Matrices in action

.

NCTM 2014

Brett Stephenson
Guilford Young College
Hobart, Australia

Matrices in action

- First developed by Sylvester & Cayley
- The term 'matrix' was first used by James Sylvester in "On a new class of theorems" (1850) to represent an array of elements that can be added and multiplied.
- The understanding and significance of the algebra of matrices was credited to Arthur Cayley (1855) and was the first to introduce matrix multiplication.
- They made formal what had been developed by Chinese, Japanese (Seki Kowa, 1683), and German (Liebniz, 1693) mathematicians.
- Matrices are now used in data encryption/decryption, solving systems of equations, quantum mechanics, graph and game theory, electrical networks, economic models and computer graphic manipulations.

Matrices in action

Definitions:
An *access* network shows the connections between points (vertices)
An adjacency matrix can be used to summarise the information from an access network.
A stochastic matrix is one where all of the entries are non-negative real numbers representing a probability and the sum of each row is equal to 1.

Matrices in action

Common Core

CCSS.N-VM.6 using matrices to represent and manipulate data

CCSS.N-VM.C8 matrix multiplication

Matrices in action

Australian Curriculum

ACMG013 - use matrices to display information

ACMG015 - perform matrix multiplication

ACMG016 - use matrices to model and solve problems

Matrices in action

- System - what we are looking at
- State - the possibilities of the system
- Stochastic process - the sequence of states that a system moves through

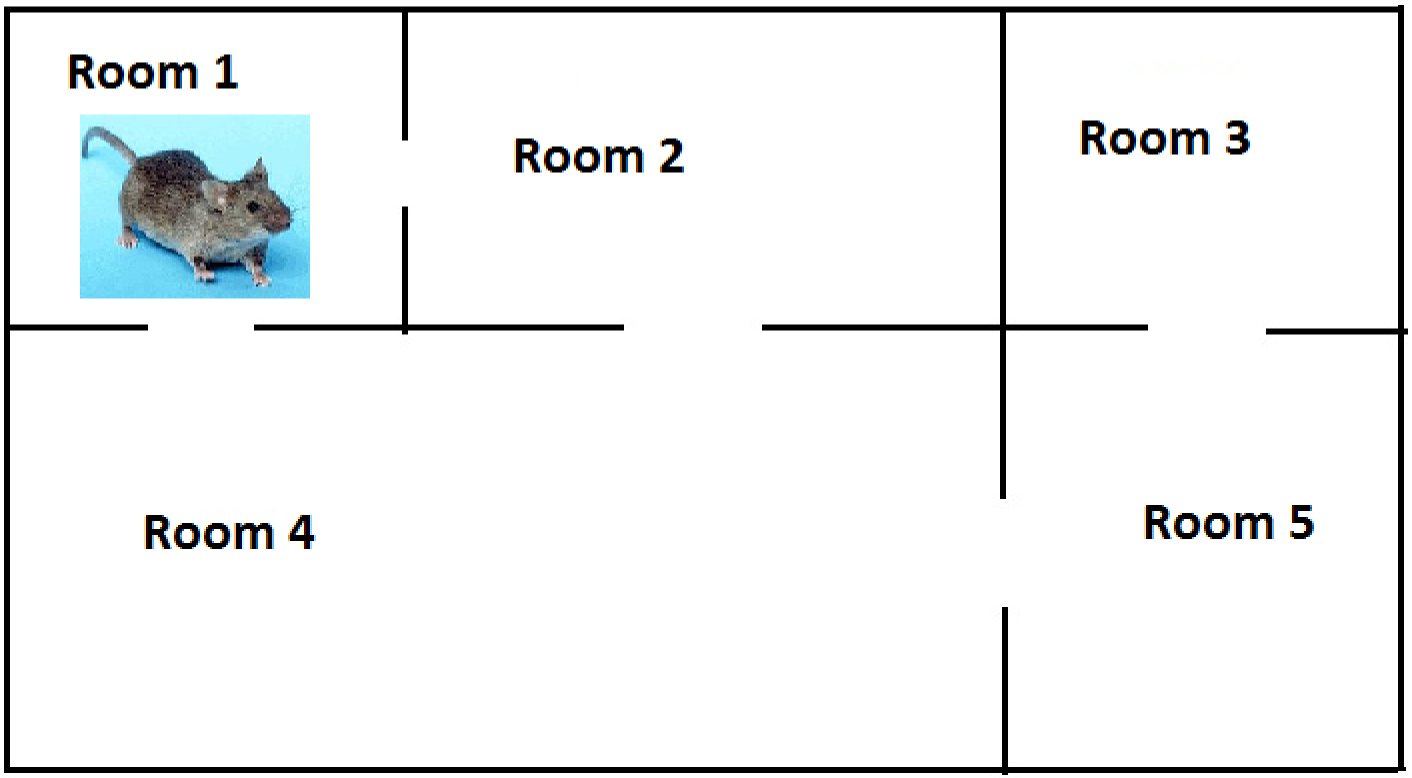
Matrices in action

5 applications

1. The lab mouse in a maze
2. Rental cars
3. Telstra market share
4. Chances of finishing a post graduate program
5. A food web in Antarctica

1. The lab mouse problem

A lab mouse is placed into Room 1 in a ‘lab maze” and the movements recorded as the mouse moved from room to room



The lab mouse problem

- The lab mouse is able to move from room to room and it is assumed that is moves randomly from room to room through one of the available doorways. This assumption means that the chance of the mouse selecting any door is equally likely.

The lab mouse problem

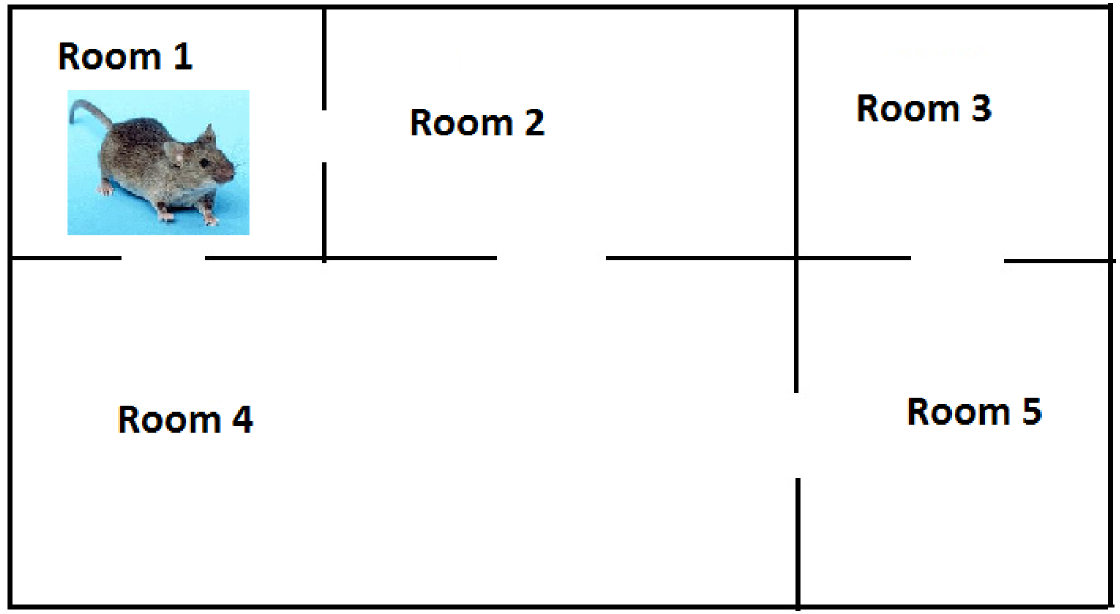
In the lab mouse problem

- system = the lab mouse in the maze
- state = the room that the mouse is in
- stochastic process = the probability that the mouse will be each of the particular rooms after each time period

The lab mouse problem

The probabilities that the mouse will move from Room 1 to any of the other rooms in the maze can be determined

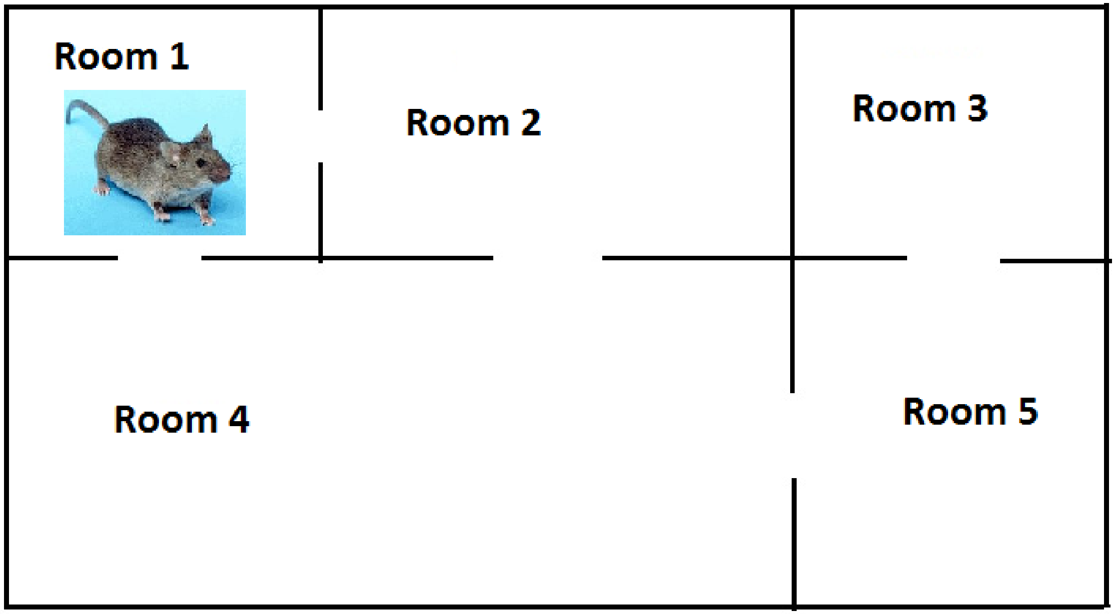
Mouse	To	1	2	3	4	5
From	***1***					



The lab mouse problem

The probabilities that the mouse will move from Room 1 to any of the other rooms in the maze can be determined

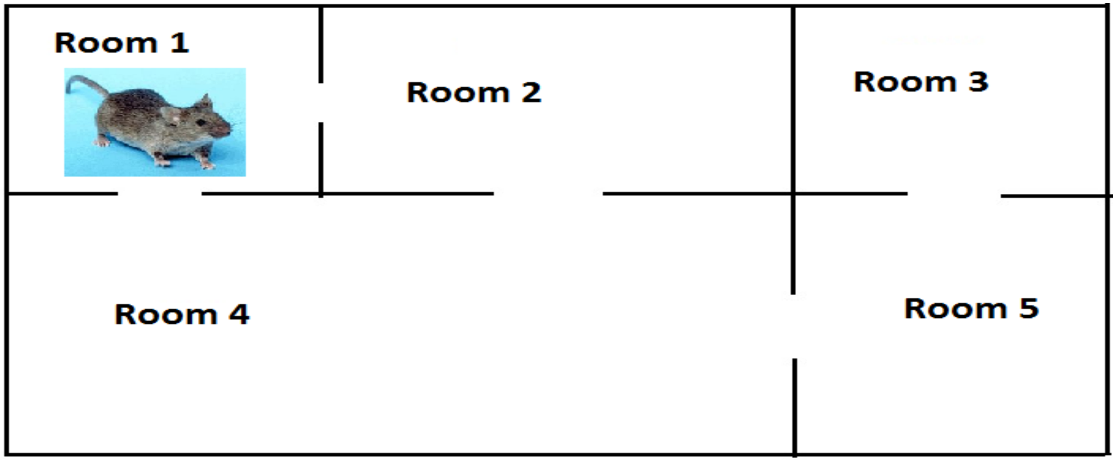
Mouse	To	1	2	3	4	5
From	***1***	***0***	***0.5***	***0***	***0.5***	***0***



The lab mouse problem

We have completed the first row of the 5x5 transition matrix for the mouse. What are the other 4 rows?

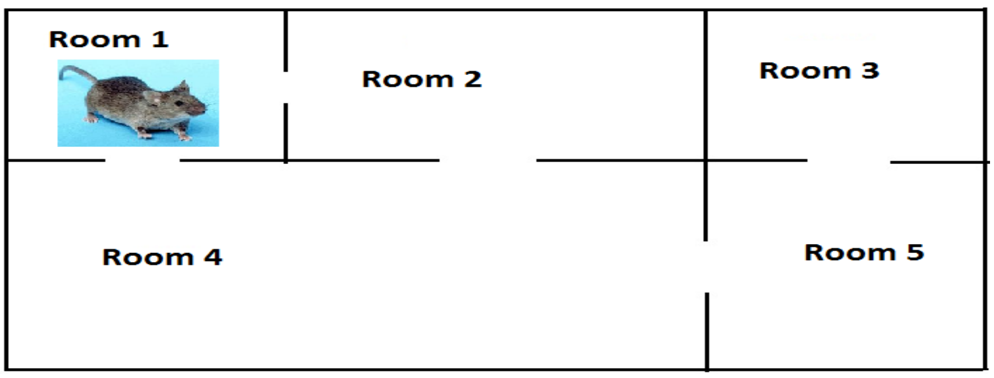
Mouse	To	1	2	3	4	5
From	*1*	*0*	*0.5*	*0*	*0.5*	*0*
	2					
	3					
	4					
	5					



The lab mouse problem

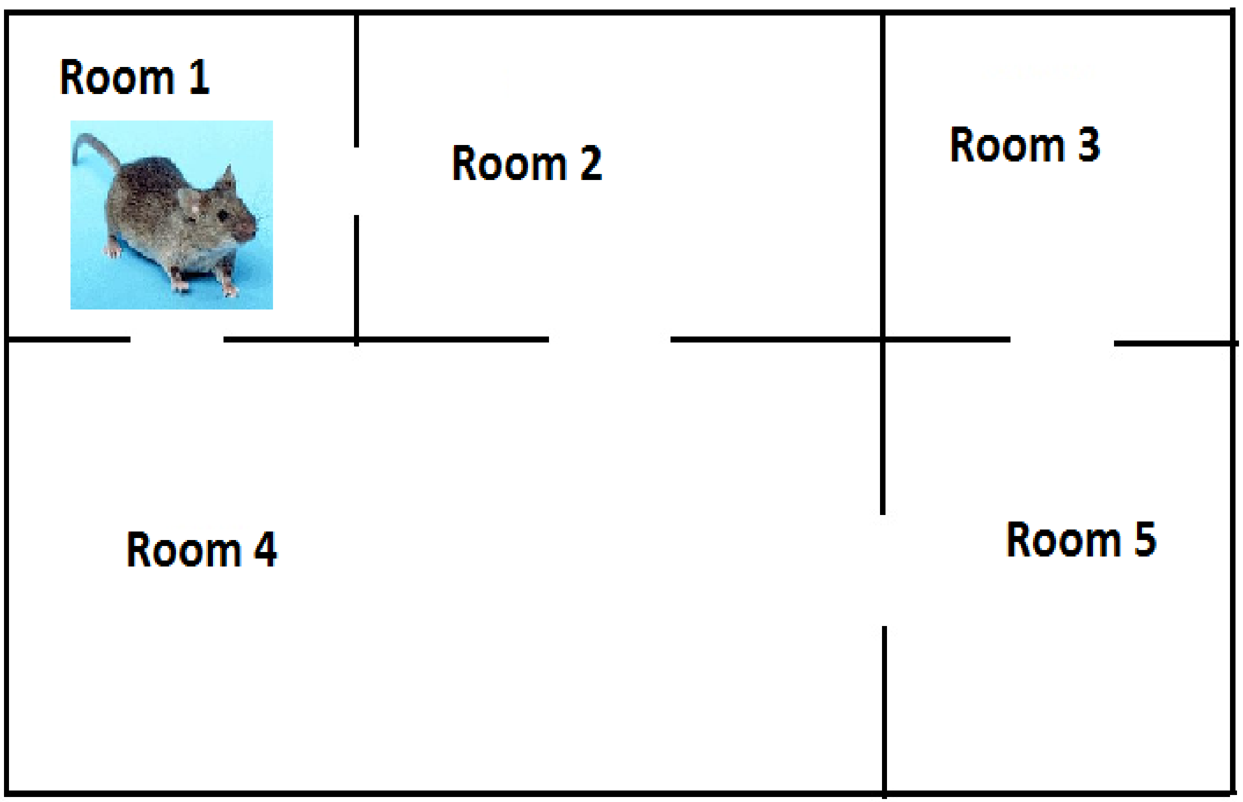
We have completed the first row of the 5x5 transition matrix for the mouse. Here are the other 4 rows?

Mous e	To	1	2	3	4	5
From	*1*	*0*	*0.5*	*0*	*0.5*	*0*
	2	*0.5*	*0*	*0*	*0.5*	*0*
	3	*0*	*0*	*0*	*0*	*1*
	4	*1/3*	*1/3*	*0*	*0*	*1/3*
	5	*0*	*0*	*0.5*	*0.5*	*0*



The lab mouse problem

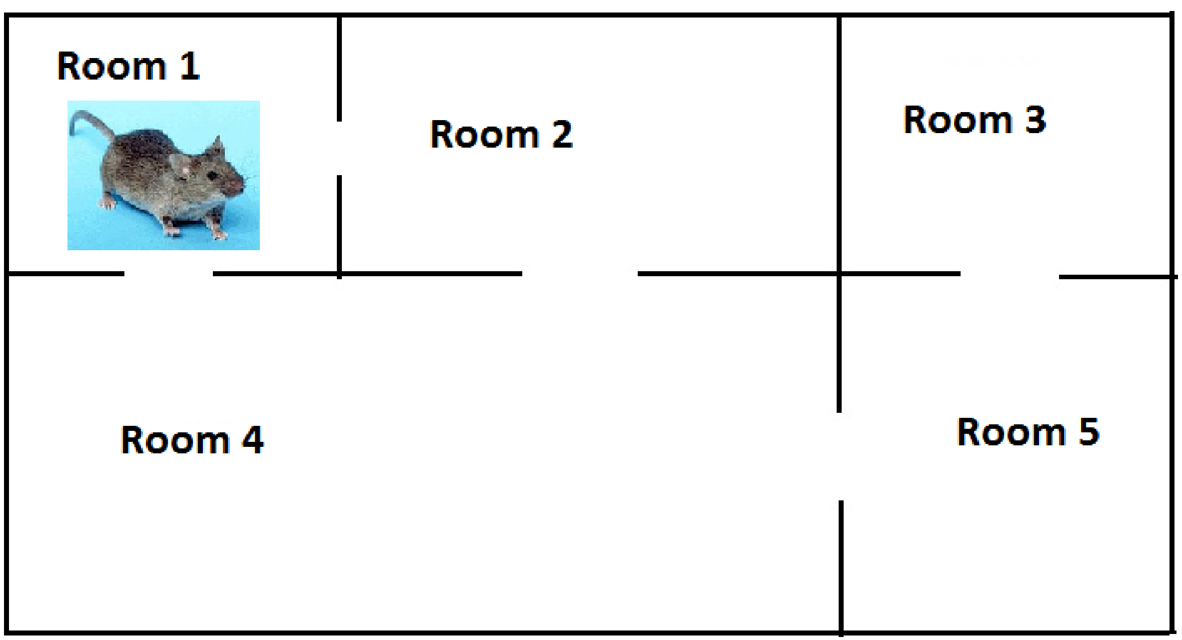
The next thing we need for this problem is the initial state describing the probability of the mouse being in each particular room at the start.



The lab mouse problem

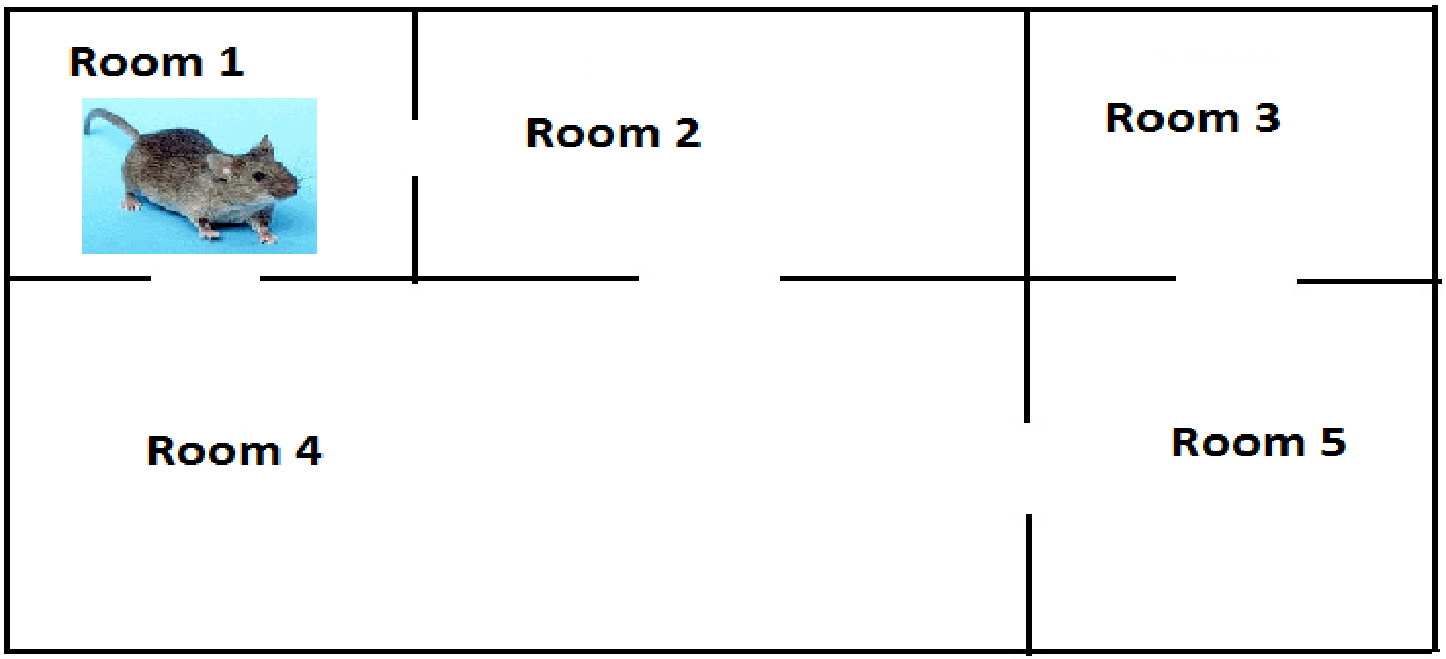
The next thing we need for this problem is the initial state describing the probability of the mouse being in each particular room at the start.

The row matrix (1 0 0 0 0) indicating that the mouse started at Room 1 should describe it well.



The lab mouse problem

It would now be interesting to look at where the mouse will be after a time period of 1, 2 and so on. We can use matrix multiplication to determine the probability of the mouse being in a particular room after a particular time



The lab mouse problem

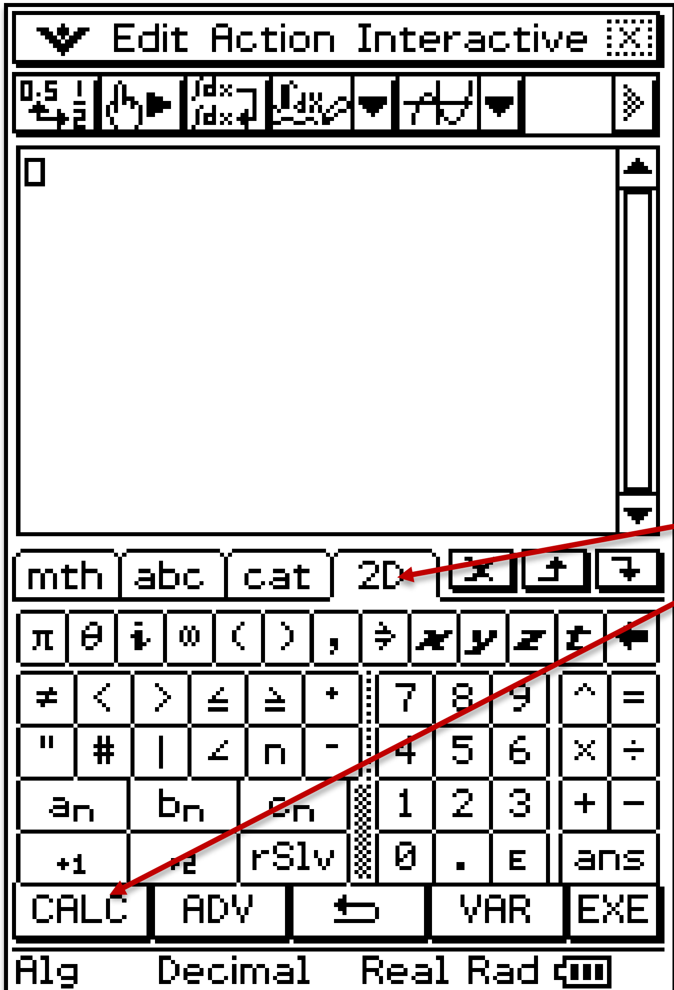
And now to the matrices

If we let T be the 5x5 transition matrix and R be the 1x5 row matrix giving the probabilities of the mouse being in each of the 5 rooms, we can calculate RT to find the probability that the mouse will be in a particular room after the first time period.

$$RT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

The lab mouse problem

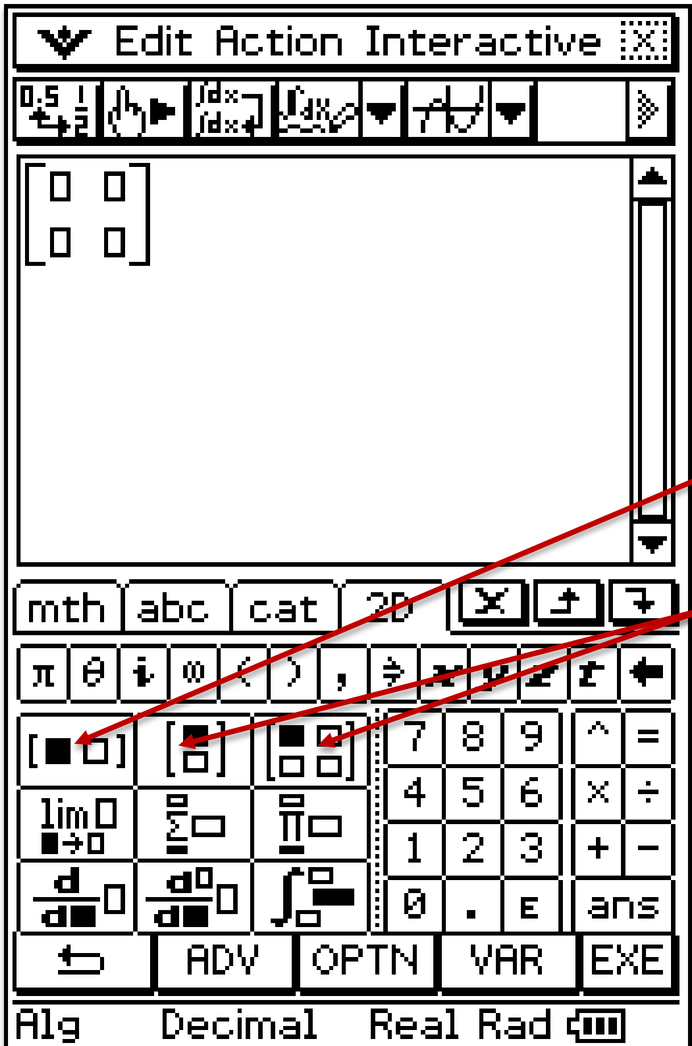
And now to the calculator



We can enter the two key matrices into the Casio Classpad to look at what happens in the future.

If we select the soft Keyboard and use the 2D TAB and the CALC TAB to get an option for entering matrices.

The lab mouse problem



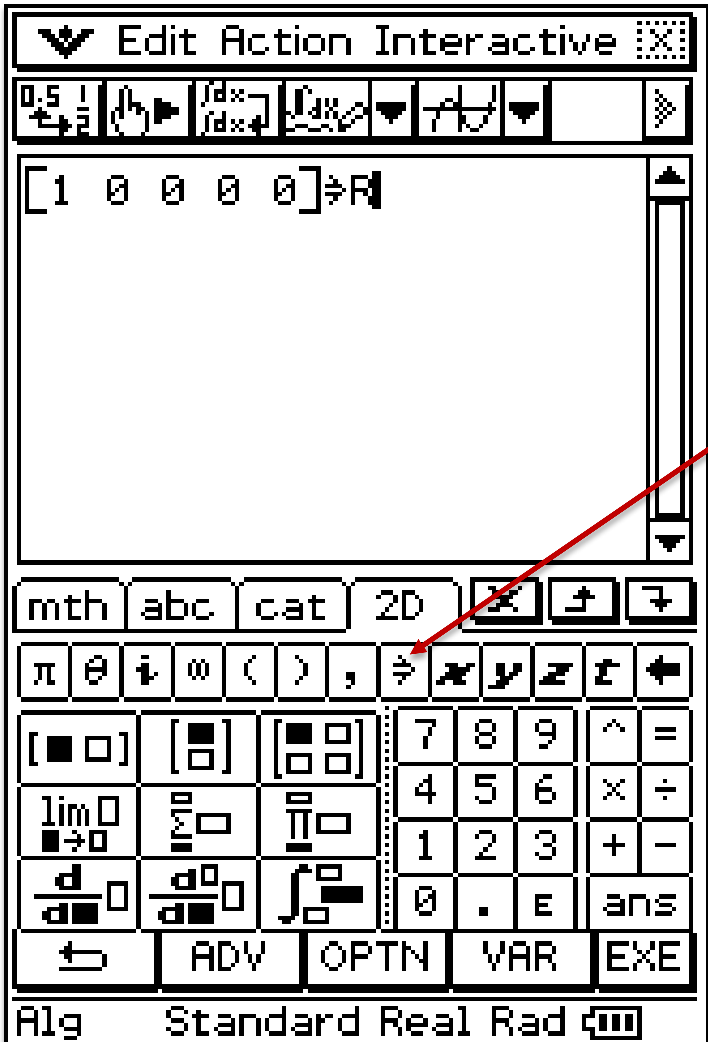
More with the calculator

The next screen will enable matrices to be entered and rows and columns added to produce the matrix size needed.

The 1st icon adds columns, the 2nd icon adds rows and the 3rd icon adds both (and creates a blank 2 x 2 matrix if it selected first.

The resultant matrix then needs only the entries to be added in the spaces provided.

The lab mouse problem



It is also a good idea at this stage to use the arrow key to store the matrices for ease of repeated use.

The lab mouse problem

Edit Action Interactive

R

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

T

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

R*T

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Alg Standard Real Rad

$$= \left(0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 0 \right)$$

50% chance to be in Room2

50% chance to be in Room 4

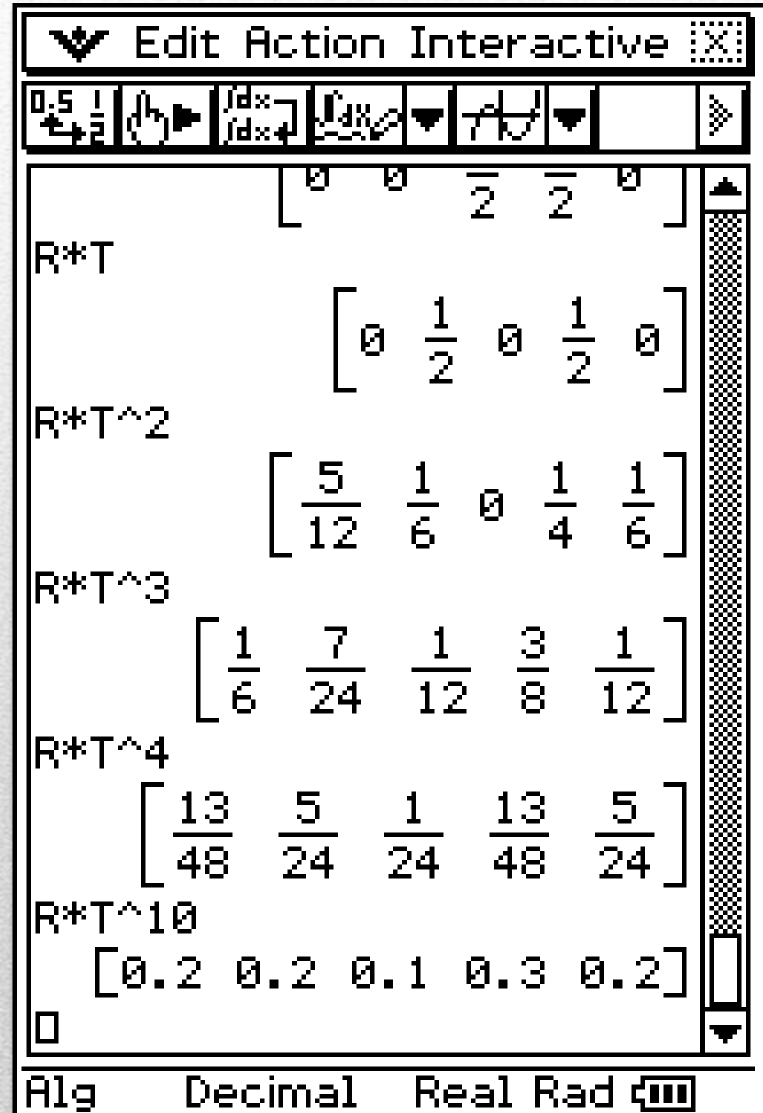
The lab mouse problem

$$RT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

50% chance to be in Room2

50% chance to be in Room 4

The lab mouse problem



We can start to observe the chance of the mouse being in each room after a particular number of time frames.

It can be observed that the mouse is less likely to be in Room 3 than any other room (from Time 2 onwards) and more likely to be in Room 4 (from Time 4 onwards) than any other room.

Is this predictable?

The lab mouse problem

Chance of room	*Room 1*	*Room 2*	*Room 3*	*Room 4*	*Room 5*
R (start)	*1*	*0*	*0*	*0*	*0*
RT *(time 1)*	*0*	*1/2*	*0*	*1/2*	*0*
RT^2 *(time 2)*	*5/12 (0.417)*	*1/6 (0.167)*	*0*	*1/4 (0.25)*	*1/6 (0.167)*
RT^3 *(time 3)*	*1/6 (0.167)*	*7/24 (0.292)*	*1/12 (.083)*	*3/8 (0.375)*	*1/12 (0.083)*
RT^{10} *(time10)*	*0.207*	*0.206*	*0.086*	*0.285*	*0.216*

2. Rental cars

- System - what we are looking at
- State - the possibilities of the system
- Stochastic process - the sequence of states that a system moves through

In the rental car problem

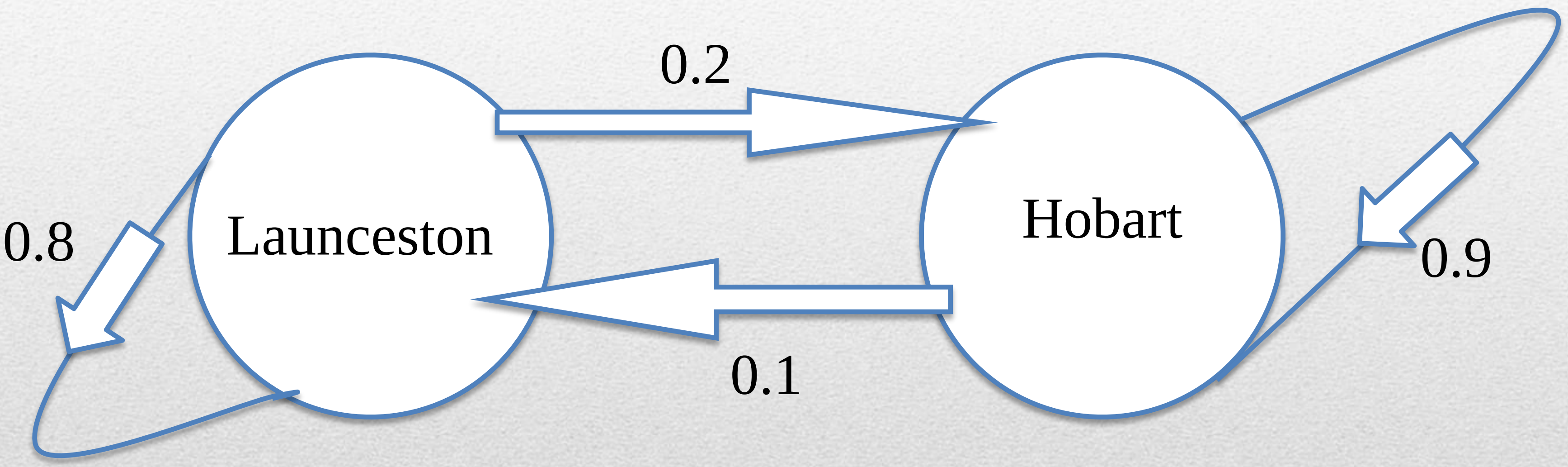
- system = the rental car
- state = Launceston or Hobart depot
- stochastic process = the probability that the rental cars will be at Hobart or Launceston at drop-off after each year

Rental car problem

A rental company has two outlets - one in Hobart and one in Launceston. The hirers of the cars usually return their cars to the same outlet as the one that they hired from. Some of the hirers prefer to pick up the car at one outlet and return it to the other. The company estimate that there is a 20% chance that a car hired in Launceston will be dropped off in Hobart and there is a 10% chance that a car picked up in Hobart will be dropped off in Launceston. There is an initial split of cars with 70% in Hobart and 30% in Launceston.

- What will be the long term split of cars between the two cities?
- What advice would you give to the company regarding the split up of cars between the two cities?

Rental car problem

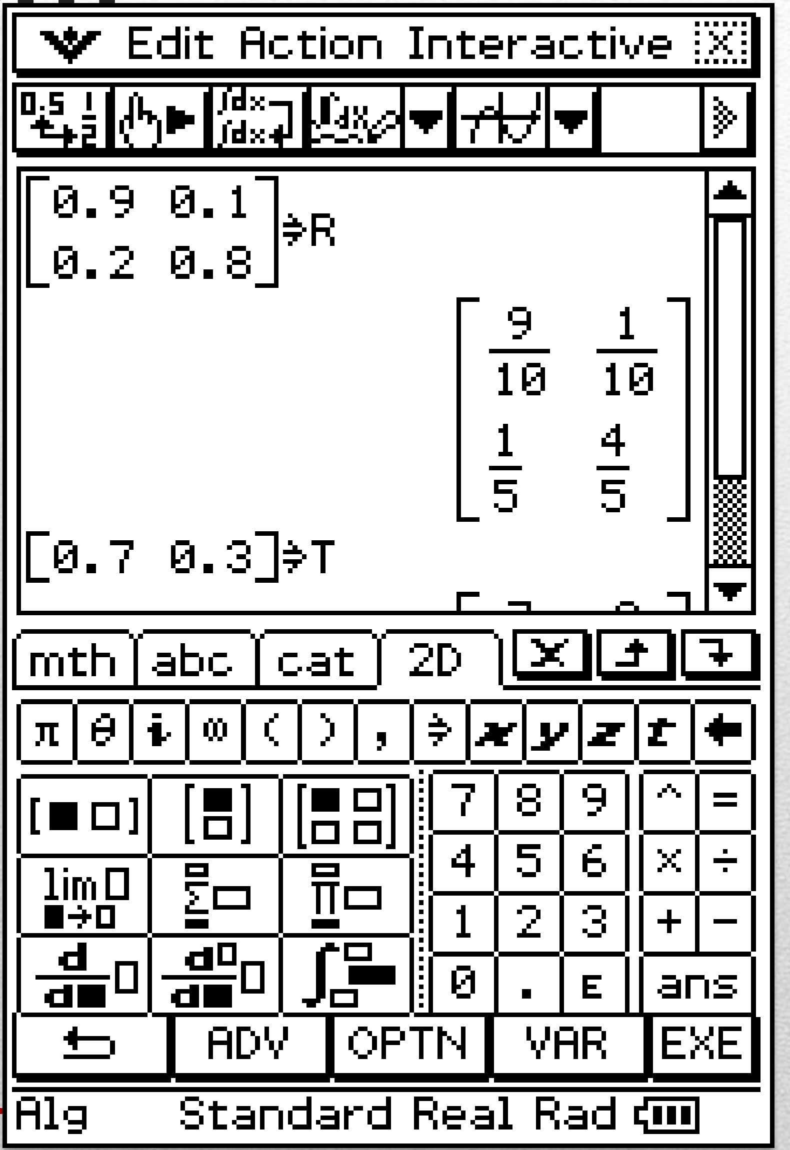


Cars	*To*	*H*	*L*
From	*H*	*0.9*	*0.1*
	L	*0.2*	*0.8*

Rental car problem

Cars	To	H	L
From	H	0.9	0.1
	L	0.2	0.8

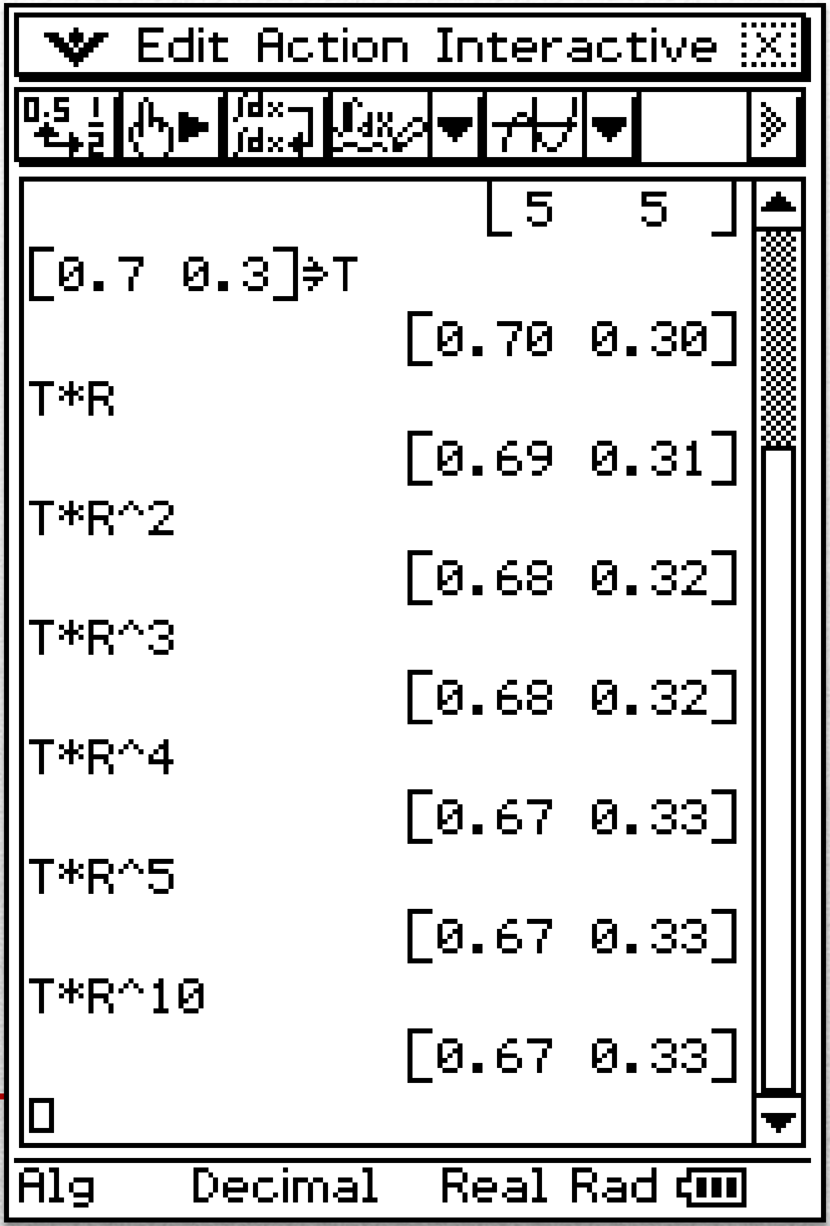
The transition matrix, R, can then be defined and the effect of the transfer of cars from the two cities measured. The initial matrix, T, will have a value of the initial placement of cars at the two cities



Rental car problem

By calculating TR, TR^2, etc. with further increases we can investigate the proportion of rental cars that will be at Launceston and Hobart over time.

What would you be able to say about the optimal placement of cars in the two cities?



Rental car problem

This can also be solved by looking at (a b) as the ideal long term car location matrix and solving

$$(a\ b)\begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} = (a\ b)$$

Expanding the matrices leaves the two equations

$0.9a + 0.2b = a$

$0.2a + 0.8b = b$

We really only 1 of these and the condition that $a+b = 1$

From the first

$9a + 2b = 10a$

$-a + 2b = 0 \quad (1)$

$a + b = 1 \quad (2)$

Now (1) + (2)

$3b = 1$

So $b = 1/3$ and $a = 2/3$

This agrees with the long range approximation used!

3. Telstra market share

- System - what we are looking at
- State - the possibilities of the system
- Stochastic process - the sequence of states that a system moves through

In the rental car problem

- system = the customer
- state = Telstra or other
- stochastic process = the probability that a customer will be using Telstra or otherwise at a particular stage

Telstra market share

Telstra finds that 90% of it's customers renew contracts after 2 years and 5% of non-Telstra customers switch to Telstra. The estimated market share is initially 40% Telstra.

Is Telstra likely to improve it's market share over time?
What matrices could we use in this process?

Telstra market share

The initial state distribution matrix is
S=(0.4 0.6)

The transition matrix is

$$T=\begin{pmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{pmatrix}$$

We could alternatively have used a tree diagram with limited choices!

Telstra market share

Looking at the combination

$$\mathbf{ST} \quad =(0.4 \quad 0.6)\begin{pmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{pmatrix}$$

$$=(0.39 \quad 0.61)$$

What has happened to the market share for Telstra?

Telstra market share

Looking at the combination

$$ST = (0.4 \quad 0.6)\begin{pmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{pmatrix}$$

$$= (0.39 \quad 0.61)$$

What is T^2, T^3, T^{10} ?

What is the stationary matrix S so that ST=S?

What does this mean for Telstra?

4. Chance of finishing a postgraduate program

- System - what we are looking at
- State - the possibilities of the system
- Stochastic process - the sequence of states that a system moves through

In the postgraduate problem

- system = enrolled student
- state = Entry Exams or Thesis or Leave or Postgraduate degree
- stochastic process = The probability of each of the states after a particular number of years from current enrolment

Chance of finishing a postgraduate program

In this example we consider a student who has enrolled in a postgraduate course. After a year it is estimated that 80% will qualify to go on to a thesis, 10% will need to repeat and 10% will leave. If a students leaves then it is assumed that they do not return. After a year a student who is doing a thesis is 60% likely to be still working on the thesis, 30% likely to have finished the postgraduate course and 10% likely to leave. A student who has finished the course will stay finished.
What is the probability that a student who starts a postgraduate course now will finish in 5 years time?

Chance of finishing a postgraduate program

To

$$\begin{array}{ccc} & & \begin{array}{cccc} \quad Q & \quad\; L & \quad\; T & \quad\; D \end{array} \\ \text{From} & \begin{array}{c} Q \\ L \\ T \\ D \end{array} & \begin{bmatrix} 0.10 & 0.10 & 0.80 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.10 & 0.60 & 0.30 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \end{array}$$

This is the transition state matrix.
What can we interpret from this?

Chance of finishing a postgraduate program

Edit Action Interactive

$$\begin{bmatrix} 0.10 & 0.10 & 0.80 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.10 & 0.60 & 0.30 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

P^2

$$\begin{bmatrix} 0.01 & 0.19 & 0.56 & 0.24 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.16 & 0.36 & 0.48 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

P^3

$$\begin{bmatrix} 0.00 & 0.25 & 0.34 & 0.41 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.20 & 0.22 & 0.59 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

Alg Decimal Real Rad

Edit Action Interactive

0.00 0.00 0.00 1.00

P^5

$$\begin{array}{c} Q \\ L \\ T \\ D \end{array} \begin{bmatrix} 0.00 & 0.30 & 0.12 & 0.57 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.23 & 0.08 & 0.69 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

P^10

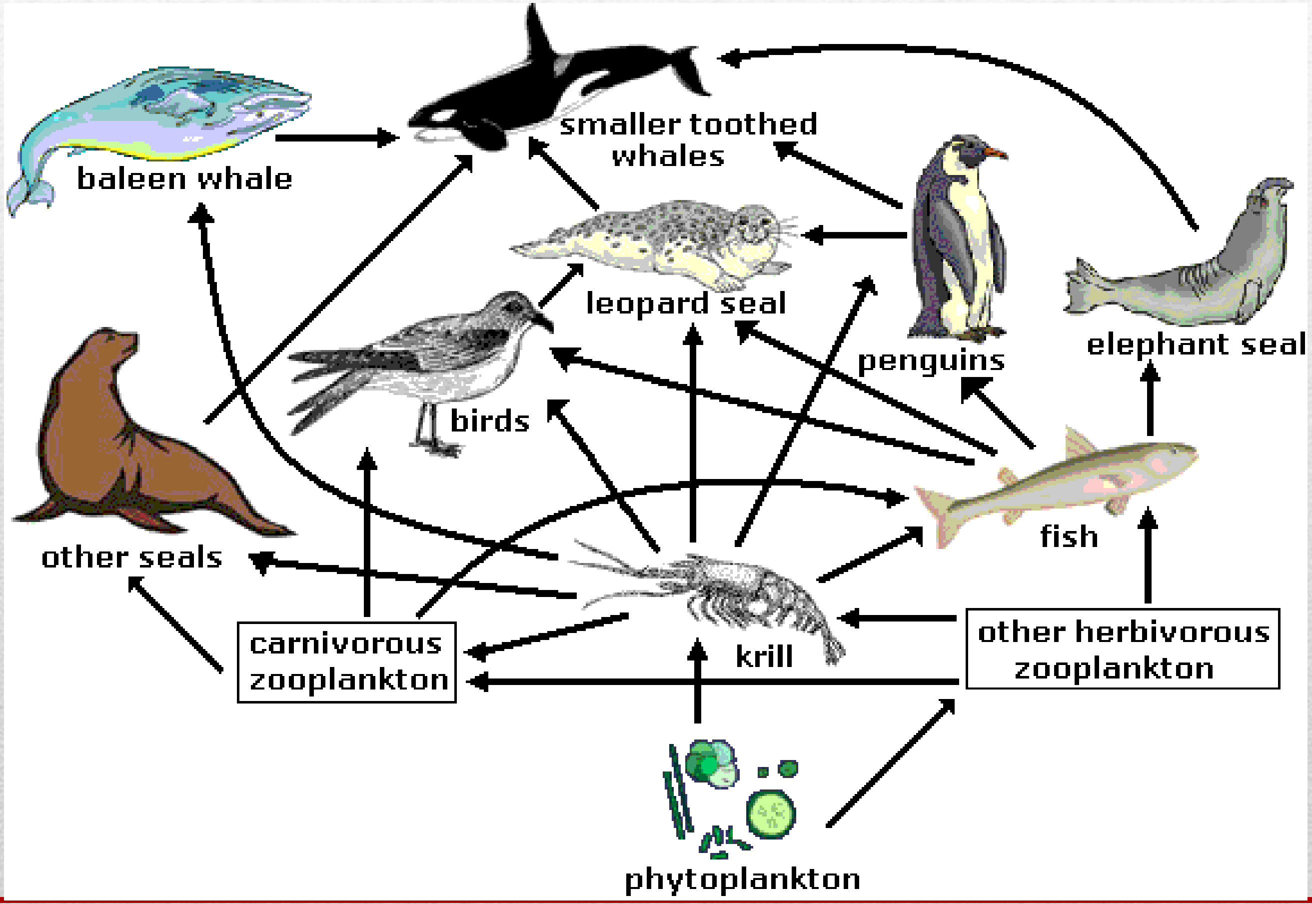
$$\begin{bmatrix} 0.00 & 0.33 & 0.01 & 0.66 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.25 & 0.01 & 0.75 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix}$$

P^20

$$\begin{bmatrix} 0.00 & 0.33 & 0.00 & 0.67 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.25 & 0.00 & 0.75 \end{bmatrix}$$

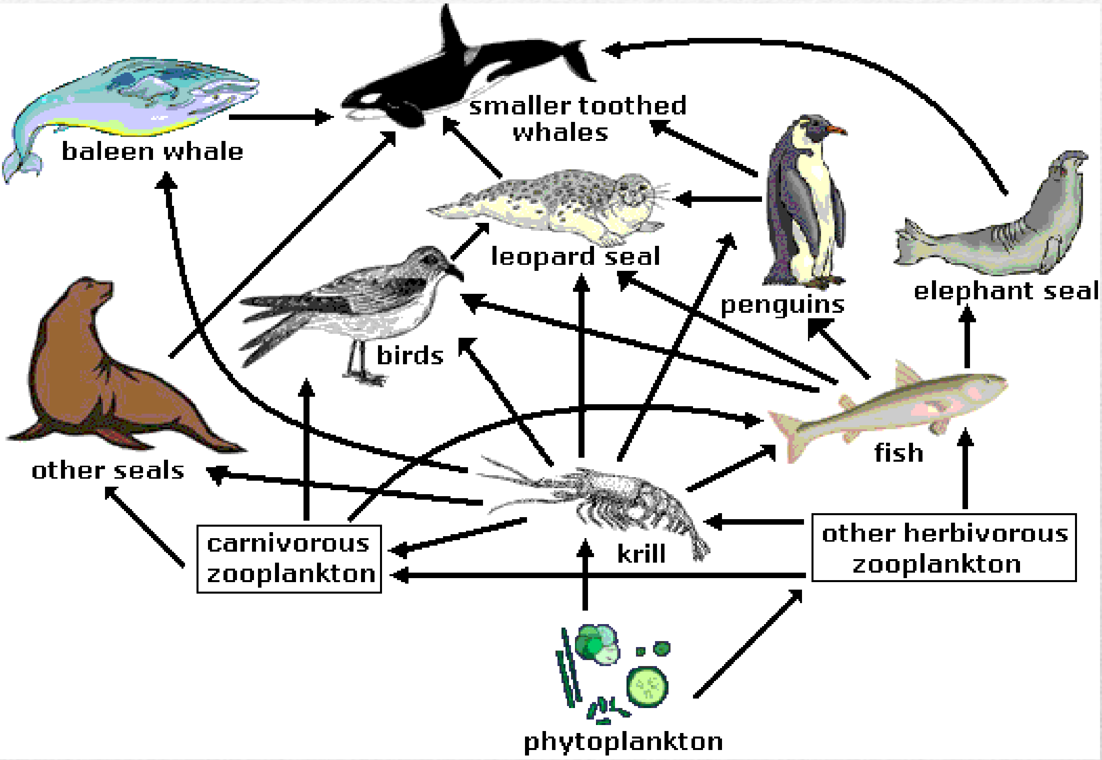
Alg Decimal Real Rad

5. A food web in Antarctica



5. A food web in Antarctica

0 = no path
1 = direct path



	Ph	Kr	HZ	CZ	Fi	OS	B	LS	Pe	ES	W	KW
Ph												
Kr												
HZ												
CZ												
Fi												
OS												
B												
LS												
Pe												
ES												
W												
KW												

Can you fill in the adjacency matrix?

5. A food web in Antarctica

	Ph	Kr	HZ	CZ	Fi	OS	B	LS	Pe	ES	W	KW
Ph												
Kr												
HZ												
CZ												
Fi												
OS												
B												
LS												
Pe												
ES												
W												
KW												

We now have a 12 x 12 adjacency matrix of length 1which can inform us of whether a path of length 1 exists from one organism to another organism. This means we can move from one organism to another by moving along 1 directed edge.

5. A food web in Antarctica

How about paths of length 2?

	Ph	Kr	HZ	CZ	Fi	OS	B	LS	Pe	ES	W	KW
Ph												
Kr												
HZ												
CZ												
Fi												
OS												
B												
LS												
Pe												
ES												
W												
KW												

- Can we find an adjacency matrix for paths of length 2?
- A path of length 2 will be where we move from one organism to another organism by moving along 2 directed edge.
- Could we find this another way?
- What could we say about the path from krill to killer whale?
- Is this the same as the corresponding value in the matrix for paths of length 2?

5. A food web in Antarctica

	Ph	Kr	HZ	CZ	Fi	OS	B	LS	Pe	ES	W	KW
Ph												
Kr												
HZ												
CZ												
Fi												
OS												
B												
LS												
Pe												
ES												
W												
KW												

What about W^2, W^3

What can this tell us about matrices?