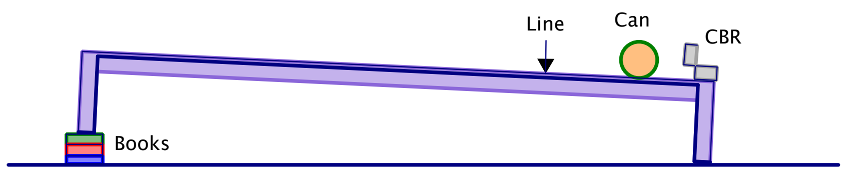
You Want Me to Factor Every Quadratic? Get Real!

Quadratic equations come in different forms: factored form: $y = a(x - r_1)(x - r_2)$, standard form: $y = ax^2 + bx + c$, and vertex form: $y = a(x - h)^2 + k$ are perhaps the most commonly known. Knowing when to use each form is important as well as being able to convert between forms in order to identify different pieces of information. When considering real world data and finding an equation to fit, we have options depending on what we can see in our data set.

Rolling Can Investigation



Prop up one end of a long table. Place a CBR at the low end of the table and draw a line on the table one meter from the CBR with tape or dry-erase marker. Set the CBR to record 50 points over 10 seconds then gently roll the can up the table. Stop the can when it returns to the start. It should cross the line twice.

Subtract one (the distance from the CBR to the line) from the distance values recorded. Plot this data. (** To be Continued **)

Let's look at the various forms of the equation and how to move from one form to another.

Factored Form → Standard Form

Step 1: On your calculator, graph the equations $y = x + 3$ and $y = x - 4$ on the same screen.

Step 2: What is the x-intercept of each equation you graphed in Step 1?

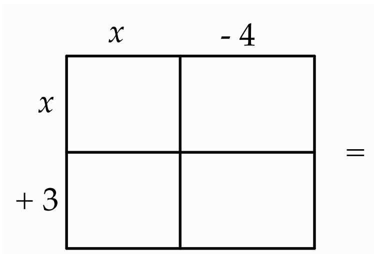
Why are these values the x-intercepts?

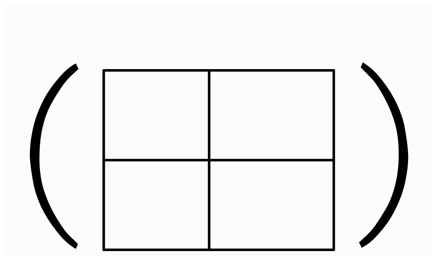
Step 3: Graph $y = (x + 3)(x - 4)$ on the same set of axes as in Step 1. Where are the x-intercepts?

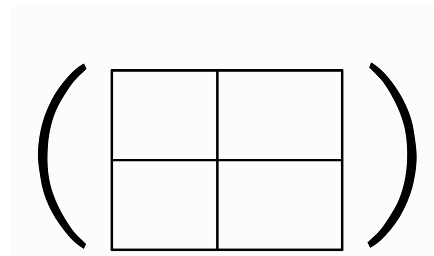
How could you have predicted these values by looking at the equation?

Step 4: Graph $y = x^2 - x - 12$ on your calculator. What do you notice?

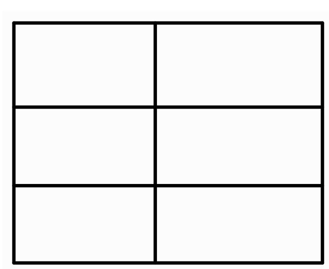
How do we convert between these two forms of a quadratic equation?

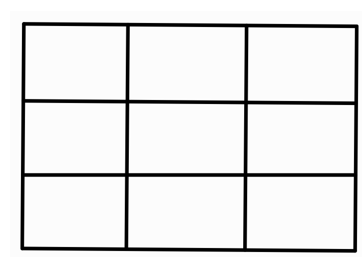






This method can be extended to higher degree polynomials.





Factored Form → Vertex Form

Step 1: Begin with an equation in the factored form $y = a(x - r_1)(x - r_2)$. Replace r_2 with the opposite of r_1. Choose two non-zero values between -3 and 3 for *a* and for r_1. On your calculator enter the equation $y = a(x - r_1)(x + r_1)$ and graph this in the window [-6.4, 6.4, 1, -12, 12, 1]. Using Trace find the coordinates of the two *x*-intercepts and the vertex. Together with those around you what can you say about these three points based on *a* and *r*?

Step 2: Pick three values for *a*, r_1, and r_2 and do the same thing. Focus on the *x*-value of the vertex (*h*). Now change the value you used for *a* but not for the roots. When you change the value of *a* how does this change the value of *h*?

Step 3: How can we use the values of the roots to find the value of *h*? Work together to see if you can create a rule that works with each person's choices.

Step 4: If the intercepts are at *x* = 2 and *x* = 8 then what will be the value of *h*? Demonstrate this is right with a graph.

Step 5: What about the value for *k* (the *y*-value of the vertex)? This value depends on both the intercepts and the stretch value. But if we think about it, *k* is just a *y*-value on the curve that goes with *h.* Since we already have an equation for *y* this becomes the equation for *k* when we replace *x* with *h.*

Step 6: Given $y = 3(x - 2)(x - 8)$ where will we find the vertex? Demonstrate this is right with a graph.

What if we are starting with standard form?

Standard Form → Factored Form

Step 1: Complete the rectangle diagram whose sum is $x^2 + 5x + 6$

x^2	
	6

How do you choose what to put in the two remaining corners?

One strategy is to organize the possibilities: Consider the expression $x^2 - 16x + 48$

Starting with products

Product	Sum
$-1 \cdot -48$	-49
$-2 \cdot -24$	
$-3 \cdot -16$	
$-4 \cdot -12$	
$-6 \cdot -8$	
$-8 \cdot -6$	

Starting with sums

Sum	Product
$-1 + -15$	15
$-2 + -14$	
$-3 + -13$	
$-4 + -12$	
$-5 + -11$	
$-6 + -10$	

x^2	
	48

With a leading coefficient other than one, the product must be *ac*, not just *c*. Note: if there is a common factor, you must first factor it out and then proceed as before.

Try this one: $3x^2 + 17x + 10$

$3x^2$	
	10

and this one: $4x^2 - 10x - 6$

2 (

$2x^2$	
	-3

)

Standard Form → Vertex Form

You can think of any parabola as the vertical dilation of some translation of $y = x^2$. The vertex form of the curve is $y = a(x-h)^2 + k$. In this investigation we want to connect the values of *h* and *k* to the values of *a, b,* and *c* when given $y = ax^2 + bx + c$

Step 1: Consider the graph of $y = ax(x-p)$. Where will this cross the *x*-axis? Explore this with random choices of *a* and *p.* Mix and match them until you are sure you know for every pair.

Step 2: If we have intercepts at both 0 and *p* then where will we find the axis of symmetry?

Step 3: If we "factor" the right side of $y = ax^2 + bx$ for all real values of *a* and *b* we can always put this into the form of $y = ax(x-p)$. Make random choices for values of *a* and *b.* Using what you know about factoring, guess the value for *p* for your choice of *a* and *b* then graph both forms to show they are indeed the same. How is *p* calculated?

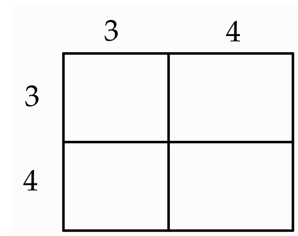
Step 4: Using what you found in step 2 and in step 3 how do you find *h* (the vertex *x*-value) if we know *a* and *b*?

Step 5: What happens to the axis of symmetry if we translate the graph vertically? Choose a pair of values to use for *a* and *b* then make a set of graphs with different values for *c* using $y = ax^2 + bx + c$.

Step 6: As before, recall that *k* is the *y*-coordinate when *x* = *h* so a way to find *k* is to put *h* into the equation $k = ah^2 + bh + c$. Select values for *a*, *b*, and *c*. Use them to find *h* and *k*. Then check the graphs to see if they match. $y = a(x-h)^2 + k$

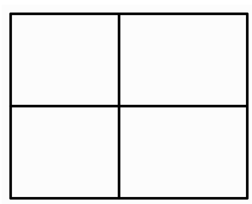
Vertex Form → Standard Form

Step 1: The rectangle diagram shows how to express 7^2 as $(3 + 4)^2$. Find the area of each inner rectangle to find the area of the overall square.



Step 2: You can do the same thing with binomials. Label a rectangle for each expression. Label each inner rectangle and find the total sum. Combine like terms, and express your answer as a trinomial.

a. $(x+5)^2$ b. $(x-7)^2$ c. $(3x+2)^2$



Step 3: What patterns do you see when you compare your final trinomials to the original binomials that were squared?

Step 4: Use what you discovered in Step 3 to convert these expressions in vertex form to standard form.

a. $(x-2)^2+5$ b. $(x+5)^2-7$ c. $2(x+6)^2-11$

Here is another look at changing from standard form to vertex form using the technique of completing the square.

Standard Form → Vertex Form

Step 1: Complete each rectangle diagram so that it is a square. How do you know which number to place in the lower-right corner?

a.

	x	?
x	x^2	$2x$
?	$2x$	?

b.

	x	?
x	x^2	$-3x$
?	$-3x$	?

Step 2: The diagram in Step 1a represents the equation $x^2 + 4x + \_\_\_ = \left(x + \_\_\_\right)^2$. For each diagram in Step 1, write an equation in the form $x^2 + bx + c = \left(x + h\right)^2$.

Step 3: Consider the equation $y = x^2 + 6x - 1$. Describe what's happening in each stage of changing the equation to vertex form.

$y = x^2 + 6x - 1$

$y + 1 = x^2 + 6x$

$y + 1 + 9 =$

	x	3
x	x^2	$3x$
3	$3x$	9

$= (x+3)^2$

$y + 10 = \left(x + 3\right)^2$

$y = \left(x + 3\right)^2 - 10$

Step 4: Repeat this process for each equation below to change it to vertex form.

a. $y = x^2 - 8x + 17$

b. $y = x^2 + 10x - 3$

What if you have no equation at all, but only data? It is easy enough to estimate some things from your graph. You can choose to estimate the intercepts or to estimate the vertex. Either of these gets you two-thirds of an equation, leaving you to find the stretch and thus completing either the factored or vertex form. There are different ways of finding *a.* We'll look at one way that may be new to some of you.

Back to the Rolling Can (finally)

Imagine a linear equation through the vertex and some other point on the curve $y = a(x-h)+k$. The value of *a* is the slope of the line. In terms of distance and time this value is the speed (change in distance over change in time). In the case of a parabola *a* is the acceleration (change in distance over change in time squared).

Step 1: Using trace you should locate the vertex (*h*, *k*). Note that this may be at a point or between two points and need to be estimated.

Step 2: Now still using trace select one other point on the curve (not too close to the vertex for better accuracy). Based on these two points what is the change in distance (y) over the change in time (x) squared. How does this value compare with others in the class who used different points?

Step 3: Using your values for *h* and *k* and the value you just found for *a,* plot the equation $y = a(x-h)^2 + k$ on the calculator along with your data. If needed, make adjustments to *a* to improve the fit of the curve to the data.

Step 4: Again using trace you should locate each of the roots (*r*, 0). Note that these may be at data points or between points and need to be estimated.

Step 5: Using your same value of *a* enter the equation $y = a(x-r_1)(x-r_2)$ and graph this with the data. Are your two equations the same? Does one fit better?

Step 6: Expand each of your equations to standard form $y = ax^2 + bx + c$ and compare all equations with your data to verify that they agree.