amp 21 applied mathematics practices for the 21st century

Waiting for Polynomials

The Activity

A new regional airport Charlene York International Airport (CYA) is planned for northern California. Based on the projected volumes of passengers, the Transportation Security Agency (TSA) is planning for two passenger scanning stations at the airport. They are trying to decide whether to build one big area with two screening stations and one waiting line or to build two separate screening stations, each with its own waiting line. If they choose the latter, passengers will not have to walk as far from the ticket counters to a security line. They estimate that two separate stations will save on average two to three minutes walking time for passengers lugging along their backpacks and/or carry-on bags.

As passengers go through the screening area, they move through a series of security screening steps. First, their boarding pass and identification are checked. Then, they wait to join a screening line. There they wait to approach a long counter where they begin unloading their personal items into bins to be passed through an x-ray detection device. As their personal carry-on items are screened, they pass through a metal detection device or a full-body scanning device. The primary backlog is created by the bins passing through the x-ray equipment. Each bin is looked at carefully. There is a lot of variability in the scanning time for a bin, because the number of bins and their contents vary from passenger to passenger. Sometimes a person's bags have to be opened and the contents specially checked.

Director Ralph Waldo is in charge of security planning for CYA. He consulted with the chief analytics officer for TSA, Dr. Gabriela Cue. Dr. Cue explained that data analysts at TSA have found that the longest component of processing was the time to pass through and screen all of an individual's carry-ons and personal property. The average time for this x-ray and personal screening was 45 seconds. However, there is significant variability around this average. Approximately, one in every four passengers takes longer than 60 seconds to process. About one in ten passengers takes at least 105 seconds, a minute longer than the average.

The number of passengers arriving to be screened varies throughout the day. At peak times, an estimated 152 passengers per hour will need to be screened. These peaks occur in the early morning and late afternoon hours. At other times in the day, the arrival rate can be as low as 100 passengers per hour.

1. If on average a single station can screen a passenger in 45 seconds, on average, how many passengers can one station screen in an hour? At 0.75 min. per passenger, $60 \div 0.75 = 80$ passengers

This number represents the station's maximum screening capacity per hour.

2. Explain why two stations would be enough to handle the peak arrival rates? $80+80 > 152$

The proportion of time the stations are busy is equal to the ratio of the number of arriving passengers to the passenger screening capacity per hour. This is also called the utilization rate.

3. What is the utilization rate during peak hours? $\frac{152}{160} = 0.95$
4. What is the utilization rate when the average arrival rate is only 100 passengers per hour? $\frac{100}{160} = 0.625$

We use the Greek letter ρ (rho) to represent the utilization rate. Its two components, the average rate of arrivals per hour and the average passenger screening capacity per hour, are represented by the Greek letters λ (lambda) and μ (mu), respectively.

5. Write an equation that expresses ρ in terms of λ and μ. $\rho = \frac{\lambda}{\mu}$

Dr. Cue explained to Director Waldo that there are queueing models that can forecast average waiting times for the two alternatives. These models involve polynomials that can be easily evaluated with scientific calculators or in a spreadsheet. The simpler formula models the two single stations as depicted in Figure 1. The formula requires as input the average server utilization rate, ρ.

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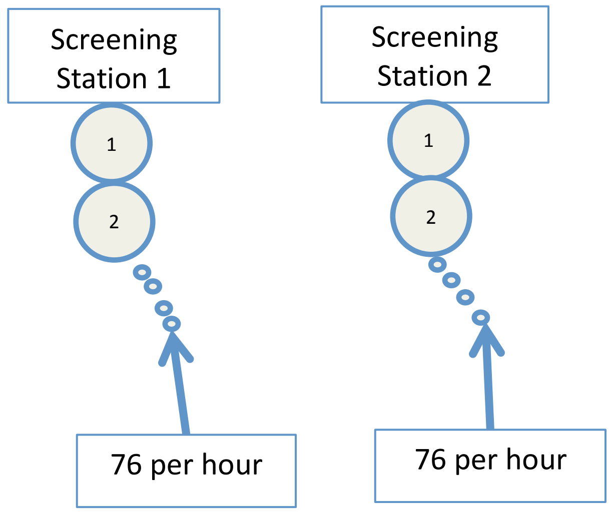

Figure 1: A Separate Line for Each Screening Station

At peak times, an average of 76 passengers per hour would approach each station. On average, the station can process a passenger in 45 seconds. This is equivalent to a capacity of 80 passengers per hour. Thus, in peak times

$$\rho = \frac{76}{80} = 0.95 .$$

6. What assumption did we make in computing this utilization rate? The arrivals will go to each station in equal numbers.

The following formula calculates, L_q, the average number of passengers waiting to be screened.

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(0.95)^2}{1-0.95} = \frac{0.9025}{0.05} = 18.05$$

This is an average of 18.05 passengers waiting in line.

The average time a customer spends waiting is defined as W_q, To determine W_q, , we manipulate a second formula known as Little's Law.

$$L_q = \lambda W_q$$

$$\frac{L_q}{\lambda} = W_q$$

$$W_q = \frac{18.05}{76} = 0.2375 \text{ hours.}$$

Finally, because most people would make more sense of the average number of minutes waiting in line, we convert 0.2375 hours to 60(0.2375) = 14.25 minutes

In summary, with two separate screening stations, each with its own line, there would, on average, be slightly more than 18 passengers waiting in line at each screening station. The average waiting time would be 14.25 minutes per passenger.

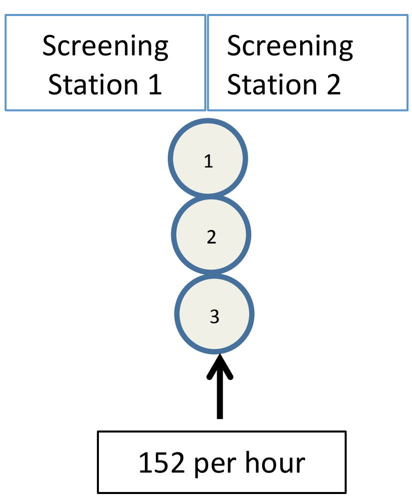

Figure 2: One Centrally Located Line for Two Screening Stations

7. On average, what is the total number of passengers who would be waiting in line at the two screeners? 2(18.05) = 36.1

The formula for L_q when there are two stations with one waiting line, as depicted in Figure 2, involves higher degree polynomials. When there is one waiting line for two screeners,

$$L_q = \frac{2\rho^3}{1-\rho^2}.$$

The average utilization rate for each station does not change with this new arrangement. On average, a total of 152 customers arrive per hour. However, working together, the two stations can process an average of 80 + 80 = 160 passengers per hour. So, in this case,

$$L_q = \frac{2(0.95)^3}{1-0.95^2} = \frac{2(0.857375)}{1-0.9025} = \frac{1.71475}{0.0975} \approx 17.59 .$$

Director Waldo observed that this total of 17.59 waiting for one of the two stations seems only slightly less than the 18.05 for the two single screening system. Dr. Cue pointed out that the director was not comparing the right numbers. To make an accurate comparison, he needs to combine the number of passengers waiting to be screened at each of the two separate stations. The differences become clearer when we calculate the average time a passenger spends waiting in line at the single line, two-sceeener system. In this instance, all of the passengers are arriving to wait in the same line. Now when we use Little's formula, λ=152, and

$$W_q = \frac{L_q}{\lambda} = \frac{17.59}{152} \approx 0.116 \text{ hours, or about 6.9 minutes.}$$

Director Waldo was surprised by the large difference. Locating the new screening stations together with one line would reduce average wait times by more than 7 minutes. This would easily balance out the extra few minutes of walking time with this design. Director Waldo wondered how large the differences would be at other times during the day. He asked Dr. Cue to have her staff complete an analysis for different arrival rates starting with 100 passengers per hour. She asked her staff to complete Tables 1 and 2 using the formulas given above. Dr. Cue also requested that her staff graph W_q for each system over the range provided in the Tables.

8. Fill in the missing values in Table 1 below.

		Average server utilization rate, ρ	One combined line	Sum of two separate lines
	Arrival rate, λ		L_q	L_q
Arrival Rates	100	0.625	$2(0.625)^3/(1-0.625^2)$=0.80	1.04+1.04=2.08
	110	0.688	1.23	3.03
	120	0.750	1.93	4.50
	130	0.813	3.16	7.04
	140	0.875	5.72	12.25
	152	0.95	17.59	18.05+18.05 =36.1

Table 1: Comparison of average number of passengers waiting in line.

9. Fill in the missing values in Table 2 below.

		One combined line	Sum of two separate lines
	Arrival rate, λ	W_q (minutes)	W_q (minutes)
Arrival Rates	100	60(0.80/100)=0.48	1.25
	110	0.67	1.65
	120	0.96	2.25
	130	1.46	3.25
	140	2.45	5.25
	152	6.94	14.25

Table 2: Comparison of average time waiting in line.

10. Using the values in Table 2 above, for which arrival rate(s) does the reduced waiting time for the single line model make up for the increased walking time from the ticket counters to the screening line. 140 and 152

11. If the single line model is used instead of the two-line model, by how many minutes is the waiting time reduced at peak times? What is the percentage reduction in waiting time? 7.31 min; 7.31/14.25≈51%

12. If the single line model is used instead of the two-line model, what is the percentage reduction in waiting time when the arrival rate is 100 passengers per hour? 0.77/1.25≈62%

13. What happens to the percentage reduction as the arrival rate decreases? It increases.

Notice that in the two-line model, the number of passengers waiting in *each* line *exceeds* the *total* number waiting in the combined single line model. Thus when you add the two lines of customers in the two-line model, the total is always more than double the centralized example.

We can use algebra to prove that statement! If $0 < \rho < 1$, then we need to show that

$$\frac{\rho^2}{1-\rho} + \frac{\rho^2}{1-\rho} > 2\left(\frac{2\rho^3}{1-\rho^2}\right)$$
$$2\left(\frac{\rho^2}{1-\rho}\right) > 2\left(\frac{2\rho^3}{1-\rho^2}\right)$$
$$\frac{\rho^2}{1-\rho} > \frac{2\rho^3}{1-\rho^2}.$$

Now, if we divide both sides by ρ^2, we get

$$\frac{1}{1-\rho} > \frac{2\rho}{1-\rho^2}.$$

Factoring the denominator of the right hand side yields

$$\frac{1}{1-\rho} > \frac{2\rho}{(1-\rho)(1+\rho)},$$

and multiplying both sides by $(1-\rho)$ gives us

$$1 > \frac{2\rho}{1+\rho}.$$

Now if we multiply both sides by $1+\rho$, the result is

$$(1+\rho) > 2\rho.$$

14. Why is the above statement true in this case? $0 < \rho < 1$.

15. When does the inequality approach equality? As $\rho \rightarrow 1$, *both* $1+\rho$ and $2\rho \rightarrow 2$

16. When is the difference the greatest? As $\rho \rightarrow 0$, $1+\rho \rightarrow 1$, but $2\rho \rightarrow 0$.