Session 617

Visualizing Patterns: Fibonacci Numbers and 1,000-Pointed Stars

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STARS

Abstract

Patterns in mathematics invite exploration and often arise in peculiar places. This talk will engage you in two such experiences appropriate for students in grades 9-12:

(1) Determining how many stars have a given number of points

(2) Discovering new relations among the Fibonacci numbers via strategic visual arrangement of them

American Invitational Math Exam (AIME) Problem 8 (2004)

How many 1,000-pointed stars are there?

Define a regular n-pointed star to be the union of n line segments P_1P_2, P_2P_3, …, P_nP_1 such that

*The points P_1, P_2, …, P_n are coplanar and no 3 of them are collinear;
*Each of the n segments intersects at least one of the other line segments at a point other than the endpoint;
*All of the angles at P_1, P_2, …, P_n are congruent;
*All of the n line segments P_1P_2, P_2P_3, …, P_nP_1 are congruent;
*The path P_1P_2, P_2P_3, …, P_nP_1 turns counterclockwise at an angle less than 180 degrees at each vertex.

Agenda

- Stars
  - AMC and AIME problems
- Fibonacci Numbers
  - Tiling
  - *k*-column arrangements of Fibonacci numbers

The American Mathematics Competitions (AMC)

http://amc.maa.org

- Series of Middle/High School Mathematics Competitions
- MAA-sponsored since 1950
- 100,000's of students participate annually from 1,000's of schools nationwide

The American Mathematics Competitions (AMC)

http://amc.maa.org

SCHEDULE:

- AMC 8 (November)
- AMC 10 and 12 (February)
- AIME (March)
- USA Math Olympiad (April)
- Math Olympiad Summer Program (June)
- International Math Olympiad (July)

The American Mathematics Competitions (AMC)

http://amc.maa.org

TOPICS:

- Algebra
- Number Theory
- Functions
- Counting and Probability
- Sequences
- Geometry

The American Mathematics Competitions (AMC)

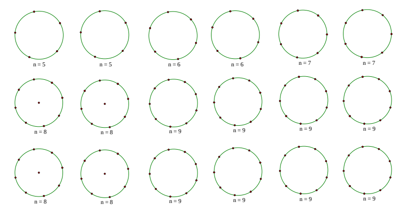
AIME

- 15 questions: each answer is an integer in the range 000-999
- 3 hours
- Calculators are not allowed
- Top ≈500 nationwide scorers are invited to take the United States Math Olympiad

American Invitational Math Exam (AIME) Problem 8 (2004)

How many 1,000-pointed stars are there?

Activity Sheet #1



American Invitational Math Exam (AIME) Problem 8 (2004)

How many 1,000-pointed stars are there?

n	Pairs of j values producing stars	Number of n-pointed stars
2	N/A	0
3	N/A	0
4	N/A	0
5	(2,3)	1
6	N/A	0
7	(2,5) and (3,4)	2
8	(3,5)	1
9	(2,7) and (4,5)	2
10	(3,7)	1
11	(2,9) and (3,8) and (4,7) and (5,6)	4
12	(5,7)	1

American Invitational Math Exam (AIME) Problem 8 (2004)

How many 1,000-pointed stars are there?

Activity Sheet #3

#1) Circle the numbers below that are relatively prime to n = 20 and determine the number of stars with 20 points. ANSWER: 1,3,7,9,11,13,17,19 ➔ 3 stars with 20 points

#2) Circle the numbers below that are relatively prime to n = 45 and determine the number of stars with 45 points. ANSWER: 1,2,4,7,8,11,13,14,16,17,19,22,23,26,28,29,31,32,34,37,38,41,43,44 ➔ 11 stars with 45 points

#3) How many numbers from 1 to 100 contain a factor of 2 ? 50
How many numbers from 1 to 100 contain a factor of 5 ? 20
How many numbers from 1 to 100 contain a factor of both 2 and 5 ? 10

How many numbers from 1 to 100 are relatively prime to 100 ? 40
How many 100-pointed stars are there? 19

#4) How many numbers from 1 to 1000 contain a factor of 2 ? 500
How many numbers from 1 to 1000 contain a factor of 5 ? 200
How many numbers from 1 to 1000 contain a factor of both 2 and 5 ? 100

How many numbers from 1 to 1000 are relatively prime to 1000 ? 400
How many 1000-pointed stars are there? 199

FIBONACCI NUMBERS

Your list of Numbers

a
b
a + *b*
a + 2*b*
2*a* + 3*b*
3*a* + 5*b*
5*a* + 8*b*
8*a* + 13*b*

- You create a list of numbers generated in the same way as the Fibonacci numbers (add the previous two numbers)
- The first two numbers are

  $J(0) = a$ and $J(1) = b$

Fibonacci Numbers

1
1
2
3
5
8
13
…

- The first two numbers are:

  $f(0) = 1$ and $f(1) = 1$

- The sequence of numbers generated by adding the two previous numbers:

  $f(n) = f(n-1) + f(n-2)$

Yellow Handout

TILING

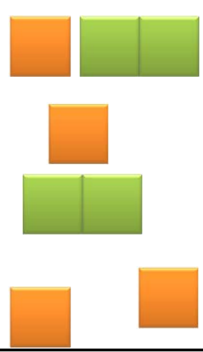
Lucas Numbers

1
3
4
7
11
18
29
…

- The first two numbers are $L(1) = 1$ and $L(2) = 3$
- The Lucas Numbers are a list of numbers generated in the same way as the Fibonacci numbers (add the previous two numbers):

  $L(n) = L(n-1) + L(n-2)$

Yellow Handout



Directions:

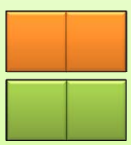
- You are being asked to arrange tiles using only two types of tiles. One type of tile is of length 1 by 1 and the other type of tile is twice as long (1 by 2 tile).
- Objective: Before you begin to tile you must discover a way to look at all possible arrangements of length 1 by *n*, where *n* is the length of the space available to tile.

Yellow Handout

- First, how many different ways can you arrange tiles if you are only required to arrange them to make 1 by 1 tile arrangement?

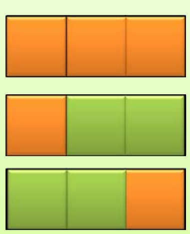
Yellow Handout

- Second, how many different ways can you arrange tiles if you are only required to arrange them to make 1 by 2 tile arrangement?



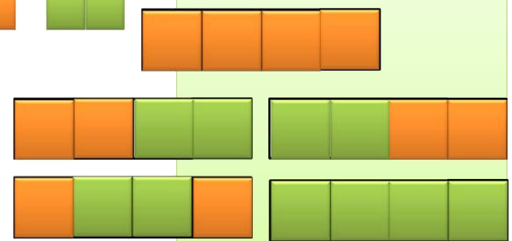
Yellow Handout

- Next, how many ways to arrange tiles to make a 1 by 3 tile arrangement?



Yellow Handout

- Finally, how many ways to arrange tiles to make a 1 by 4 tile arrangement?



Yellow Handout

- What do you predict will be the number of ways to arrange tiles for a 1 by 5? Or a 1 by 6? Or a 1 by *n*?

Number of Arrangements

Yellow Handout

$f(n)$	Number of different ways	Picture of Arrangements
$f(0)$		
$f(1)$	1	
$f(2)$	2	
$f(3)$	3	
$f(4)$	5	
$f(5)$	8	
…	…	…
$f(n)$		

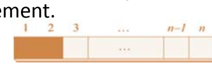
Combinatorial Interpretation

Yellow Handout

How many ways to arrange tiles of 1 by n length with 1 by 1 tiles and 1 by 2 tiles?

- If the first tile is a 1 by 1 tile, then there are $f(n-1)$ ways to complete the arrangement.

- If the first tile is a 1 by 2 tile, then there are $f(n-2)$ ways to complete the arrangement.



- Hence,

$$f(n) = f(n-1) + f(n-2).$$

The Arrangement

Peach Handout

1	1
2	3
5	8
13	21
34	55
89	144
…	…

- What about using two columns?
  - What is the linear combination for this two columns case?

$f(n) =$ ___ $f(n-2) +$ ___ $f(n-4)$

Peach Handout

K-COLUMN ARRANGEMENTS OF FIBONACCI NUMBERS

The Arrangement

Peach Handout

1	1	2
3	5	8
13	21	34
55	89	144
233	377	610
987	1597	2584
…	…	…

- Continue through the three columns case.
  - What is the linear combination for this three columns case?

$f(n) =$ ___ $f(n-3) +$ ___ $f(n-6)$

The Arrangement

Peach Handout

1
1
2
3
5
8
13
21

- We want to arrange the Fibonacci numbers into various numbers of columns.
- We are familiar with the 1 column case
  - We know the linear combination to list these numbers in 1 column is:

$$f(n) = f(n-1) + f(n-2)$$

The Arrangement

Peach Handout

- Can we stop here and predict the formula for $f(n)$ case when k=4, k=5, and so on?
- Is there a pattern?
- Is this enough to say there is a pattern?

Results

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Results			
Number of Columns	Linear Combination		
1	_1_ *f(n-1)	+	_1_ *f(n-2)
2	_3_ *f(n-2)	+	_(-1)_*f(n-4)
3	_4_ *f(n-3)	+	_1_ *f(n-6)
4	_7_ *f(n-4)	+	_(-1)_*f(n-8)
5	_11_ *f(n-5)	+	_1_ *f(n-10)
6	_18_ *f(n-6)	+	_(-1)_*f(n-12)
7	_29_ *f(n-7)	+	_1_ *f(n-14)

Conclusion of our pattern

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- We can now take our results from the table above and our description of the patterns that arise from the coefficients, to generalize all our linear combination to one recurrence relation for every $n>2k$, where $k\geq 1$:

$$f(n) = \underline{\;L(k)\;}\, f(n-k) + \underline{\;(-1)^{k+1}\;}\, f(n-2k)$$

The pattern of the linear combination

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- Study the linear combinations in the table as a function of *k,* the number of columns.
- We observe that a pattern arises among the coefficients. We describe this pattern:

The coefficient of $f(n-k)$ in the k^{th} row	
The coefficient of $f(n-2k)$ in the k^{th} row	

Thank you!

Questions?

The pattern of the linear combination

Peach Handout

The coefficient of $f(n-k)$ in the k^{th} row	The coefficient is the Lucas number, $L(k)$.
The coefficient of $f(n-2k)$ in the k^{th} row	+1 if k is odd. −1 if k is even. In other words, $(-1)^{k+1}$

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