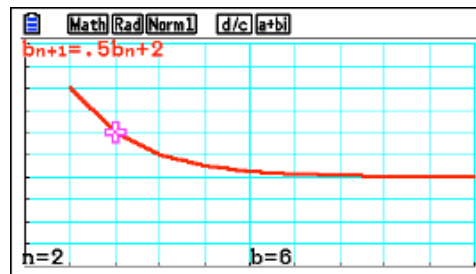
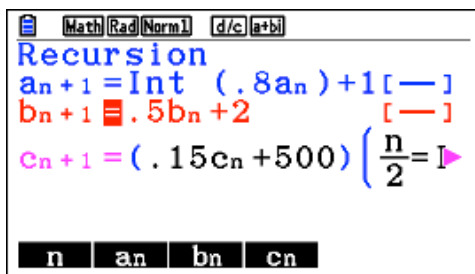


# RECURSION

## On The Casio PRIZM Graphing Calculator

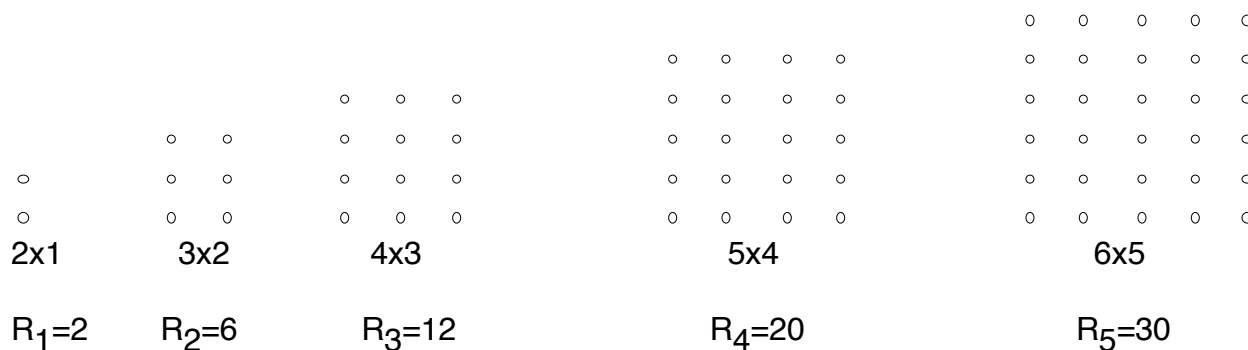


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## Recursion:

A rectangular array of dots can generate a sequence. Recursion is a method by which equations can be used to generate the terms of a sequence.

Consider an array of dots. The array is of size  $n+1$  by  $n$ .



An **explicit formula** for the **Casio PRIZM** in function mode would be  $Y1 = (x+1)x$

$a_{n+1} = (n+1)(n+2)$  determines the  $n$ th term for this sequence **explicitly** in **recursive mode**.

A **recursive formula** to generate any term of this sequence on the **Casio PRIZM** is  $b_n + 1 = b_{n-1} + 2(n+1)$ .

These sequential equations can be graphed on the PRIZM. The graphs can be traced and tables of values generated. Be sure that the calculator is in **Recursion mode**. See figures 1 through 7 below.

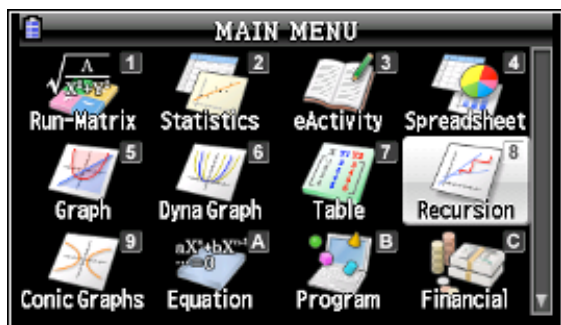


Figure 1 (mode)

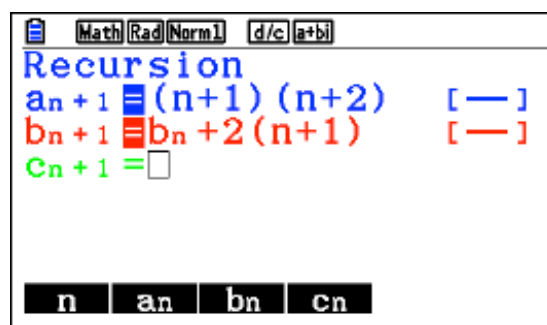


Figure 2 (Formulas)

Press Set (F5). Set  $a_0$  and  $b_0$  and  $a_1$  and  $b_1$  as follows in figure 3 and figure 4.

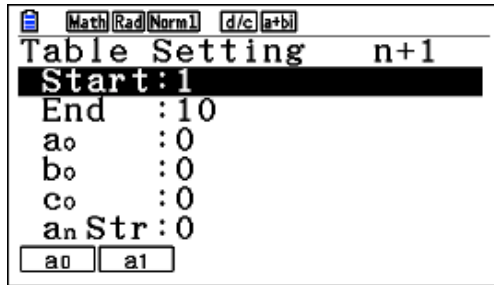


Figure 3 (Table Setting  $a_0$ )

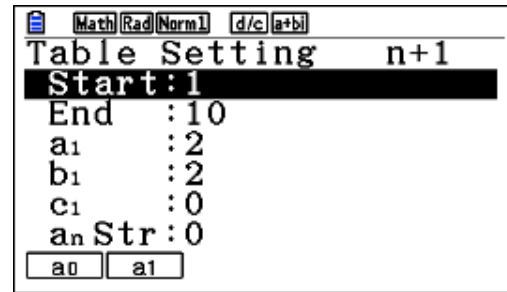


Figure 4 (Table Setting  $a_1$ )

**Press Table (F6) see Figure 5.**

**Press GPH-Plt (F6) see Figure 6**

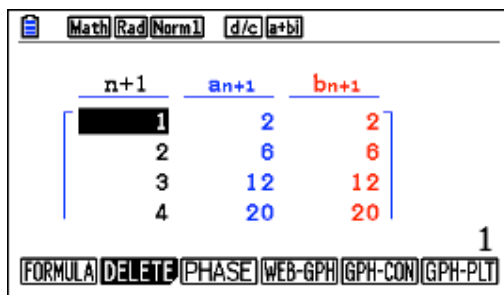


Figure 5 (table)

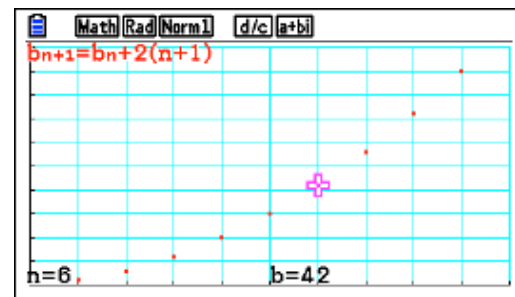


Figure 6 (Graph)

### Tower of Hanoi:

The fable of the Tower of Hanoi states that the Buddhist Monks of Hanoi are to solve a 100 tile puzzle. When the monks have completed the puzzle the world is to come to an end.

The Tower of Hanoi puzzle generates a sequence of numbers based on the number of tiles to move and the number of moves required.

Only one disk could be carried at a time. A disk could not be placed on top of a smaller disk. Only three locations were available on which to place the disks and the disk must always rest on one of those locations.

The monks began to move disks back and forth from the original pile to the pile at the new location, always keeping the piles in order (largest on the bottom, smallest on the top). According to the legend, before the monks could make the final move to complete the new pile in the new location, the temple would turn to dust and the world would end.

Is there any truth to this legend?

To answer, you will need some idea how long it will take the monks to finish their task, so you play a game using a small collection of disks and three piles into which you can put them. Play the game to find the smallest number of moves necessary to move all the disks. We will be using the mathematical pattern to generate a list of the number of moves. We will model the list explicitly and recursively. Use this to predict how long it will take the monks to move their 100 disks. Assume at least one second per move.

Go to <http://www.mazeworks.com/hanoi/> for a chance to try your hand at solving the puzzle.

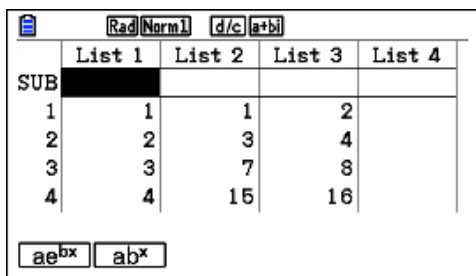
Also check out <http://www.lhs.berkeley.edu/Java/Tower/towerhistory.html> for further information on the Tower of Hanoi.

The table below lists the number of tiles and the corresponding number of moves required.

Tiles	Moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127

A recursive formula for this sequence would be  $a_1 = 1$ ,  $a_{n+1} = 2 a_n + 1$ . An explicit formula would be  $2^{n+1} - 1$ .

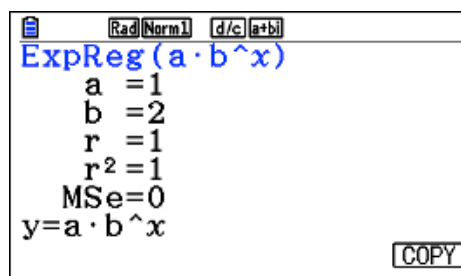
The calculator screens below will assist you in finding the number of tiles and associated moves. Using **Statistics** on the **PRIZM**, where List 1 = number of tiles, List 2 = number of moves, List 3 = List 2 + 1. **F2 CALC Reg F6 F2** then subtract 1 from the resulting equation. This yields  $2^x - 1$ .



	List 1	List 2	List 3	List 4
SUB				
1	1	1	2	
2	2	3	4	
3	3	7	8	
4	4	15	16	

ae<sup>bx</sup>   ab<sup>x</sup>

Figure 7 (List of tiles and moves)



ExpReg ( $a \cdot b^x$ )

a = 1  
b = 2  
r = 1  
 $r^2 = 1$   
MSe = 0  
 $y = a \cdot b^x$

COPY

Figure 8 (ExpReg)

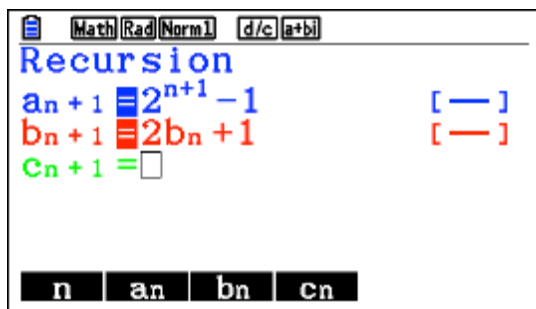


Figure 9 (Y= menu)

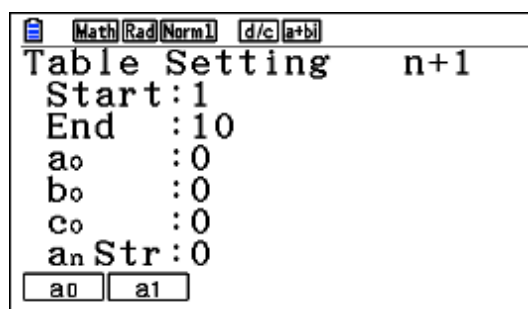


Figure 10 (Table setting  $a_0$ )

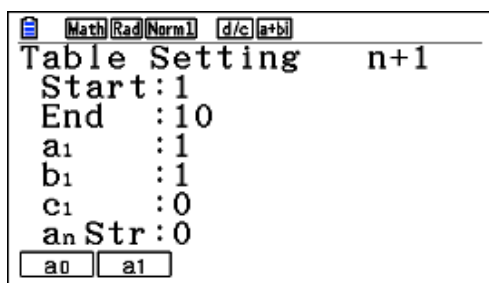


Figure 11 (Table setting  $a_1$ )

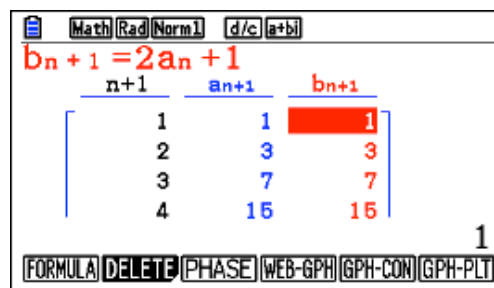


Figure 12 (Table)

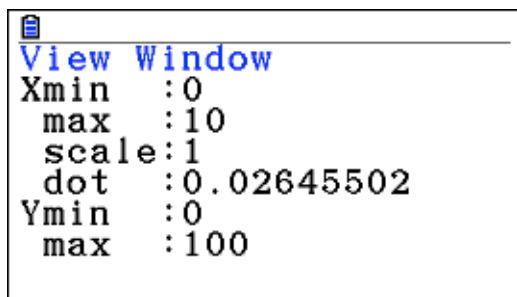


Figure 13 (Graph Window)

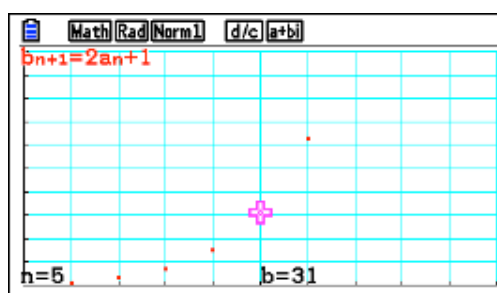


Figure 14 (Graph)

## David Kapolka Recursion on the Casio PRIZM

### Forest for the Trees:

Imagine yourself a county commissioner facing a vote on a forestry proposal. A forest near you contains approximately 4000 trees. A local forester proposes to harvest 20% of the trees each year but replace that harvest with 1000 new trees each year. Replacement of the 1000 trees is guaranteed until harvest. What is the future of the forest? Will it eventually disappear? Will it flourish? Evaluate the data and make a

determination as to how you would vote if you were on the local planning commission. Each year 80% of the trees will remain and 1000 new trees will be added to the forest. No fraction of a tree will be harvested so the following equation will generate the number of trees remaining after each harvest.

On your PRIZM, press Menu, 8 (Recursion) see below for the key strokes:

$\boxed{8}$   $\boxed{\text{OPTN}}$   $\boxed{\text{F5}}$   $\boxed{\text{F2}}$   $\boxed{(\text{ )}}$   $\boxed{\cdot}$   $\boxed{8}$   $\boxed{\text{F2}}$   $\boxed{)} +$   $\boxed{1}$   $\boxed{0}$   $\boxed{0}$   $\boxed{0}$   $\boxed{\text{EXE}}$   $\boxed{\text{F5}}$   
 $\boxed{1}$   $\boxed{\text{EXE}}$   $\boxed{5}$   $\boxed{0}$   $\boxed{\text{EXE}}$   $\boxed{4}$   $\boxed{0}$   $\boxed{0}$   $\boxed{0}$   $\boxed{\text{EXE}}$

Figure 1

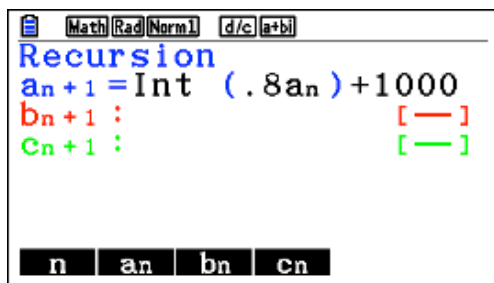


Figure 2

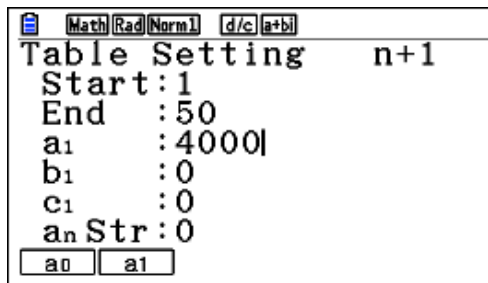


Figure 3

Tracing on the graph or using the Table produces some interesting results. What happens to the forest over time? When?

Press F6 for **Table** and scroll down using the control wheel. Examine what happens over time using the table.

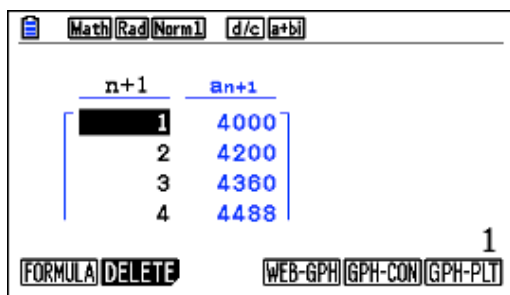


Figure 4

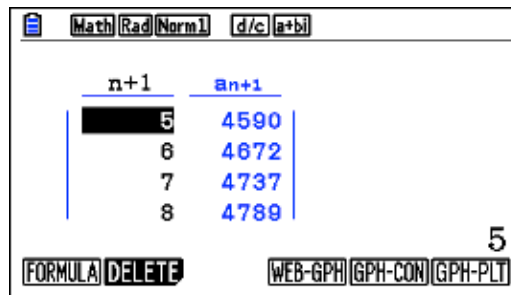


Figure 5

To adjust the **graphing window** press shift-F3. Set as indicated in Figure 6.

To **Plot the function** press Exit F6.

To **Trace**, press F1 and use arrow wheel to move the cursor.

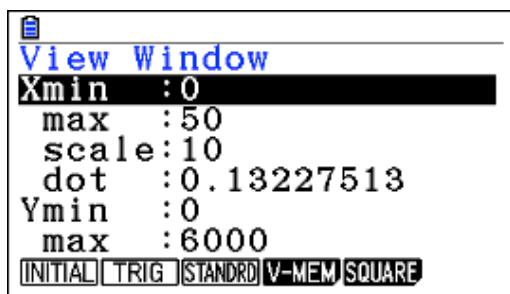


Figure 6

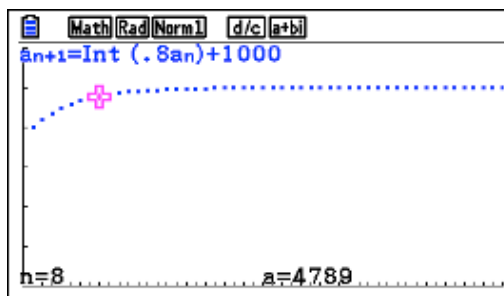


Figure 7

## Recursion and medicine:

### Dentistry:

Often an oral surgeon prescribes a mega dose of antibiotics prior to a procedure to prevent infection. Following the procedure, a lesser amount of antibiotics is taken for several days, also to prevent infection. The amount and percent of the medicine remaining in the body is of concern to the dentist and patient. A recursive equation can model this situation effectively.

Let us suppose that 50% of the medicine remains in the body after 24 hours. The patient takes 8 capsules of Trimox on the morning of the procedure and 2 each day after that for several days. What level of Trimox is in the patient's body if this dosage continued for 2 weeks?

Note: Deselect  $a_n + 1$  by pressing F1 (may need to press F1 twice).

Set  $b_{n+1}$  and Table Setting (F5) as in Figure 8 and Figure 9.

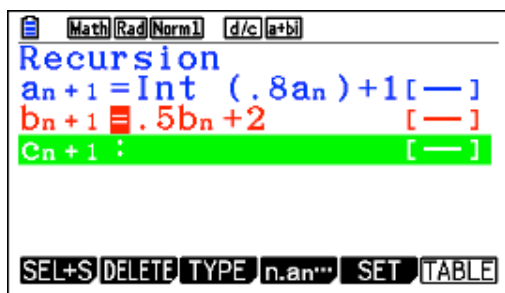


Figure 8

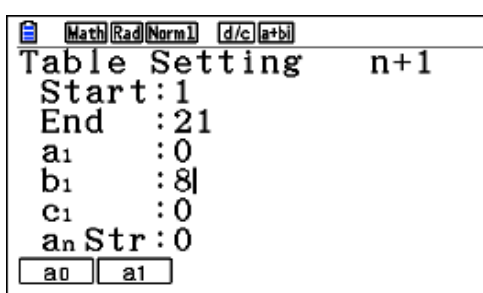


Figure 9

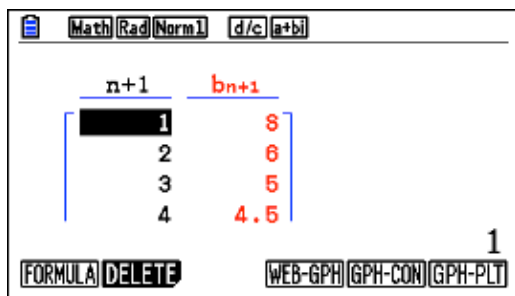


Figure 10

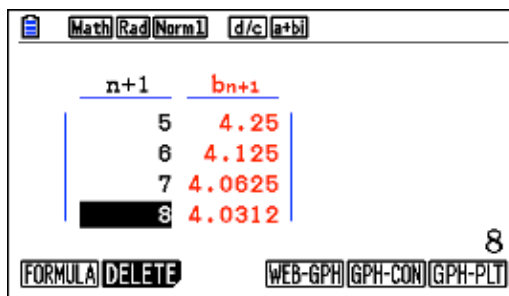


Figure 11

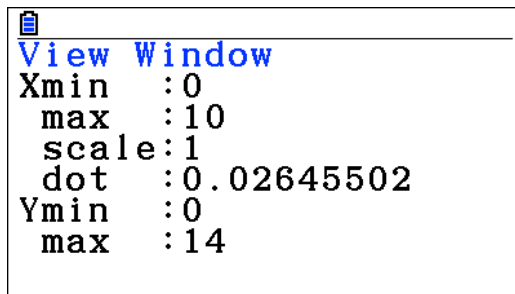


Figure 12

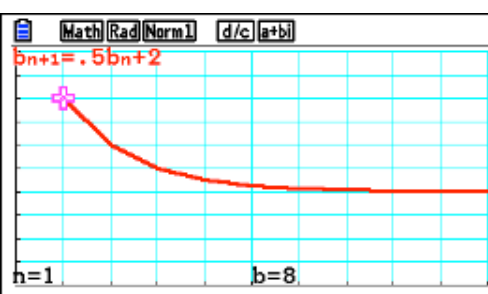


Figure 13

### Maintenance Drugs:

Many people take drugs to regulate such things as blood pressure, heart rate, or to thin their blood to avoid a stroke. These drugs are taken daily. Often, the patient is required to have blood drawn and evaluated to see if the level of the drug dosage needs to be adjusted so as to stay within a certain range.

Let us suppose that 15% of a maintenance drug remains in the body after 24 hours. The patient takes 500mg of the drug daily. How many mg of the drug remain in the body if this daily regimen continues for many days?

Change  $c_{n+1}$  as follows:

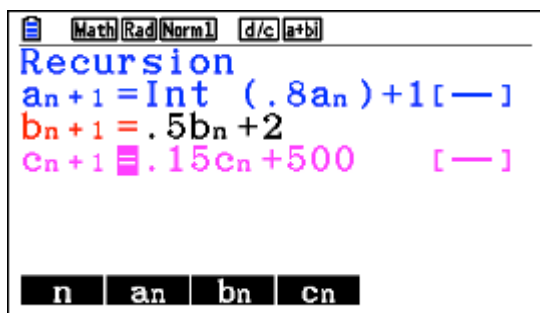


Figure 14

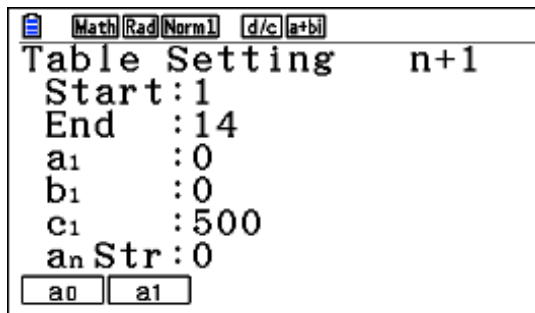


Figure 15



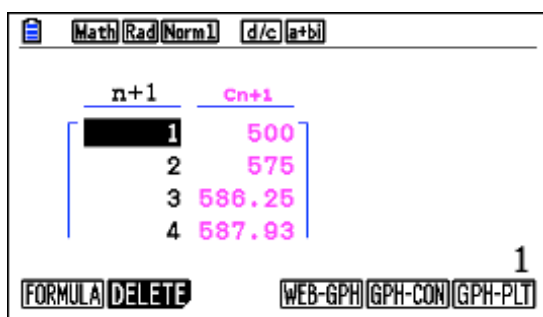


Figure 16

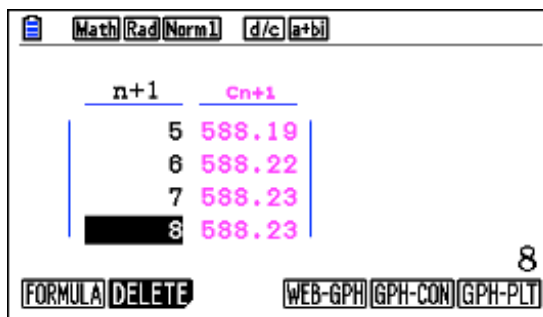


Figure 17

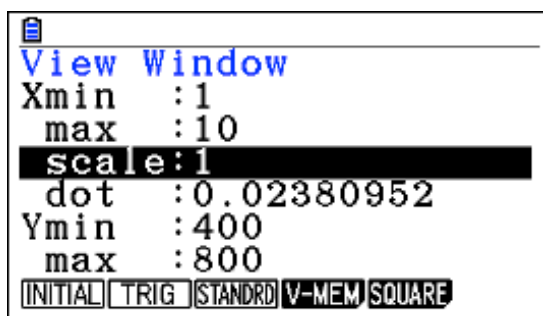


Figure 18

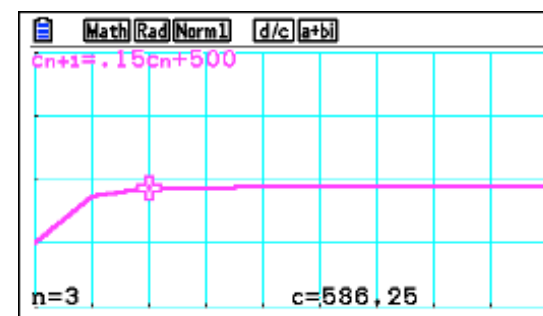


Figure 19

The limit appears to be very close to 588.24 mg.

Change the percent remaining in the body to 50%, 25%, 5% and investigate the results.

### Odd/Even Day Doses

Suppose after a recent checkup, the doctor decides to increase the dosage in the following manner: on even days you take one pill and on odd days you take one and one half pills. In milligrams, that will work out to 500mg on even days and 750mg on odd days. What would that look like graphically? Can you alter the equation to model this situation? A computer technique for determining odd/even is needed here. One such technique is to use the “TEST” feature of the calculator as follows:

If  $\text{INT}(n/2) = n/2$  then the day is an even numbered day, otherwise the day would be an odd numbered day. Also note that if the “tested” statement is true, the calculator returns a 1, if false, the calculator returns a 0. These values are then used in the equation to determine the appropriate dosage. Try for yourself with  $\text{Int}(5/2) = 5/2$  and  $\text{Int}(6/2) = 6/2$ . The first should yield a 0 and the second a 1.

Below are the equations and graph.

$$c_{n+1} : (.15c_n + 500)\left(\frac{n}{2} = \text{int} \frac{n}{2}\right) + (.15c_n + 750)\left(\frac{n}{2} \neq \text{int} \frac{n}{2}\right)$$

Start : 1

End : 14

$c_n$  : 500

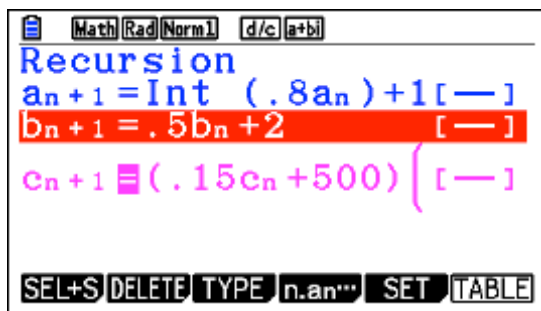


Figure 20

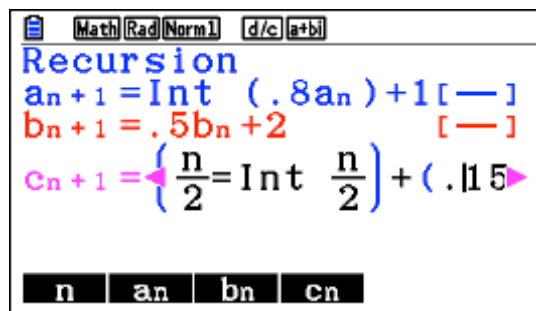


Figure 21

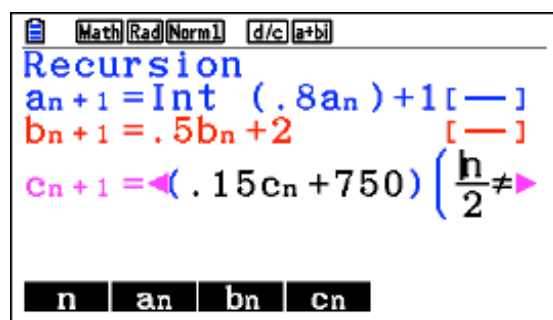


Figure 22

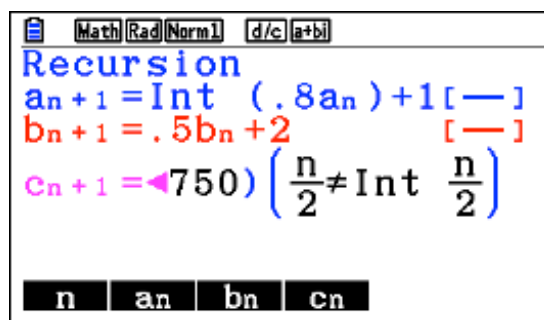


Figure 23

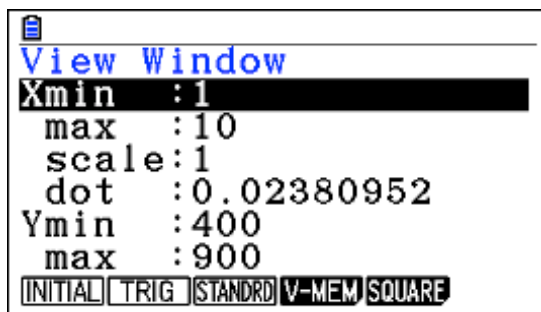


Figure 24

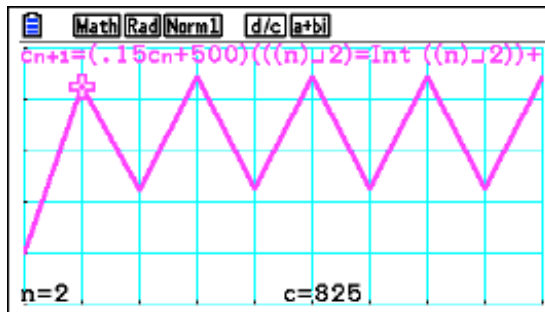
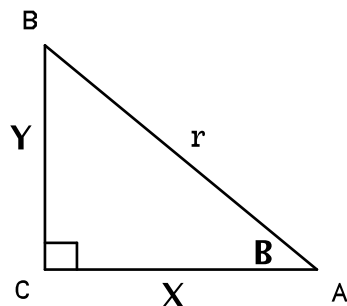


figure 25

# Parametrics For Beginners

Applications from Algebra to Calculus

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$$\cos \beta = \frac{x}{r} \quad \sin \beta = \frac{y}{r} \quad \text{so} \quad \begin{aligned} x &= r \cos \beta \\ y &= r \sin \beta \end{aligned}$$

To graph lines such as  $y = ax + b$  in parametric form use the following

$$\begin{aligned} X_{t1} &= t \\ Y_{t1} &= at + b \end{aligned}$$

For example to plot  $y = 2x + 1$ , set

$$\begin{aligned} X_{t1} &= t \\ Y_{t1} &= 2t + 1 \end{aligned}$$

The differences between using parametric equations versus standard function equations include setting a range of values for “t”. Both x and y depend on t in order to plot points as (x,y) values. Setting the viewing window in parametric mode includes setting the Tmin and Tmax. For  $y = 2x + 1$  typical settings might be the “INITIAL” settings of:

$$\begin{array}{lll} X_{\min} : -6.3 & Y_{\min} : -3.1 & T_{\min} : -6 \\ X_{\max} : 6.3 & Y_{\max} : 3.1 & T_{\max} : 6 \\ \text{Scale} : 1 & \text{Scale} : 1 & \text{Tpitch} : .05 \text{ (basically } \Delta T) \end{array}$$

On the PRIZM calculator press MENU 5 (Graph) TYPE and choose Param.

Set  $X_{t1} = T$  by pressing the  $x, \theta, T$  key, and  $Y_{t1} = 2T + 1$ .

To set the Window, press SHIFT F3 (V-WIN) and fill in the values listed above using the REPLAY wheel to move down for the next entry.

Once that is completed, press EXE and DRAW (F6)

### Plotting a parabola in vertex form:

$$\begin{aligned} X_{t1} &= t \\ Y_{t1} &= a(t-h)^2 + k \end{aligned}$$

Or Standard form:

$$Xt1 = t$$

$$Yt1 = at^2 + bt + c$$

Choose values for a,h, and k, or a,b, and c. Graph normally. For example plot:

$$Xt1 = t$$

$$Yt1 = 2(t-1)^2 + 2$$

One nice advantage of parametric equations is the ability to plot the inverse of a function by switching the X and Y parameters:

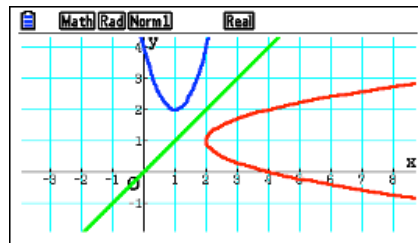
$$Xt2 = 2(t-1)^2 + 2$$

$$Yt2 = t$$

We can also plot the reflection line  $y = x$  as:

$$Xt3 = t$$

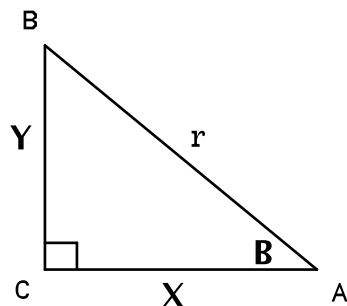
$$Yt3 = t$$



By tracing on the parabolas we can demonstrate some of the properties of inverses and their graphical relationships. For example:

1. The graphs are reflected over the line  $y = x$
2. The coordinates  $(x,y)$  are reversed for the function and its inverse for the same  $t$ -value.
3. Can you think of other benefits?
4. How would you plot  $y^2 = x$  ?

**Conics:**



$$\cos \beta = \frac{x}{r} \quad \sin \beta = \frac{y}{r} \quad \text{so} \quad \begin{aligned} x &= r \cos \beta \\ y &= r \sin \beta \end{aligned}$$

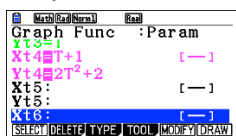
**Standard form of conics:**

**Parabola:**  $y - k = a(x - h)^2$

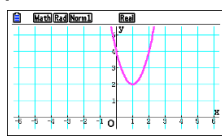
**Ellipse:** 
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

**Hyperbola:** 
$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

**Sample Parabolic equation:**



Equations



Graph

**Ellipse:** 
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$
 To put an ellipse into parametric mode we will need an equation using trigonometry that is the sum of two squares that equals 1.

$$\cos^2 \theta + \sin^2 \theta = 1 \quad \left\{ \begin{array}{l} \text{Set } \frac{(x - h)^2}{a^2} = \cos^2 \theta \text{ and solve for } x; \quad x = a \cos \theta + h \\ \text{Set } \frac{(y - k)^2}{b^2} = \sin^2 \theta \text{ and solve for } y; \quad y = b \sin \theta + k \end{array} \right.$$

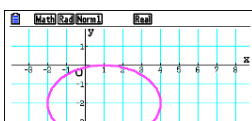
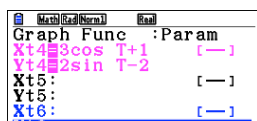
**Ellipse parametrically:**

$$X_2T = a \cos T + h$$

$$X_2T = 3 \cos T + 1$$

$$Y_2T = b \sin T + k$$

Sample equation:  $Y_2T = 2 \sin T - 2$



**Hyperbola:**  $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$  To put a hyperbola into parametric mode we will need an equation using trigonometry that is the difference of two squares that equals 1.

$$1 + \tan^2 \theta = \sec^2 \theta \Rightarrow \sec^2 \theta - \tan^2 \theta = 1$$

$$\Rightarrow 1/\cos^2 \theta - \tan^2 \theta = 1$$

$$\left\{ \begin{array}{l} \text{Set } \frac{(x-h)^2}{a^2} = 1/(\cos^2 \theta) \text{ and solve for } x; \quad x = a/(\cos \theta) + h \\ \text{Set } \frac{(y-k)^2}{b^2} = \tan^2 \theta \text{ and solve for } y; \quad y = b(\tan \theta) + k \end{array} \right.$$

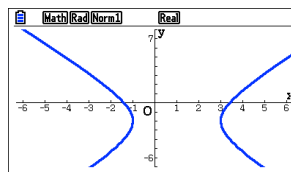
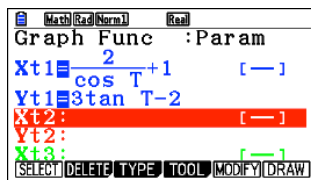
**Hyperbola parametrically:**

$$X_3T = a/(\cos T) + h$$

$$X_3T = 2/(\cos T) + 1$$

$$Y_3T = b \tan T + k$$

Sample equation:  $Y_3T = (3 \tan T) - 2$



**Polar Graphs:**

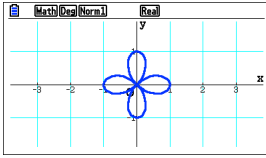
Put the calculator in degree mode using SETUP (Shift MENU)

Let  $r = \cos(2T)$  the parametric equations would be:

$$Xt4 = \cos 2T \cos T$$

$$Yt4 = \cos 2T \sin T$$

Set the window to be INITIAL and  $T = \{0, 360, 5\}$



How would you center the plot at (2,-2)?

Now try:

1)  $r = 3\cos 4T$

2)  $r = 1 + 2\cos T$

### Lissajou Figures (experiments for fun)

$Xt5 = \cos 5T$  Try T – coefficients of (7,5) , (3,7) , (6,4) etc on your own

$Yt5 = \sin 3T$

### Chrysanthemum:

$r = \sin .7T \quad 0 < T < 20\pi$

$Xt6 = \sin .7T \cos T$

$Yt6 = \sin .7T \sin T$

$-2.52 < X < 2.52$

$-1.24 < Y < 1.24$

$0 < T < 20\pi$

X Scale = .5

Y Scale = .5

ptch: .1

### Sine Wave Tracer:

Radian Mode. This will plot a circle and corresponding sine wave. Use the trace feature to compare locations on each plot for the same T-value.

$Xt6 = 2 \cos T$

$Xt7 = T$

$Yt6 = 2 \sin T$

$Yt7 = 2\sin T$

$-2\pi < X < 2\pi$

$-4 < Y < 4$

$0 < T < 2\pi$

X Scale =  $\pi/2$

Y Scale = 1

ptch: .1

### Classic Train Problem:



A train leaves Chicago and heads toward New York at 50 mph. An hour later a second train leaves Chicago on a parallel track heading to New York traveling at 55 mph. When does the second train overtake the first? The X values will represent the distance of the train and the Y value will represent the track number.

$$\begin{array}{ll} X_{t6} = 50T & X_{t7} = 55(T-1) \\ Y_{t6} = 1 & Y_{t7} = 2 \end{array}$$

$$\begin{array}{lll} X_{\min} : 0 & Y_{\min} : -2 & T_{\min} : 0 \\ X_{\max} : 850 & Y_{\max} : 4 & T_{\max} : 17 \\ \text{Scale} : 100 & \text{Scale} : 1 & T_{\text{pitch}} : .01 \end{array}$$

Now we'll use the Table mode to examine where the trains meet.

How would you change the functions so that the trains was heading in toward each other on parallel tracks? When would they meet?

### Baseball:

Suppose a batter hits a pitched ball with an initial velocity of 150 ft/sec at an angle of 20 degrees. The batted ball will slow down at a rate of 8.8 feet/sec after being hit. The pitch comes in at a height of 3 feet. Model this situation parametrically and determine if the ball will clear a 10 foot fence 400 feet from home plate.

$$\begin{array}{l} X_{t7} = 150T\cos 20 - 8.8T \\ Y_{t7} = -16T^2 + 150T\sin 20 + 3 \end{array}$$

$$\begin{array}{lll} X_{\min} : 0 & Y_{\min} : 0 & T_{\min} : 0 \\ X_{\max} : 410 & Y_{\max} : 100 & T_{\max} : 5 \\ \text{Scale} : 50 & \text{Scale} : 10 & T_{\text{pitch}} : .1 \end{array}$$

### Roots of Complex Numbers:

Find the three cube roots of 8.

The roots are  $[8^{1/3}, 0 + 360n/3]$ , where  $n = 0, 1, 2$

In trigonometric and complex form:

$$2(\cos(0) + i\sin(0)) = 2(1) + i(0) = 2 + 0i$$

$$2(\cos(120) + i\sin(120)) = 2(-1/2 + i\sqrt{3}/2) = -1 + i\sqrt{3}$$

$$2(\cos(240) + i\sin(240)) = 2(-1/2 + i\sin(240)) = -1 - i\sqrt{3}$$

Parametrically they would be:

$$X_{t1} = 2(\cos(2\pi/3 \cdot T))$$

$$Y_{t1} = 2(\sin(2\pi/3 \cdot T))$$

$$X_{\min} : -2.5 \quad Y_{\min} : -2.5 \quad T_{\min} : 0$$

$$X_{\max} : 2.5 \quad Y_{\max} : 2.5 \quad T_{\max} : 3$$

$$\text{Scale} : 1 \quad \text{Scale} : 1 \quad \text{Tpitch} : 1$$

Next plot:

- 1) The 4<sup>th</sup> roots of 16
- 2) The 6<sup>th</sup> roots of 64
- 3) The cube roots of  $1 + i$

### Powers of Complex Numbers:

To raise a complex number such as  $-1 + i$  to any power parametrically, start with  $[r, \theta]$  form.  $-1 + i = [\sqrt{2}, 135]$ , then the general form would be:

$$X_{t1} = (\sqrt{2})^T (\cos 135T)$$

$$Y_{t1} = (\sqrt{2})^T (\sin 135T)$$

Mode: Degree

$$X_{\min} : -4.7 \quad Y_{\min} : -3.1 \quad T_{\min} : 0$$

$$X_{\max} : 4.7 \quad Y_{\max} : 3.1 \quad T_{\max} : 6$$

$$\text{Scale} : 1 \quad \text{Scale} : 1 \quad \text{Tpitch} : 1$$

Graph and trace.

### Projectile Examples:

**Bullet:** If you fire a projectile horizontally in a vacuum and drop an object at the same time from the same height as the projectile, they will collide with each other every time, regardless of the initial velocity of the projectile. This can be demonstrated parametrically.

$$X_{t1} = 700T \quad X_{t2} = 300$$

$$Y_{t1} = -4.9T^2 \quad Y_{t2} = -4.9t^2$$

Xmin : 0      Ymin : -2      Tmin : 0  
 Xmax : 310   Ymax : 2      Tmax : .6  
 Scale : 100   Scale : .5      Tpitch : .001

**Ball Toss:** This example will model a ball tossed in the air vertically and also plot the associated height versus time graph.

$X_{t1} = T$                        $X_{t2} = 14$   
 $Y_{t1} = -4.9T^2 + 50T$          $Y_{t2} = -4.9t^2 + 50T$

Xmin : -3      Ymin : -100   Tmin : 0  
 Xmax : 16      Ymax : 200   Tmax : 12  
 Scale : 1       Scale : 50      Tpitch : .01

**Moving Car:** Suppose you have a car moving horizontally at a constant rate and a ball is tossed vertically (ignore air resistance). Then the ball will land in the car every time. Plot only the first and third graphs.

$X_{t1} = 3T$   
 $Y_{t1} = 2$

$X_{t2} = 3T$   
 $Y_{t2} = -4.9(T-.2)^2 + 10(T-.2) + 2$

$X_{t3} = 3T$   
 $Y_{t3} = Y_{t2} / (Y_{t2} \geq 2)$

Xmin : 0      Ymin : 0      Tmin : 0  
 Xmax : 10      Ymax : 12      Tmax : 14  
 Scale : 1      Scale : 1      Tpitch : .005