Cool Math from Cool Graphs

Mark Howell
Gonzaga College High School
Washington, DC
mhowell@gonzaga.org

It has been more than 25 years since hand-held graphing calculators became available to students and teachers. Over that entire time, machines have been limited to graphing functions, or inequalities with boundaries that were graphs of functions. A new hand-held graphing engine in the HP-Prime graphing calculator expands the graphing arena significantly. HP-Prime is a color, touch screen graphing calculator with a built in computer algebra system (CAS), dynamic geometry environment, spreadsheet, networking capabilities, and other unique features. HP-Prime has been approved by the College Board for use on Advanced Placement exams and the SATs.

Using the Advanced Graphing App with HP-Prime, you can graph arbitrary relations in two variables. HP Prime is the first graphing calculator that allows the user to graph any relation in two variables.

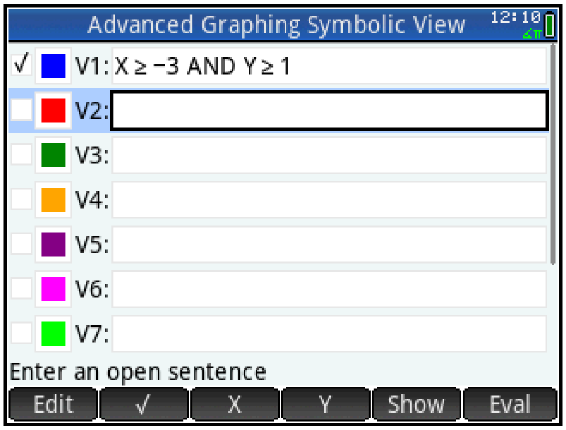
There are several types of classroom opportunities this capability affords:

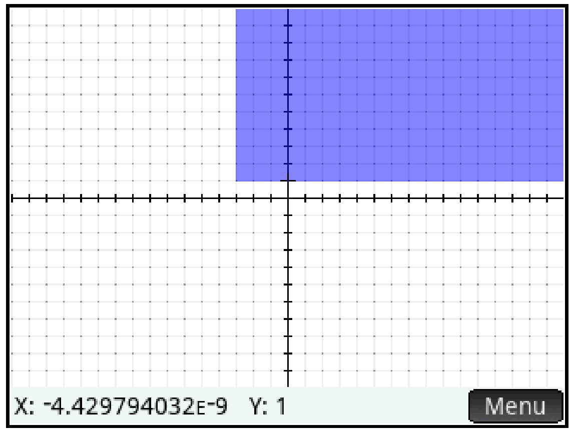
1. Graph familiar relations that previously had to be rewritten in an unfamiliar form. For example, on other graphing calculators, you could not enter the equation for a parabola in vertex form, $y-k=a\cdot(x-h)^2$. You'd have to first solve for y.
2. Graph new types of relations that might illustrate concepts in new ways. For example, looking at the graph of $\sin(x)=\sin(y)$ might surprise both teachers and students!
3. Illuminate important mathematical concepts with new visualizations. Being able to graph all points where $x+x=2x$, $\sin^2(x)+\cos^2(x)=1$, or $\frac{x+y}{3}=\frac{x}{3}+\frac{y}{3}$ in order to reveal the nature of an identity is gratifying. At the same time, seeing all points where $\sqrt{x^2}=x$, or where $\sqrt{x^2+y^2}=x+y$ can be equally revealing.
4. Challenge students to combine their artistic and mathematical creativity to produce cool looking graphs.
5. Graph implicit relations in Calculus to visualize derivatives in a new way.

In this paper, we'll take a look at a few examples with HP Prime. These might hopefully serve as a launching point for further investigations by the reader.

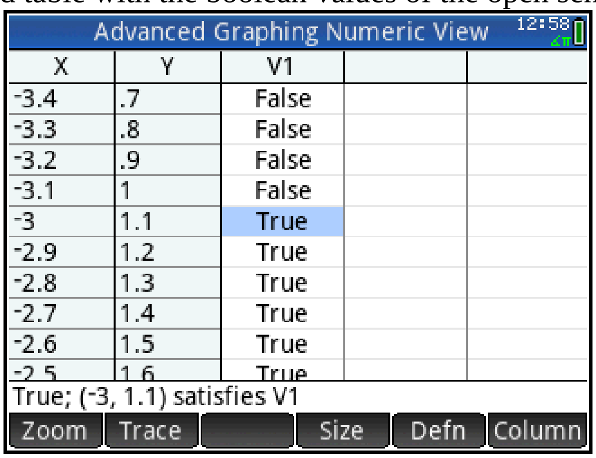
I. Visualizing solutions of inequalities in the coordinate plane

To graph an arbitrary relation in either one or two variables, the Advanced Graphing App is used. Up to ten open sentences can be graphed simultaneously. Both boolean and relational operators can be used to construct the open sentences. Here, we graph the open sentence $x\geq -3 \text{ AND } y\geq 1$.

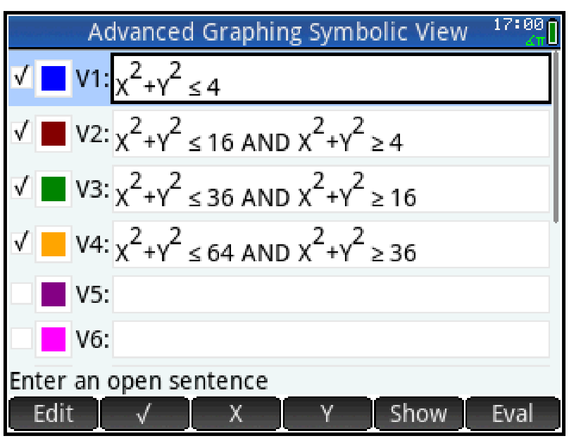


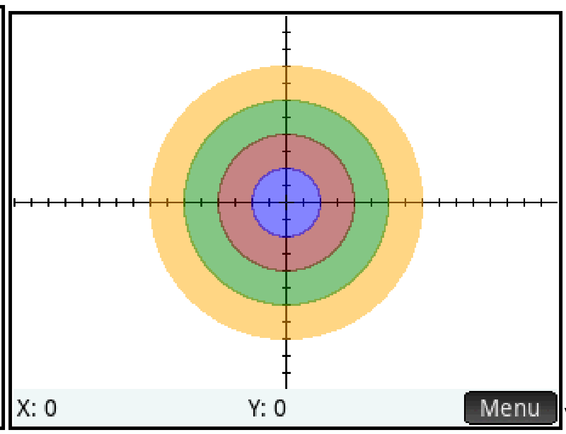


Relations are defined in the Symbolic View. The Plot View shows the graph. The Numeric View shows a two-d table with the boolean values of the open sentence.

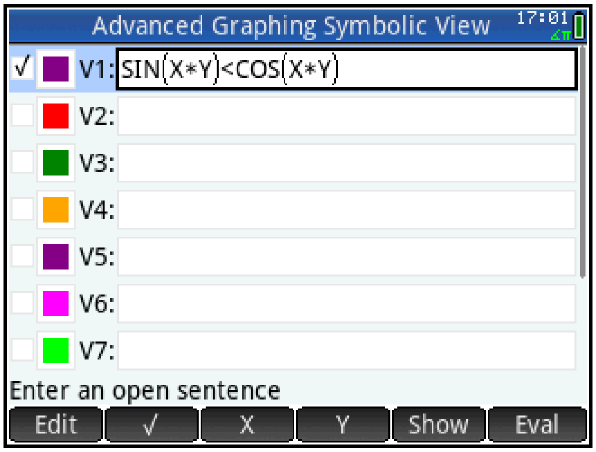


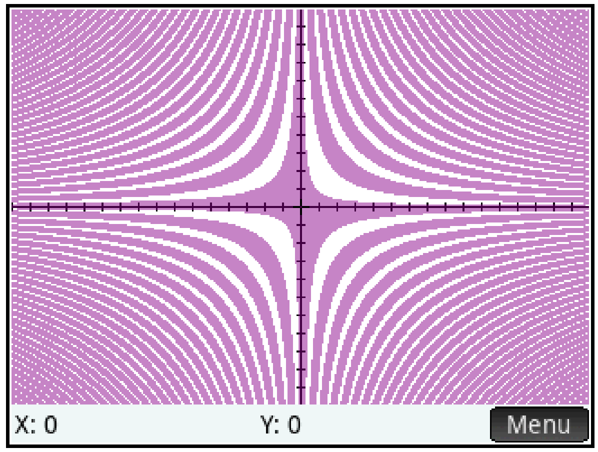
Here's a target formed by graphing several inequalities in different colors:





Or, have some fun with a trig inequality:





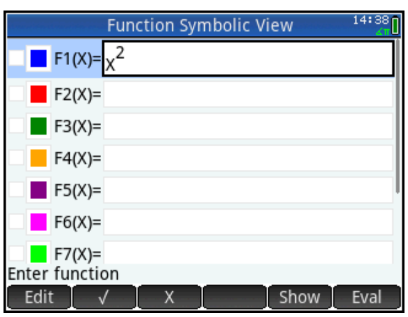
II. Equivalent Equations and one-to-one functions

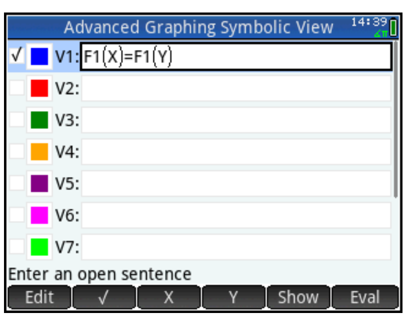
Two equations are equivalent when they have the same solution set. In the presence of machine tools that produce numeric and exact symbolic solutions of equations, an understanding of what it means for two equations to be equivalent gains importance. (Even in the absence of machine solvers, the idea of equivalent equations has earned less attention than it merits. Instead, we focus all too often on the methods of solving equations, without paying attention to when such methods may result in equations that are not equivalent!)

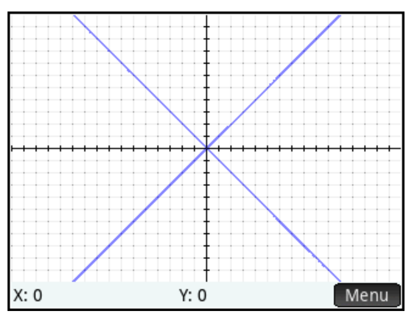
This graphic exploration of equivalent equations might help some students understand important aspects of the process of solving equations. In particular, the idea of "extraneous roots" should become plain.

So, if X = Y, under what circumstances (or, for what values if X and Y and for what functions F) will F(X) = F(Y)? Conversely, for what values of X and Y and for what functions F is it true that if F(X) = F(Y), then X = Y?

The Function App in HP Prime allows the user to graph up to 10 functions, in an environment that anyone using graphing calculators is familiar with. With HP Prime, though, you can refer to functions defined in the Function App inside the Advanced Graphing App. See the screen shots below:



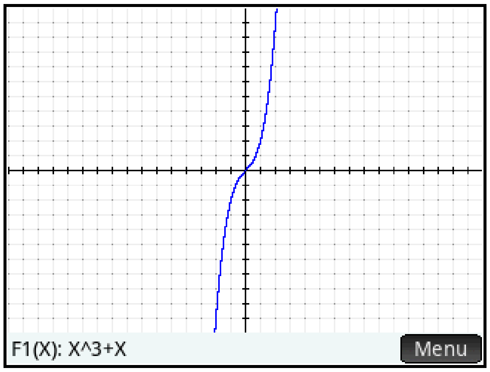


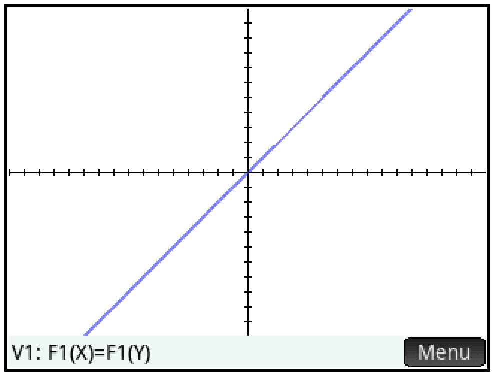


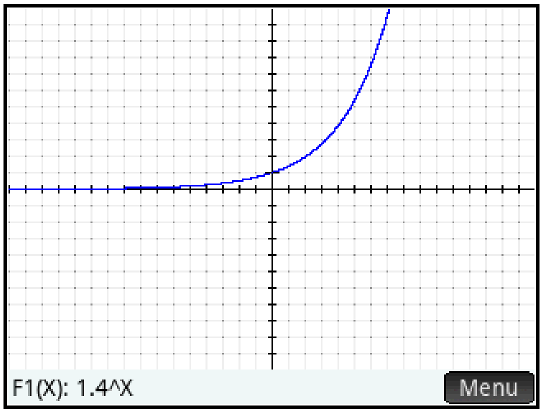
Here, it is apparent that for the squaring function, it is not the case that F(X) = F(Y) implies X = Y. Rather, it seems that X^2 = Y^2 implies X = Y or X = -Y.

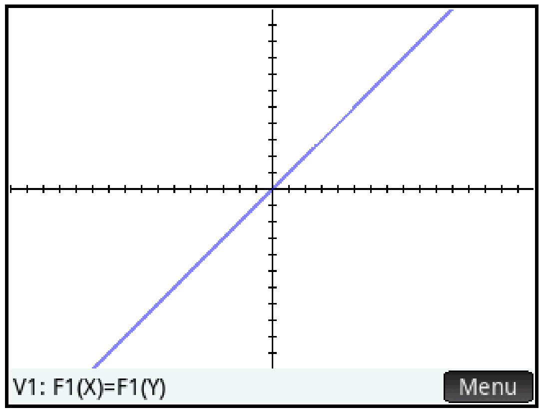
This example points to a discovery activity that leads to the conclusion that F(X) = F(Y) implies X = Y whenever F is a one-to-one function. If the domain of F is a subset of the reals, then the implication holds on that subset. Of course, we could have begun our example by simply graphing the relation $x^2 = y^2$. The setup above was chosen as a starting point for an investigation that asks students to ry various functions for F1(X), and see which ones result in the graph of X when you graph the relation F1(X) = F1(Y).

Here are some screen shots to illustrate the key steps in such an activity.

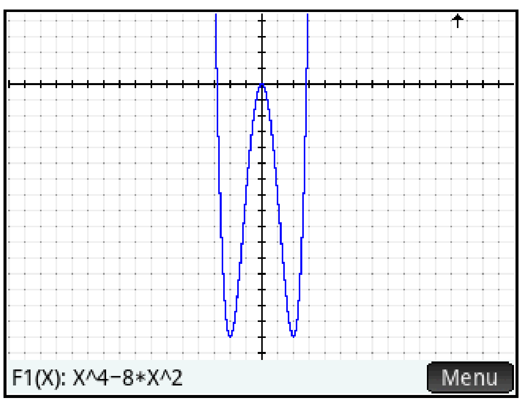


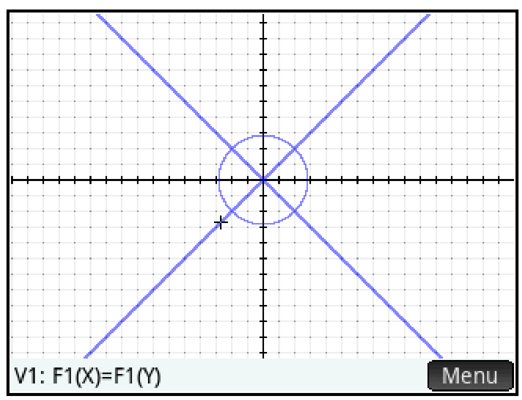






What happens if the function F1 is NOT one-to-one? There's a treasure chest of riches to explore. Consider these two gems.



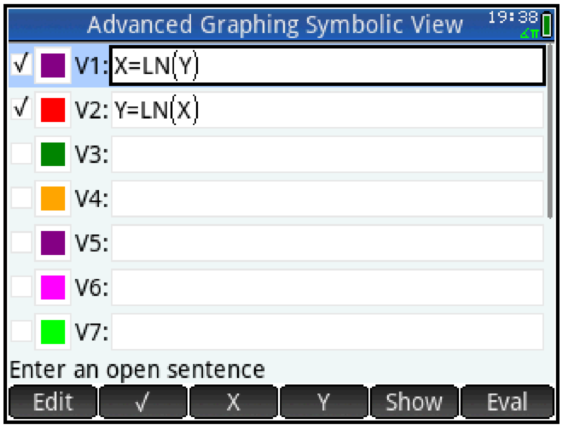


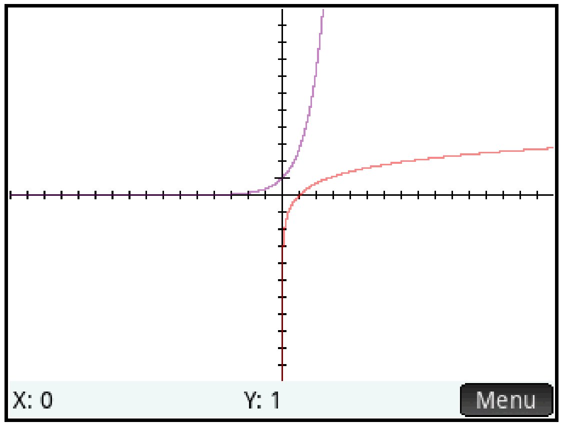
Whoa! Surprised to see the circle? I was! Tracing on the circle reveals it has radius $\sqrt{8}$. So, its equation appears to be $x^2 + y^2 = 8$.

Now, the figure above and on the right shows the graph of all ordered pairs (*x*, *y*) such that $x^4 - 8x^2 = y^4 - 8y^2$. Subtracting, we have $x^4 - y^4 - 8x^2 + 8y^2 = 0$. Factoring, we get $\left(x^2 - y^2\right)\left(x^2 + y^2\right) - 8\left(x^2 - y^2\right) = 0$ and then $\left(x^2 - y^2\right)\left(\left(x^2 + y^2\right) - 8\right) = 0$. The first factor gives rise to the lines $y = x$ and $y = -x$, the second factor gives the circle, $x^2 + y^2 = 8$. Cool!

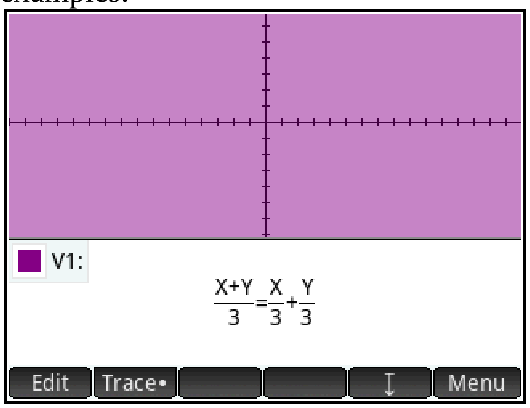
Note that replacing *y* with either *x* or –*x* in $x^4 - 8x^2 = y^4 - 8y^2$, we get a sentence that is obviously true. Try substituting $y^2 = 8 - x^2$ and see what happens!

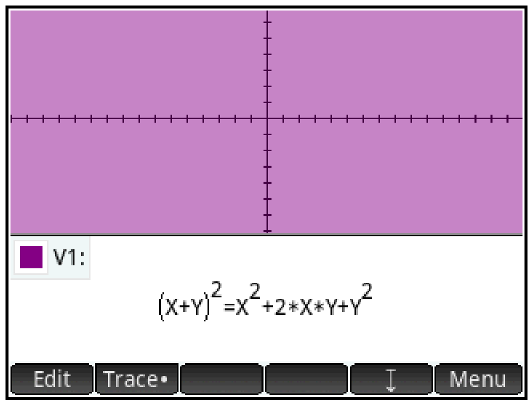
I remember how excited I was when I saw how to graph a function and its inverse using parametric mode on first generation graphing calculators. But now, this can be done in a more natural way. Check out the example.

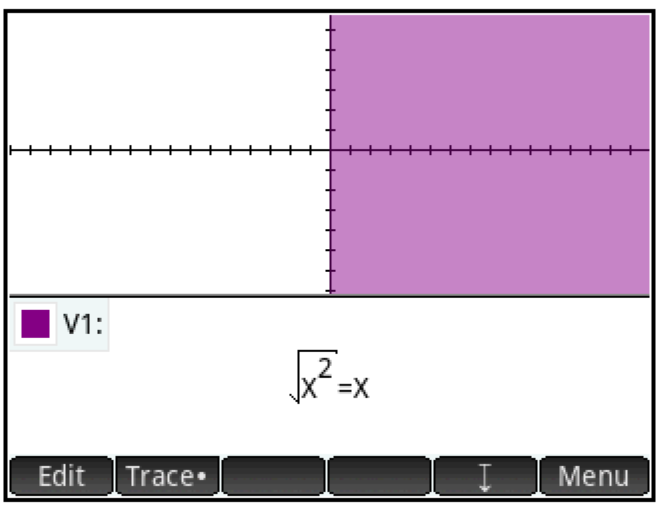




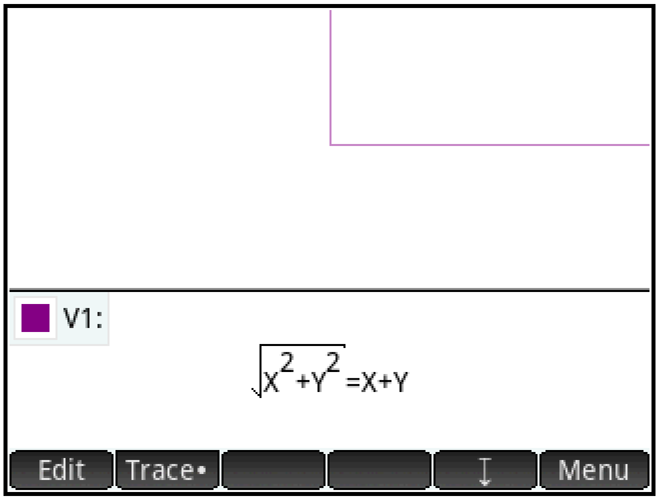
Graphing identities and equations that result from common algebraic pitfalls can be persuasive. Consider these examples:







To see this one, you have to turn off the coordinate axes:

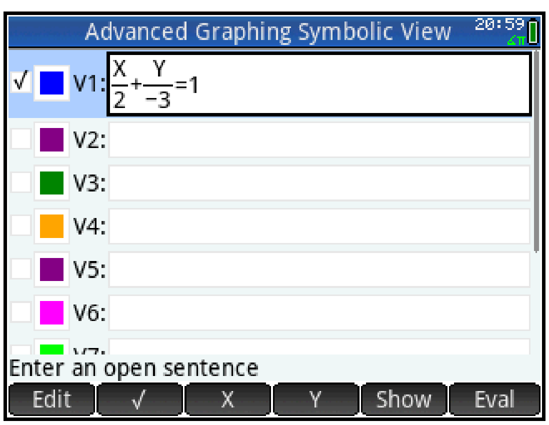


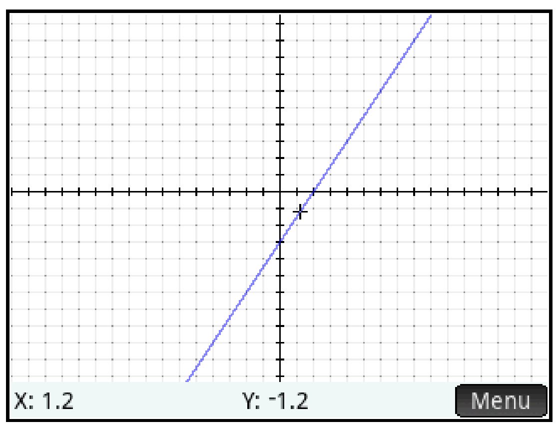
Pretty persuasive.

III. Transformations

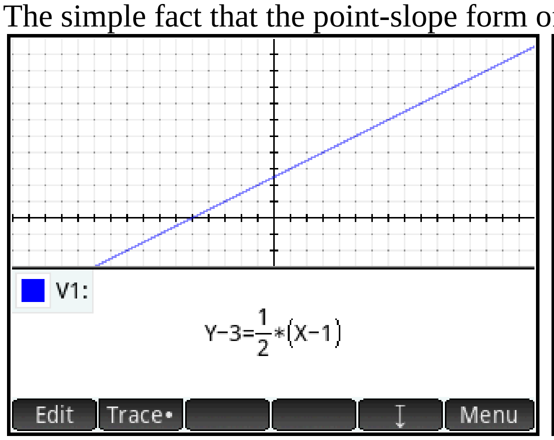
The fact that we can graph any equation with *x* and *y* in any form can be leveraged for pedagogic advantage. Consider the intercept-intercept form of the equation of a line:

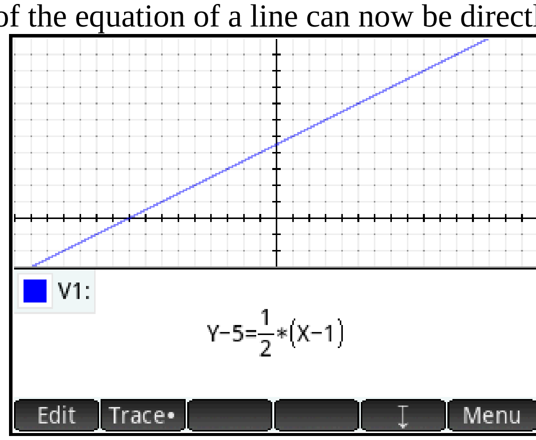
$\frac{x}{a}+\frac{y}{b}=1$. Note that the line has an *x* intercept of $(a,0)$ and a *y*-intercept of $(0,b)$. On "classic" graphing calculators, there's no direct way to investigate such equations, because before you can graph a function, you have to solve for the dependent variable.





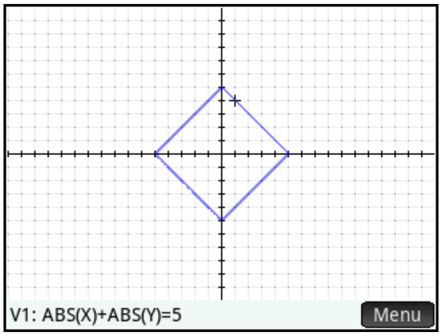
The simple fact that the point-slope form of the equation of a line can now be directly graphed is also helpful:

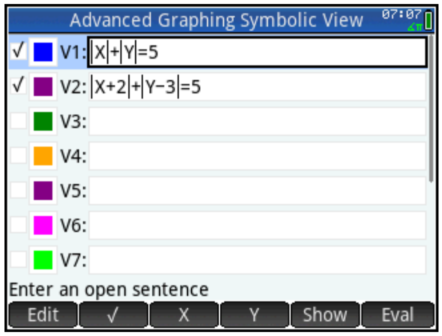


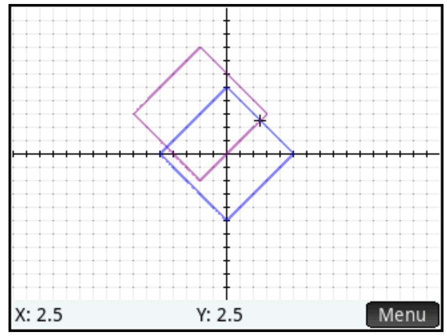


The definition can be edited right from the graph screen, so an update to the definition can be visualized directly.

It's now easier to generalize transformations of arbitrary graphs in the plane. Here are just a couple of examples.

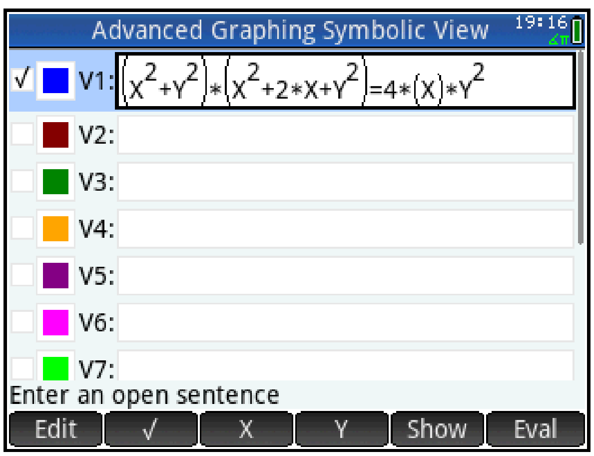


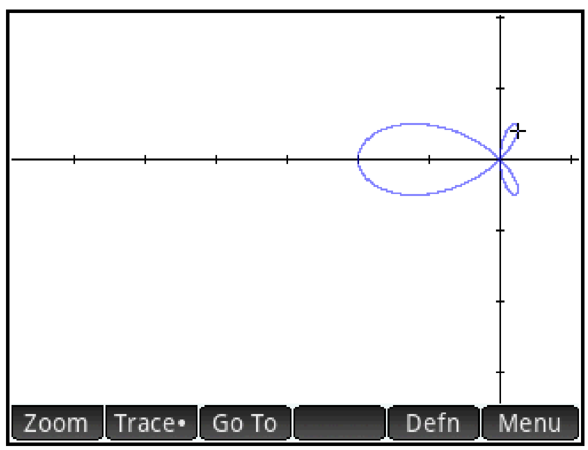




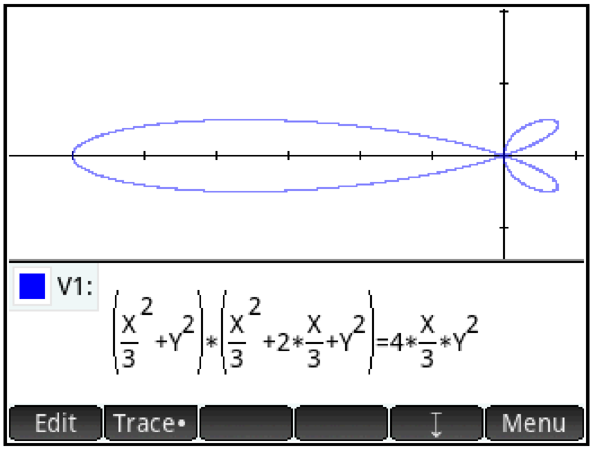
It wasn't until I started thinking about graphing equations in two variables with a graphing engine like the one in HP-Prime that I became aware of a consistent error I'd been making when explaining graphs to my students. I have been in the habit of describing dilations using words like this: "Multiply the function by two and this is a vertical stretch of the graph by a factor of two. Multiple every X by two and this is horizontal stretch by a factor of two. The effect in the horizontal direction is opposite to the effect in the vertical…that is, multiplying the X's by two squeezes the graph, multiplying the Y's by two stretches it out."

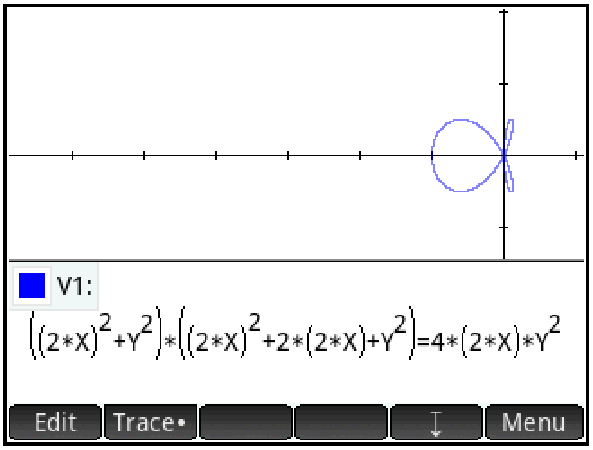
This really is misleading. In fact, replacing every instance Y by 2Y has the same effect in the vertical direction as replacing every X by 2X does in the horizontal direction! Of course this is true! This example, a graph of the folium, should illustrate the point.



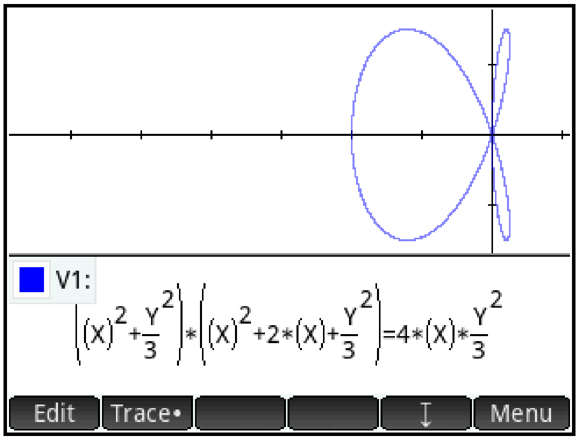


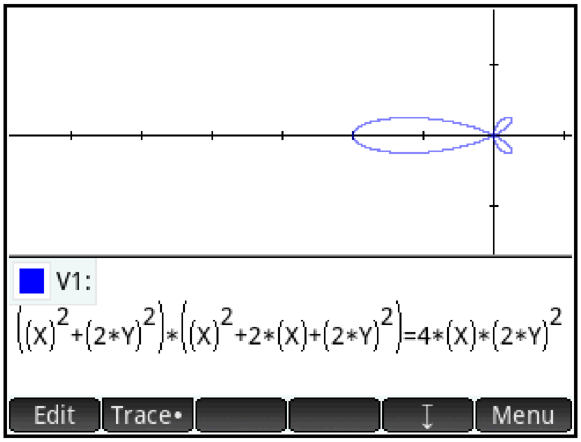
Stretching and squeezing horizontally:





Stretching and squeezing vertically:

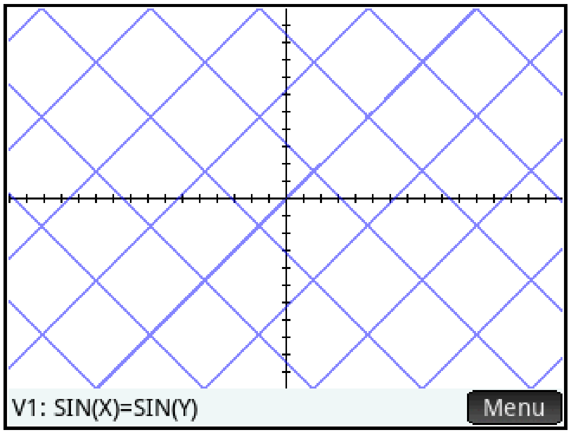




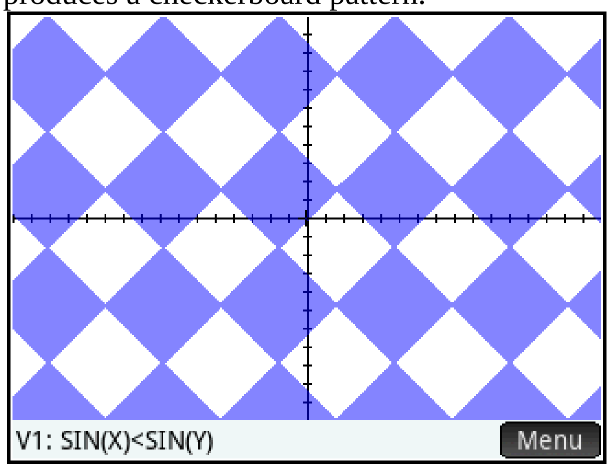
Now, we can use the same language to talk about vertical and horizontal transformations.

IV. New Visualizations of Old Content

The sine function is about as far from one-to-one as a function can be. Take a look at this graph and try to understand it.

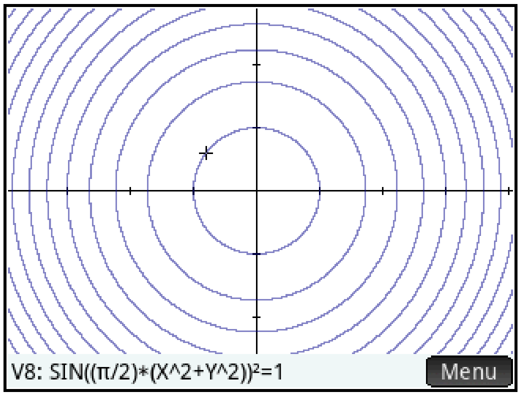


Clearly, when X = Y, SIN(X) = SIN(Y). But that's obviously not the only time. Notice how far apart the *y*-intercepts of the lines with slope 1 are. So, whenever X = Y+2π or X = Y+4π and so on, it's true that SIN(X) = SIN(Y). This explains the lines with positive slope. How about the lines with negative slope? It looks like any point on the line Y = π – X solves SIN(X) = SIN(Y). A glance at the unit circle shows why this is so. And of course if Y = π – X solves it, so should Y = 3π – X, Y = 5π – X, and so on. Cool! An inequality produces a checkerboard pattern.

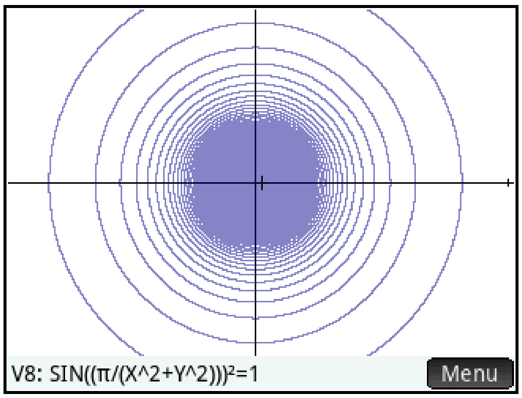


Here are a couple of challenges for students (or teachers!) that leverage the periodicity of the sine or cosine function to make interesting graphs.

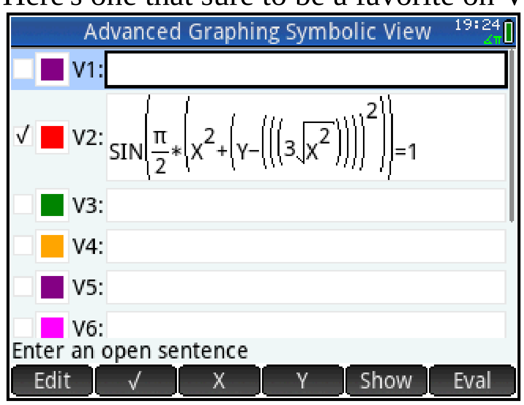
Create, using one equation, the graph of an infinite number of concentric circles. Well, $x^2 + y^2 = 1$ gives us a circle. And $\left(\sin\left(\frac{\pi}{2}x\right)\right)^2 = 1$ has solutions $x = 1, x = 3, x = 5$, etc. So, $\left(\sin\left(\frac{\pi}{2}\left(x^2 + y^2\right)\right)\right)^2 = 1$ has solutions whenenver $x^2 + y^2 = 1$, $x^2 + y^2 = 3$, etc. Check it out:

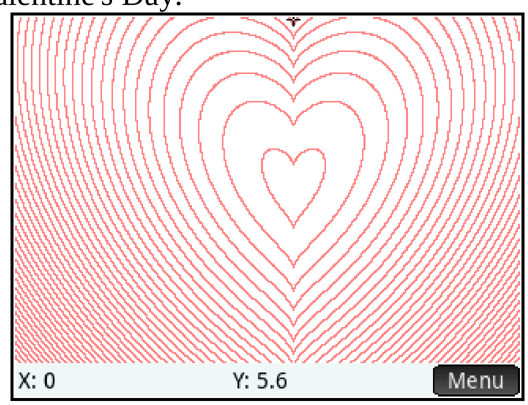


What if you wanted the circles to be closer together close to the origin, and spread farther apart as the distance from the origin gets greater? The reciprocal function does the trick!



Here's one that sure to be a favorite on Valentine's Day.



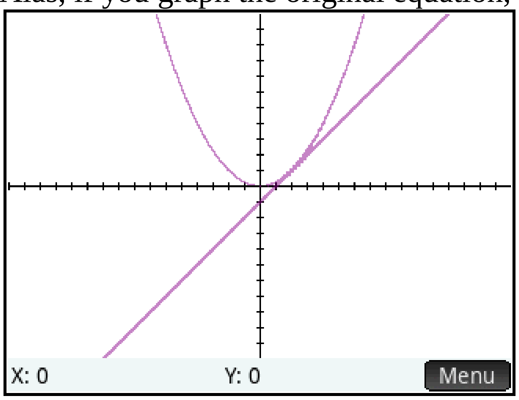


Maybe you can tell that we're just scratching the surface here. Using transformations, different kinds of functions, and a dash of creativity, the possibilities for using mathematics to build compelling graphs seem endless.

V. Implicit Differentiation

Accidentally, I stumbled on a trick that I'm sure is used by some clever problem writers for Calculus textbooks. I had assigned a homework problem that involved implicit differentiation with an equation in two variables in order to calculate the slope of the curve at a certain point. After we had worked out the problem symbolically, I remembered that my HP Prime would let us graph the original curve. When we did, we were all a bit surprised by the result. Here's an example inspired by that experience.

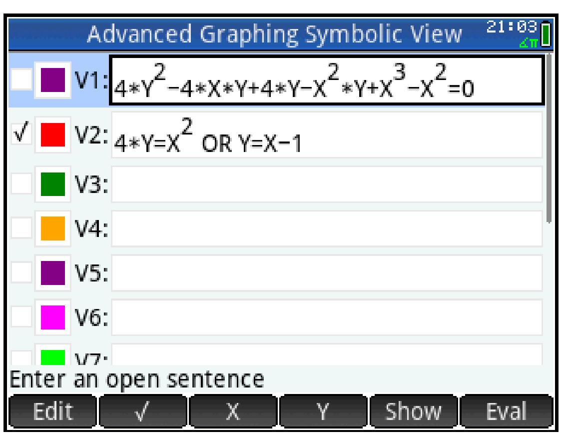
Start with the equation $4y^2-4xy+4y-x^2y+x^3-x^2=0$. Differentiate implicitly and solve for y'. You get $y'=\dfrac{2x-3y^2+2xy+4y}{8y-4x-x^2+4}$. A homework problem I assigned asked students to evaluate the slope of the line tangent to the graph of $4y^2-4xy+4y-x^2y+x^3-x^2=0$ at the point where $x = 4$ and $y = 3$. What a mess! Shockingly, the slope turns out to be 1. "How do they make these problems come out so nice?", I wondered. Alas, if you graph the original equation, you see this:

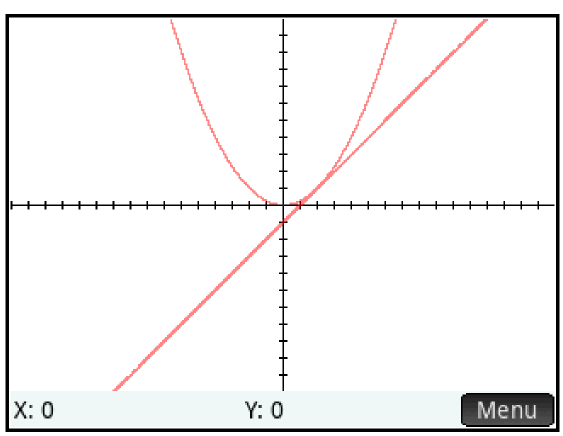


This seems to indicate that there is a linear and a quadratic factor for the original equation. In fact, by careful rearranging and grouping of terms, you can produce this result:

$$\begin{aligned}4y^2-4xy+4y-x^2y+x^3-x^2 &= \left(4y^2-x^2y\right)-\left(4xy-x^3\right)+\left(4y-x\right)\\ &= \left(4y-x^2\right)y-\left(4y-x^2\right)x+\left(4y-x\right)1\\ &= \left(4y-x^2\right)\left(y-x+1\right)\end{aligned}$$

So, if the product is 0, either factor can be zero, and we see that the original equation is equivalent to the expression $\left(4y=x^2\right)$OR $\left(y=x-1\right)$. In fact, you could graph exactly that with HP-Prime:





Those clever problem writers devised a trick that lets them build these ghastly implicit relations that have equally ghastly derivatives, but very simple and easy to find points on them, and easy to find slopes as well! This particular example was crafted so that the relation graphed contains function and its tangent line at $x = 2$. Notice that if you substitute the point (2, 1) into the derivative, you get the indeterminate form, 0/0. Alas, this must be so whenever the relation has two intersecting branches, a fact that can be easily verified by calculating the derivative using the factored form of the relation. In fact, you can verify that, in general, the relation $\left(y-f\left(x\right)\right)\left(y-g\left(x\right)\right)=0$ will have a derivative that is an indeterminate form at the point where $f\left(x\right)=g\left(x\right)$. Very cool.