values in the bin _____.

b. As $\left|\frac{\Delta y}{\Delta x}\right|$ increases, the frequency of the y-values in the bin _____.

c. As the magnitude of the slope of the secant line decreases, the frequency of the y-values in the bin _____.

d. As the magnitude of the slope of the secant line increases, the frequency of the y-values in the bin _____.

e. As the absolute value of the average rate of change of the y-values with respect to the change in x-values for an interval approaches zero, the probability that a randomly selected function value is contained in that interval _____.

f. As a maximum or minimum value of the function is approached, the probability that a randomly selected function value is contained in that interval _____.

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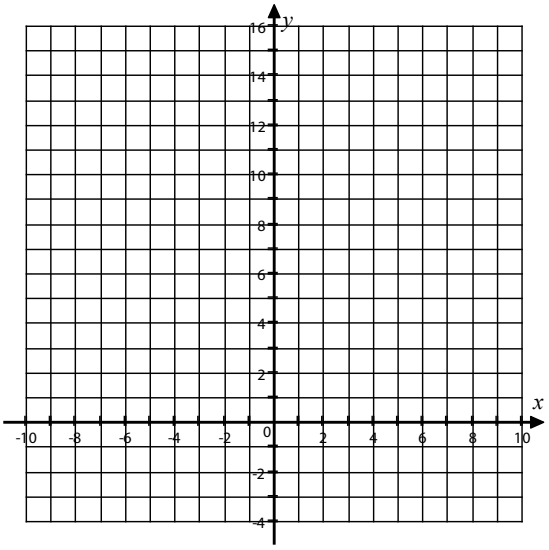
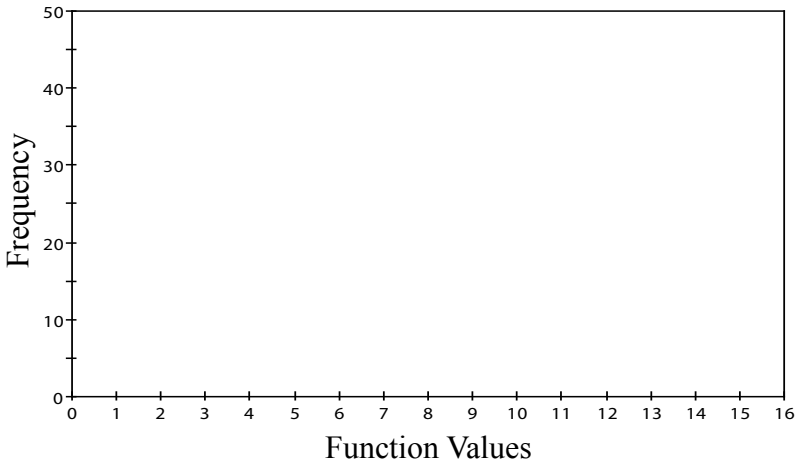
3. Sketch $f(x) = x^2$ on the interval [–2, 4]. Using the graphing calculator instructions at the end of the lesson, select 100 domain values at random from the interval [–2, 4] and evaluate the function at those domain values. Draw a histogram of your function values using bin widths of 1.





a. Describe the shape of the distribution.

b. What characteristics of the function affect the distribution of the data? Explain.

c. Which bin contains the median function value?

d. Based on the shape, will the mean or median be larger?

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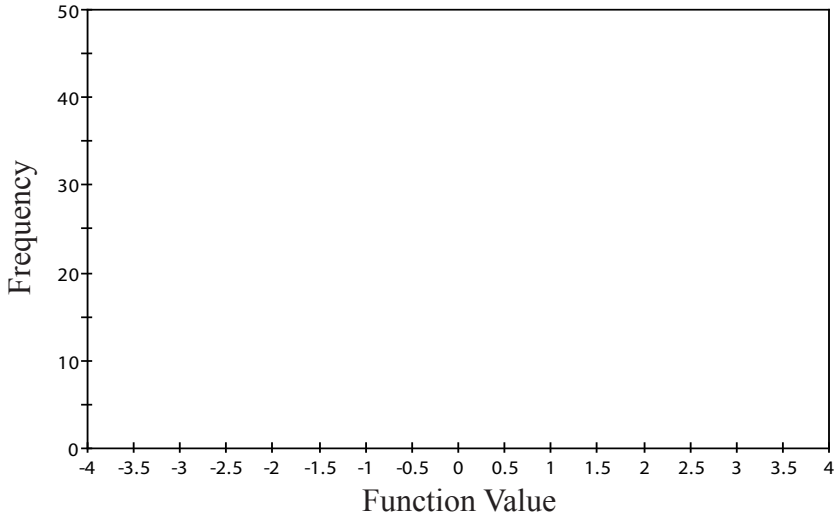
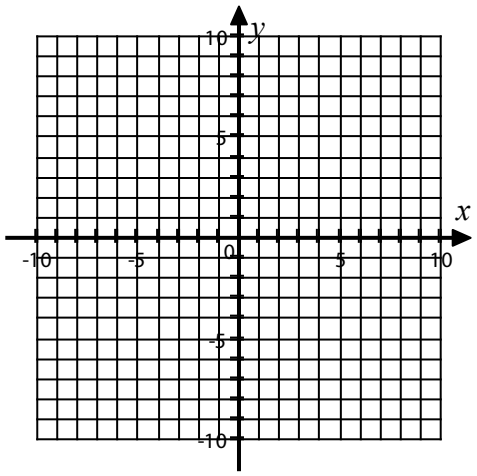
4. Select 100 domain values at random from the interval [–2, 2] and evaluate the function, $g(x) = \frac{1}{2}x^3$, for those values.

   a. Create a histogram of the function values using bin widths of 0.5 and describe the shape of the distribution of function values.



   b. Based on the shape of the distribution in part (a), make a statement about the changes in magnitude of the slope of the secant line for the intervals defined by the bin widths.

   c. Draw the function on the interval [–2, 2] and use the graph of the function to justify your answer in part (b).



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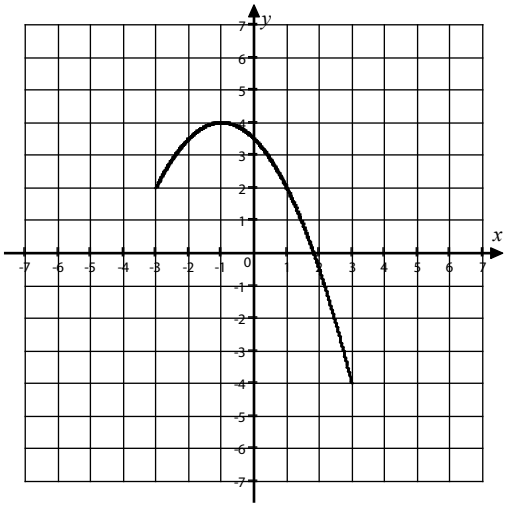
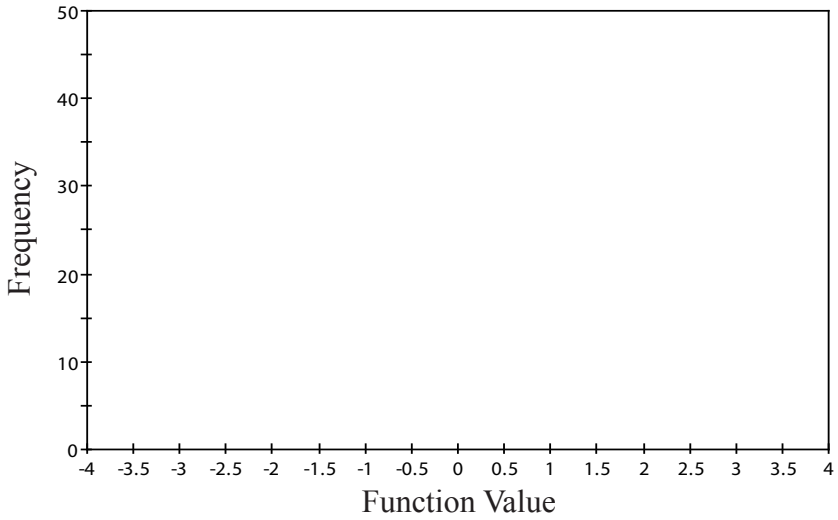
5. A quadratic function $h(x)$ is shown on the domain [–3, 3].



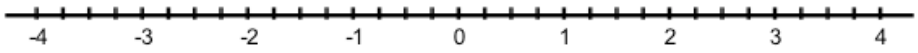


a. What is the equation for $h(x)$? Using a graphing calculator, generate 100 random values on the given domain, evaluate the function at each of those values, and sketch the histogram that shows the distribution of function values.

b. Determine the five-number summary and calculate the range and IQR for the data generated in part (a).

c. Are there any outliers? Show the work that leads to your answer.

d. Draw a modified boxplot of your function values, using your answers to parts (b) and (c).



e. How is your boxplot similar to the histogram of function values?

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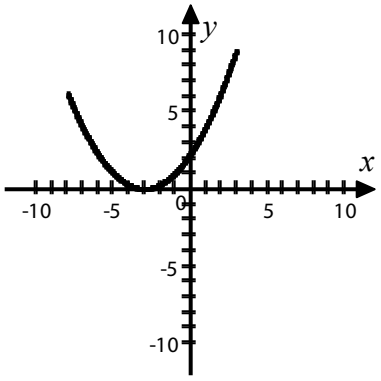
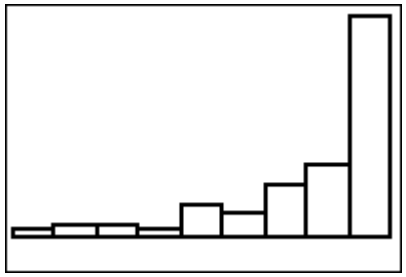
6. Match one boxplot and one histogram of function values with the graph of the given function.

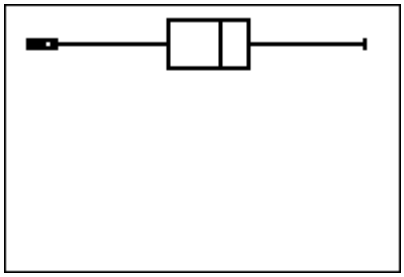
a. _____; _____



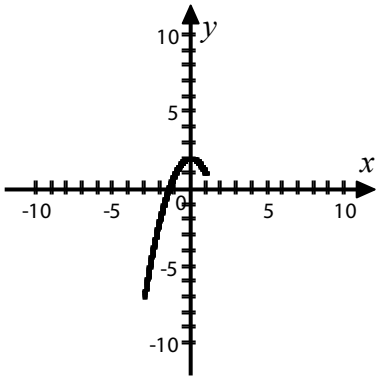
I.



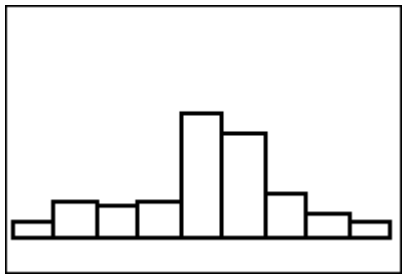
i.



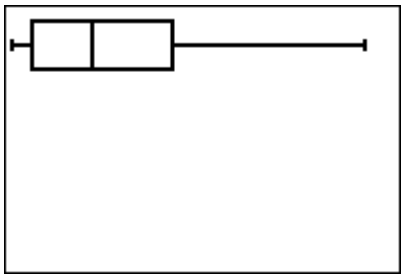
b. _____; _____



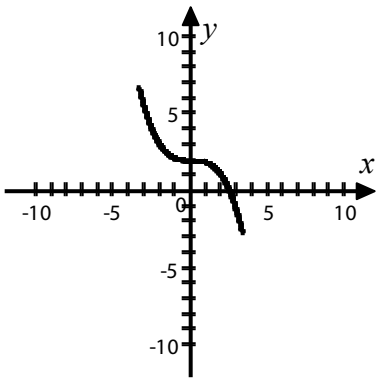
II.



ii.



c. _____; _____



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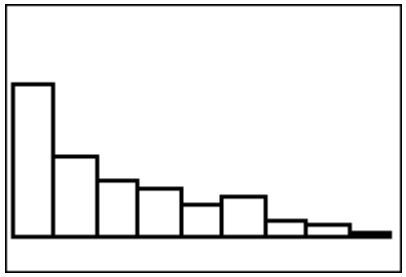
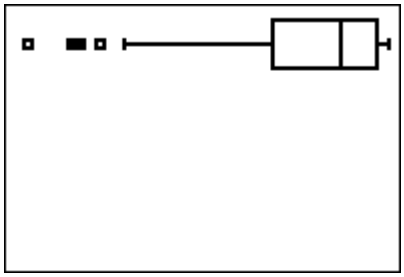


iii.



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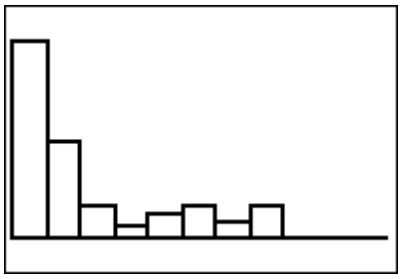
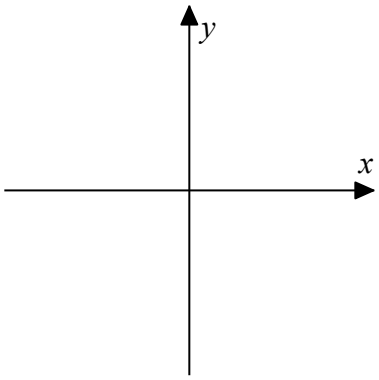
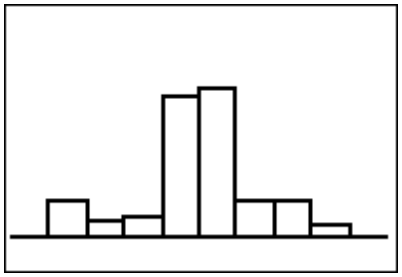
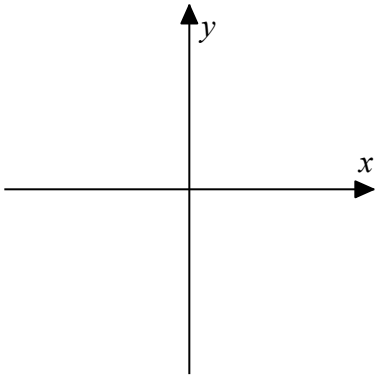
7. For each histogram, sketch a possible function with a distribution of y-values that matches the histogram. Justify your answer.

a. 

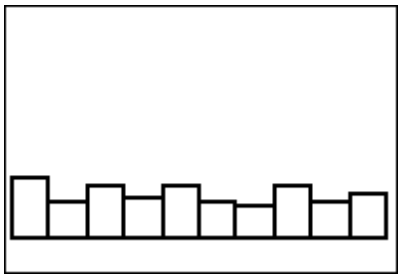


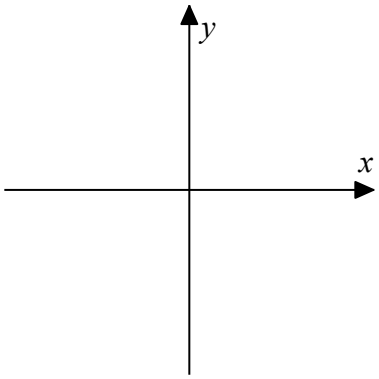
Justification:

b. 

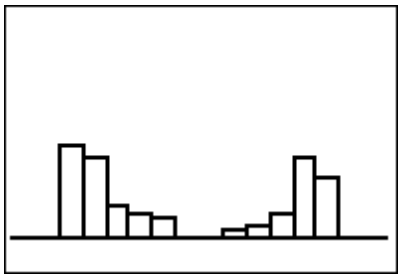


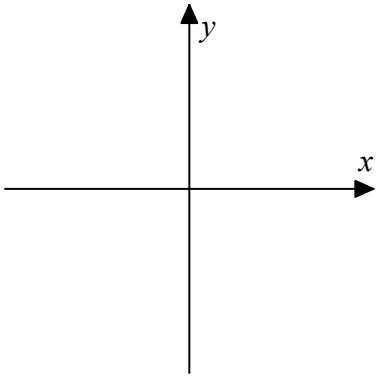
Justification:

c. 



Justification:

d. 



Justification:

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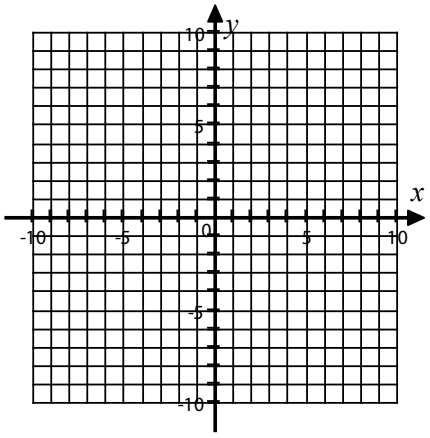
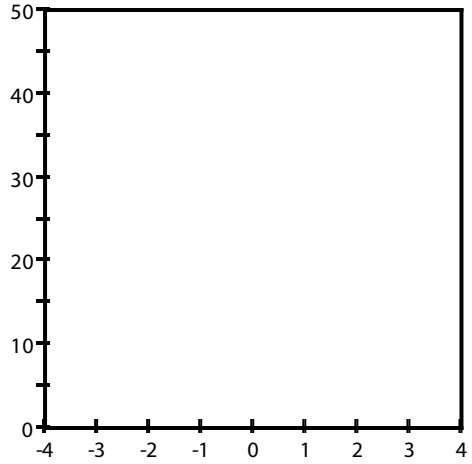
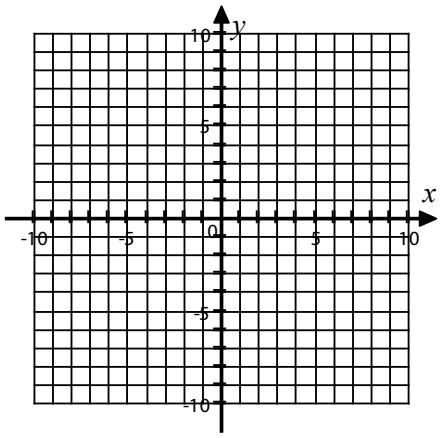
8. State the range for each function on the specified domain. Without a calculator, sketch the function and a histogram with bin widths of 1 that shows a likely distribution of 100 function values generated by randomly selecting 100 x-values and calculating the corresponding y-values.

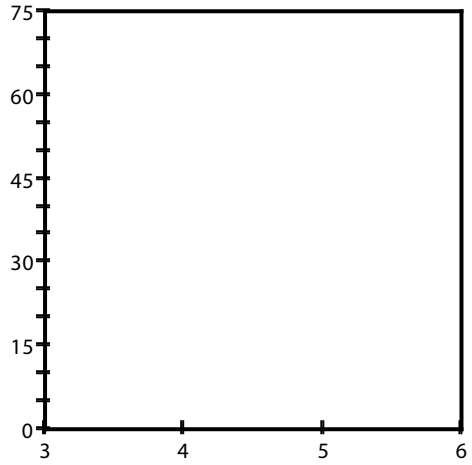
a. $f(x) = \frac{1}{2}(x-3)^3$, [1, 5]



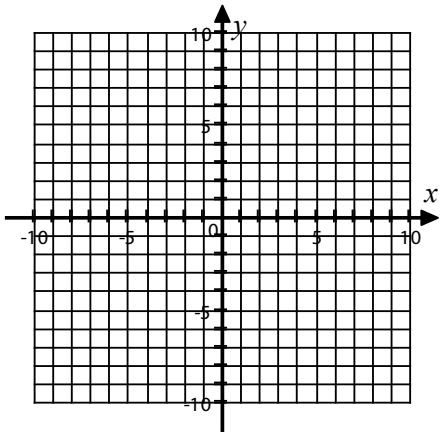


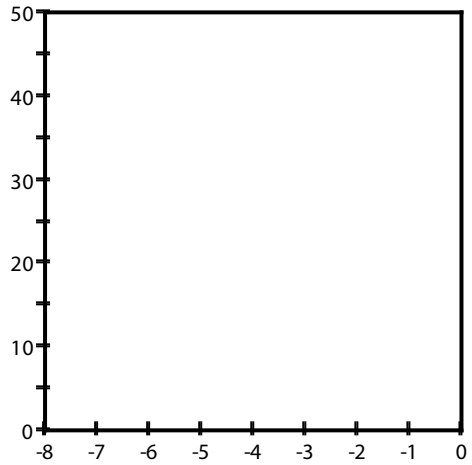
b. h(x) = log(x – 2) + 4, [2.1, 102]





c. $f(x) = \frac{1}{2}(x+4)^2 - 8$, [–8, 0]





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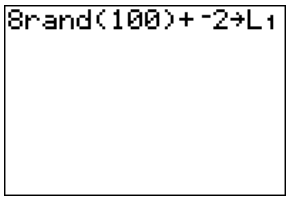
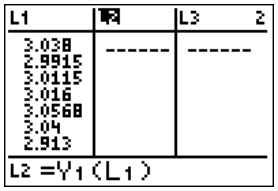
To generate a large quantity of random numbers at one time over a limited domain and store these values in List 1 on the calculator, use the following steps:
Step 1: Enter the width of the domain
Step 2: Press [MATH], arrow over to PRB, press [1]
Step 3: Press [(], enter the number of random values required, press[)]
Step 4: Press [+], enter the smallest value in the domain
Step 5: Press [STO▶], [2nd], [1]

The syntax is W*rand(n)+A→ L1 where *W* is the width of the domain, A is the smallest value in the domain, and *n* is the number of values you want to generate.
For example to generate 100 random values on the domain [–2, 6] on the calculator, the display would be:



To evaluate a function that is in Y1 at the values in a list, use the following steps:
Step 1: Press [STAT], press [1], arrow to the header of L2 and press [CLEAR], press [ENTER]
Step 2: Arrow to the header of L2, press [VARS], arrow to [Y-VARS], press [1], press [1]
Step 3: Press [(], [2nd], [1], [)] then press [ENTER]
The calculator display would be:



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