Mathematical Games
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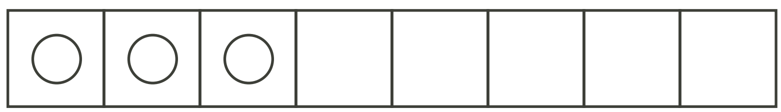
Let the Games Begin!

(Solutions to selected games appear at the end)

1. Alice and Beth are playing a game with three coins on a 1 unit by 8 unit strip of paper divided up into eight squares as shown.



The rules of the game are as follows:

(i) On a player's turn, she must move a coin any number of squares to the right.

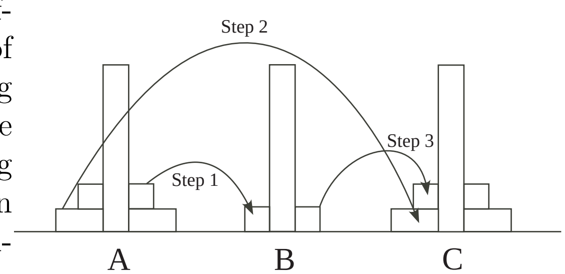
(ii) The coin may not pass over or land on a square that is occupied by another coin.

(iii) If it is a player's turn and she has no legal move she loses the game.

It is Alice's turn to go first. Describe her first move and then her winning strategy.

2. Erin and Fran are playing a game with four stacks of cards. On each player's turn, they must remove some number of cards from any one stack. The player who takes the last card wins. It is Erin's turn and the four stacks have 11, 11, 14, and 16 cards, respectively. Show that Erin has a winning strategy.

3. Mourad is playing a game called "The Three Towers of Hanoi". The game consists of disks of different sizes that fit on three pegs. The goal of the game is to move all of the disks from one peg to another in as few moves as possible. A move consists of removing a single disk from one peg and placing it onto another. The only restriction is that a disk cannot be placed on a peg that already contains any disks of a smaller width. The three steps required to complete the game when there are two disks are shown.



(a) Show how Mourad can move three disks from peg A to peg C in 7 moves.

(b) If the game started with 10 disks, how many moves would be required?

4. Emilia and Omar are playing a game in which they take turns placing numbered tiles on the grid shown.

Emilia starts the game with six tiles: $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, and $\boxed{6}$

Omar also starts the game with six tiles: $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, and $\boxed{6}$

Once a tile is placed, it cannot be moved.
After all of the tiles have been placed, Emilia scores one point for each row that has an even sum and one point for each column that has an even sum. Omar scores one point for each row that has an odd sum and one point for each column that has an odd sum. For example, if the game ends with the tiles placed as shown below, then Emilia will score 5 points and Omar 2 points.

3	1	2	4
5	5	2	4
1	3	6	6

(a) In a game, after Omar has placed his second last tile, the grid appears as shown below. Starting with the partially completed game shown, give a final placement of tiles for which Omar scores more points than Emilia. (You do not have to give a strategy, simply fill in the final grid.)

1	3		5
3	1	2	2
	4	4	

(b) Explain why it is impossible for Omar and Emilia to score the same number of points in any game.

(c) In the partially completed game shown below, it is Omar's turn to play and he has a 2 and a 5 still to place. Explain why Omar cannot score more points than Emilia, no matter where he places the 5.

1		3	6
	5		4
3		1	6

5. Candace and Darryl are playing a game with two piles of beans. On each player's turn, they must remove at least 1, but at most 7 beans from one of the piles. The winner is the player to take the last bean.

   (a) It is Candace's turn and the two piles have 27 and 25 beans, respectively. What is Candace's best move?

   (b) If Candace is choosing from two piles with x and y beans, for what values of x and y does she have a winning strategy?

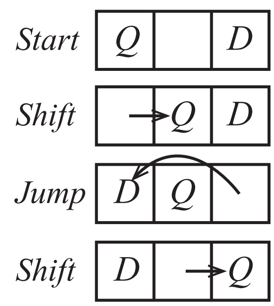
   (c) Who should win if we reverse the outcome of the game, that is if the loser is the player to take the last bean?

6. In the game "Switch", the goal is to make the dimes (D) and quarters (Q) switch spots. The starting position of the game with 1 quarter and 1 dime is shown below. Allowable moves are:

   (i) If there is a vacant spot beside a coin then you may *shift* to that space.

   (ii) You may jump a quarter with a dime or a dime with a quarter if the space on the other side is free.

   The game shown in the diagram takes three moves.



   (a) Complete the diagram to demonstrate how the game of "Switch" that starts with 2 quarters and 2 dimes can be played in 8 moves.

   (b) By considering the number of required *shifts* and *jumps*, explain why the game with 3 quarters and 3 dimes cannot be played in fewer than 15 moves.

   (c) Explain why the game with n quarters and n dimes cannot be played in fewer than $n(n+2)$ moves.

Q	*Q*		*D*	*D*
D	*D*		*Q*	*Q*

7. Two players, Alicia (A) and Bettina (B) begin a game with 33 toothpicks in a pile. On each player's turn, she must take 1, 2, 3 or 4 toothpicks away. Whoever takes the last toothpick wins. Show that Alicia has a winning strategy assuming she plays first.

   How does the game change if the winner is now the player who does *not* take the last toothpick?

8. Ian and Jen are playing a game with three jars of marbles. On each player's turn, they must remove the same number of marbles from each of two different jars. If a player is unable to do so, they lose the game. If the jars are labeled A, B and C, legal moves would include removing 1 from both A and B, removing 3 from both A and C, etc.

   (a) If it is Ian's turn and the jars contain 2, 3 and 5 marbles respectively, which player has a winning strategy?

   (b) If it is Ian's turn and the jars contain 2, 4 and 5 marbles respectively, which player has a winning strategy?

   (c) Suppose that the jars contain 2, b and c marbles respectively, with $2 < b < c$. Determine all pairs (b, c) for which Ian *does not* have a winning strategy.

9. Carolyn and Paul are playing a game starting with a list of the integers 1 to n. The rules of the game are:

   - Carolyn always has the first turn.
   - Carolyn and Paul alternate turns.
   - On each of her turns, Carolyn must remove one number from the list such that this number has at least one positive divisor other than itself remaining in the list.
   - On each of his turns, Paul must remove from the list all of the positive divisors of the number that Carolyn has just removed.
   - If Carolyn cannot remove any more numbers, then Paul removes the rest of the numbers.

   For example, if $n = 6$, a possible sequence of moves is shown in this chart:

Player	Number(s) removed	Number(s) remaining	Notes
Carolyn	4	1, 2, 3, 5, 6	
Paul	1, 2	3, 5, 6	
Carolyn	6	3, 5	She could not remove 3 or 5
Paul	3	5	
Carolyn	None	5	She can't remove any number
Paul	5	None	

   In this example, the sum of the numbers removed by Carolyn is $4 + 6 = 10$ and the sum of the numbers removed by Paul is $1 + 2 + 3 + 5 = 11$.

   (a) Suppose that $n = 6$ and Carolyn removes the integer 2 on her first turn. Determine the sum of the numbers that Carolyn removes and the sum of the numbers that Paul removes.

   (b) If $n = 10$, determine Carolyn's maximum possible final sum. Prove that this sum is her maximum possible sum.

   (c) If $n = 14$, prove that Carolyn cannot remove 7 numbers.

10. (a) Al and Bert are playing a game, starting with a pack of 7 cards. Al begins by discarding at least one but not more than half of the cards in the pack. He then passes the remaining cards in the pack to Bert. Bert continues the game by discarding at least one but not more than half of the remaining cards in the pack. The game continues in this way with the pack being passed back and forth between the two players. The loser is the player who, at the beginning of his turn, receives only one card. Show, with justification, that there is always a winning strategy for Bert.

    (b) Al and Bert now play a game with the same rules as in (a), except that this time they start with a pack of 52 cards, and Al goes first again. As in (a), a player on his turn must discard at least one and not more than half of the remaining cards from the pack. Is there a strategy that Al can use to be guaranteed that he will win?

11. Two players alternate writing numbers on the blackboard. On each move, a player can increase the tens digit or the ones digit but not both. The starting number is 01 and the player who writes 99 wins. The sequence 07, 47, 87, 88, 89, 99 is an instance of the game won by the second player.

    (a) Which player has a winning strategy and why?

    (b) If we play 999 using analogous rules, starting with 001, which player should win?

12. Gwen and Chris are playing a game. They begin with a pile of toothpicks, and use the following rules:

    - The two players alternate turns
    - On any turn, the player can remove 1, 2, 3, 4, or 5 toothpicks from the pile
    - The same number of toothpicks cannot be removed on two different turns
    - The last person who is able to play wins, regardless of whether there are any toothpicks remaining in the pile

    For example, if the game begins with 8 toothpicks, the following moves could occur:

    Gwen removes 1 toothpick, leaving 7 in the pile
    Chris removes 4 toothpicks, leaving 3 in the pile
    Gwen removes 2 toothpicks, leaving 1 in the pile

    Gwen is now the winner, since Chris cannot remove 1 toothpick. (Gwen already removed 1 toothpick on one of her turns, and the third rule says that 1 toothpick cannot be removed on another turn.)

    (a) Suppose the game begins with 11 toothpicks. Gwen begins by removing 3 toothpicks. Chris follows and removes 1. Then Gwen removes 4 toothpicks. Explain how Chris can win the game.

    (b) Suppose the game begins with 10 toothpicks. Gwen begins by removing 5 toothpicks. Explain why Gwen can always win, regardless of what Chris removes on his turn.

    (c) Suppose the game begins with 9 toothpicks. Gwen begins by removing 2 toothpicks. Explain how Gwen can always win, regardless of how Chris plays.

13. In a game, Xavier and Yolanda take turns calling out whole numbers. The first number called must be a whole number between and including 1 and 9. Each number called after the first must be a whole number which is 1 to 10 greater than the previous number called.

    (a) The first time the game is played, the person who calls the number 15 is the winner. Explain why Xavier has a winning strategy if he goes first and calls 4.

    (b) The second time the game is played, the person who calls the number 50 is the winner. If Xavier goes first,how does he guarantee that he will win?

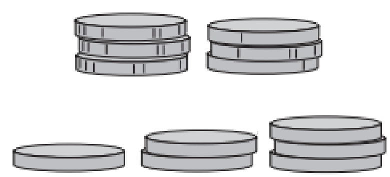
    (c) In the game described in (b), the target number was 50. For what different values of the target number is it guaranteed that Yolanda will have a winning strategy if Xavier goes first?

14. Xavier and Yolanda are playing a game starting with some coins arranged in piles. Xavier always goes first,and the two players take turns removing one or more coins from any one pile. The player who takes the last coin wins.

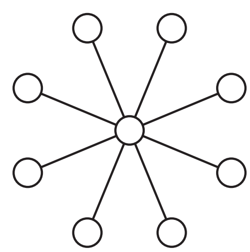
    (a) If there are two piles of coins with 3 coins in each pile, show that Yolanda can guarantee that she always wins the game

    (b) If the game starts with piles of 1, 2 and 3 coins, explain how Yolanda can guarantee that she always wins the game.

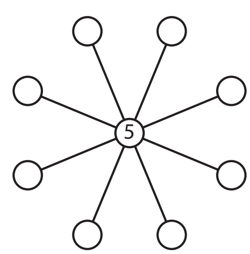
    (c) If the game starts with piles of 2, 4 and 5 coins, which player wins if both players always make their best possible move? Explain the winning strategy.



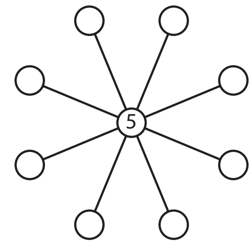
15. In "The Sun Game", two players take turns placing discs numbered 1 to 9 in the circles on the board. Each number can only be used once. The object of the game is to be the first to place a disc so that the sum of the 3 numbers along a line through the centre circle is 15.



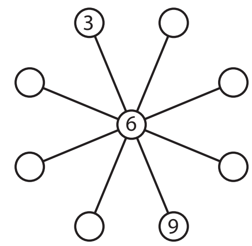
    (a) If Avril places a 5 in the centre circle and then Bob places a 3, explain how Avril can win on her next turn.



(b) If Avril starts by placing a 5 in the centre circle, show that whatever Bob does on his first turn, Avril can always win on her next turn.



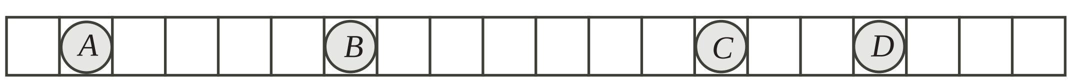
(c) If the game is in the position shown and Bob goes next, show that however Bob plays, Avril can win this game.



16. Consider a matchstick game, where the players are asked to choose 1, 2, 3 or 4 matches from one of two piles of 13 and 17 matchsticks, the winner being the player who takes the last matchstick. Describe a winning strategy.

17. Consider a rook starting on the lower left hand corner of a standard 8 by 8 chess board. The rook is a chess piece that on any turn may be moved to the right or upwards, but not both, as many squares as the player wishes. Al and Bert take turns moving the rook with Al going first, the winner being the player who advances the rook to the upper right square diagonally opposite the starting square. What is Bert's winning strategy? Who will win if the board is 8 by 10?

18. A game for two players uses four counters on a board which consists of a 20×1 rectangle. The two players alternate turns. A turn consists of moving any one of the four counters any number of squares to the right, but the counter may not land on top of, or move past, any of the other counters. For instance, in the position shown below, the next player could move D one, two or three squares to the right, or move C one or two squares to the right, and so on.



The winner of the game is the player who makes the last legal move. (After this move the counters will occupy the four squares on the extreme right of the board and no further legal moves will be possible.)
In the position shown above, it is your turn. Which move should you make and what should be your strategy in subsequent moves to ensure that you will win the game?

Solutions to Selected Games

4. (a) The tiles which remain to be placed are one 5 and two 6s. There are only 3 different ways in which these tiles can be placed. Here are the three ways along with the totals of each column and row:

1	3	6	5
3	1	2	2
5	4	4	6

(Column totals: 9, 8, 12, 13; Row totals: 15, 8, 19)

1	3	5	5
3	1	2	2
6	4	4	6

(Column totals: 10, 8, 11, 13; Row totals: 14, 8, 20)

1	3	6	5
3	1	2	2
6	4	4	5

(Column totals: 10, 8, 12, 12; Row totals: 15, 8, 19)

Of these the ones which give Omar more points than Emilia is the first

1	3	6	5
3	1	2	2
5	4	4	6

where Omar gets 4 points and Emilia gets 3.

(b) The total number of points available is 7, since there are 4 columns and 3 rows and each column and each row gives 1 point to one of the players.
So Emilia's score and Omar's score must add up to 7.
Since 7 is odd, Emilia's score and Omar's score cannot be the same.

(c) *Solution 1*
The tiles that remain to be placed are 2, 2, 4 and 5, ie. three even numbers and one odd number.
In the grid, the fourth column is already complete and has an even sum (16), so Emilia already has one point.
Consider rows 1 and 3 and columns 1 and 3.
Each has only one space left open, and each has an even sum so far.
In any of these cases, if an even number is placed in the empty space, the row or column will be complete and the sum will be even, giving Emilia a point. If an odd number is placed in the empty space, the row or column will be complete and the sum will be odd, giving Omar a point.
But since there is only one odd-numbered tile left to play, then only one of these four rows and columns can end up with an odd score, and three are guaranteed to be even.
So no matter where Omar places the 5, Emilia will get 3 more points, for a total of at least 4 points.

Since there are only 7 points available, then Emilia is guaranteed to have more points than Omar, because Omar will get at most 3 points.

Solution 2
The tiles that remain to be placed are 2, 2, 4 and 5, ie. three even numbers and one odd number.
Suppose Omar places the 5 in the leftmost empty box. Then the numbers which go into the other three empty spaces are the 2, 2 and 4, so are all even.
The sum of the top row will then be $1+3+6$ plus either 2 or 4, so will be even.
The sum of the bottom row will be $3+1+6$ plus either 2 or 4, so will be even.
The sum of the fourth column is already 16, so is even.
The sum of the third column will be $3+1$ plus either 2 or 4, so will be even.
Therefore, in this case Emilia will get at least 4 of the 7 possible points, so will have more points than Omar.

If Omar places the 5 in the rightmost empty box, the argument is exactly the same, except we look at the first column instead of the third. In this case, again Emilia gets at least 4 points.

Suppose Omar places the 5 in the uppermost empty box. Then the numbers which go into the other three empty spaces are the 2, 2, and 4, so are all even.
The sum of the bottom row will be $3+1+6$ plus either 2 or 4, so will be even.
The sum of the fourth column is already 16, so is even.
The sum of each of the first and third columns is $3+1$ plus either 2 or 4, so will be even.
Therefore, in this case Emilia will get at least 4 of the 7 possible points, so will have more points than Omar.

If Omar places the 5 in the lowermost empty box, the argument is exactly the same as this previous case, except we look at the top row instead of the bottom row. In this case, again Emilia gets at least 4 points.

Therefore, having looked at all of the possible cases, Emilia will always get at least 4 points, so will always have more points than Omar, no matter where Omar places the 5.

6. (a) We complete the chart, including at each step a description of the step. This is one possible way for the game to be played in 8 moves.

Start	Q	Q		D	D
Shift	Q		Q	D	D
Jump	Q	D	Q		D
Shift	Q	D	Q	D	
Jump	Q	D		D	Q
Jump		D	Q	D	Q
Shift	D		Q	D	Q
Jump	D	D	Q		Q
Shift	D	D		Q	Q

There is in fact only one other possibility (which simply reverses whether a dime or quarter is being moved at each step).
(We can notice that, at each step, there is only one possible move that can be made to avoid having to backtrack.)

Start	Q	Q		D	D
Shift	Q	Q	D		D
Jump	Q		D	Q	D
Shift		Q	D	Q	D
Jump	D	Q		Q	D
Jump	D	Q	D	Q	
Shift	D	Q	D		Q
Jump	D		D	Q	Q
Shift	D	D		Q	Q

(b) In a game starting with 3 quarters and 3 dimes, the game board will have 7 squares.
Since there are only two possible types of moves that can be made ("Jumps" and "Shifts"), then it is impossible for the 3 quarters to switch their order (ie. we are not allowed to jump one quarter over a second quarter).
Thus, the quarter that starts in the third square ends in the seventh square, the quarter that starts in the second square ends in the sixth square, and the quarter that starts in the first square ends in the fifth square.

Each quarter then moves 4 spaces, making a total of 12 squares moved. Similarly, each dime moves 4 spaces, making a total of 12 squares moved. In total, the coins move 24 squares. If this was done using only shifts, this would require 24 moves.

However, the game cannot be played in this fashion, because it is necessary to make jumps. Since we want the dimes and quarters to change positions, each dime needs to jump over (or be jumped over by) each of the 3 quarters. In other words, there need to 9 jumps made. Since each jump results in a move of 2 spaces, this "saves" 9 shifts.
Therefore, the number of required moves is at least $249 = 15$.
We have assumed here that no "backtracking" is done, so the game cannot be played in fewer than 15 moves.
(Can you construct the diagram to show how the game can be played in 15 moves?)

(c) We use the same strategy as in (b).
In a game starting with n quarters and n dimes, the game board will have $2n + 1$ slots.
Since there are only two possible types of moves that can be made ("Jumps" and "Shifts"), then it is impossible for the n quarters to switch their order (ie. we are not allowed to jump on quarter over another).
Thus, the quarter that starts in the first square ends in the $(n+2)$ square, the quarter that starts in the second square ends in the $(n+3)$ square, and so on, with the quarter that starts in the nth square ending in the $(2n + 1)$ square.
Each of the n quarters has moved a total of $n + 1$ squares, so the quarters have therefore moved a total of $n(n + 1)$ squares.
Similarly, the dimes have moved a total $n(n + 1)$ squares.
So the coins have moved a total of $2n(n + 1) = 2n2 + 2n$ squares. If this was done using only shifts, this would require $2n^2 + 2n$ moves.
Using exactly the same reasoning as in (c), the required number of moves is
$2n^2 + 2n - n^2 = n^2 + 2n = n(n + 2)$.
(A question that should be asked is "Can the game be played in exactly $n(n + 2)$ moves?" In order to answer this question, we would need to come up with a general strategy that would allow us to play the game in this number of moves, no matter what value n takes.)

7. First we can see that there will be fewer toothpicks on each turn, so someone will eventually take the last one. Therefore, a tie can never occur and thus one player must have a winning strategy.
We will examine this game by analysing the winning and losing positions in terms of the number of toothpicks that a player chooses from on a particular turn.
Choosing from 1 to 4 toothpicks is a winning position since whoever is choosing can remove all of the toothpicks.

Choosing from 5 toothpicks is a losing position since choosing 1, 2, 3 or 4 toothpicks will leave the other player with 4, 3, 2 or 1 toothpicks, each of which is a winning position as discussed above.
Choosing from 6, 7, 8 or 9 toothpicks is a winning position since by removing 1, 2, 3 or 4 toothpicks, the other player is forced to choose from 5 which is a losing position.
Choosing from 10 is a losing position since choosing 1, 2, 3 or 4 toothpicks leaves the other player with 9, 8, 7 or 6 toothpicks, each of which is a winning position.
If we extend this idea, we can see that choosing from any multiple of 5 toothpicks is a losing position.
Thus, a winning strategy is to leave your opponent with a multiple of 5 toothpicks to choose from.
Since our game starts with 33 toothpicks, A should begin by removing 3, thus leaving 30 for player B. How does the game continue from this point?
No matter how many toothpicks B takes, A can always remove some number that will reduce the entire pile by 5 over the pair of turns, thus leaving B with a multiple of 5 on her next turn.
If this is repeated every turn, A will always take the last toothpick.

We can generalize this game and start with any number of toothpicks.
As long as a player can make the number of toothpicks remaining a multiple of 5, she is guaranteed to win.
This is always possible for A who goes first, unless the game begins with a multiple of 5 toothpicks.
In this case, B can follow the same winning strategy and force A to always be left with multiples of 5 on her turn.

How does the game change if the winner is now the player who does *not* take the last toothpick?

9. (a) The list starts as 1, 2, 3, 4, 5, 6.
If Carolyn removes 2, then Paul removes the remaining positive divisor of 2 (that is, 1) to leave the list 3, 4, 5, 6.
Carolyn must remove a number from this list that has at least one positive divisor other than itself remaining.
The only such number is 6, so Carolyn removes 6 and so Paul removes the remaining positive divisor of 6 (that is, 3), to leave the list 4, 5.
Carolyn cannot remove either of the remaining numbers as neither has a positive divisor other than itself remaining.
Thus, Paul removes 4 and 5.
In summary, Carolyn removes 2 and 6 for a sum of $2 + 6 = 8$ and Paul removes 1, 3, 4, and 5 for a sum of $1 + 3 + 4 + 5 = 13$.

(b) Since Carolyn removes a single number on each turn in such a way that Paul must be able to remove a number from the list, then Carolyn can remove at most half of the numbers of the list. In this case, Carolyn can remove at most five numbers.
The maximum possible five numbers that Carolyn could remove are the largest five numbers from the list (that is, 6, 7, 8, 9, 10), whose sum is 40.

This is the maximum possible without referring to all of the rules of the game. So is it possible for her to remove these five numbers?
In order to do so, she must remove them in an order which forces Paul to remove only one number on each of his turns.
If Carolyn removes 7 first, Paul removes only 1.
If Carolyn removes 9 next, Paul removes only 3.
If Carolyn removes 6 next, Paul removes only 2.
If Carolyn removes 8 next, Paul removes only 4.
If Carolyn removes 10 next, Paul removes only 5.
(Carolyn could have switched her last two turns.)
Therefore, Carolyn can indeed remove the five largest numbers, so her maximum possible final sum is 40.

(c) As in (b), Carolyn can remove at most half of the numbers from the list, so can remove at most 7 numbers.
Can she possibly remove 7 numbers?
If Carolyn removes 7 numbers, then Paul must also remove 7 numbers since he removes at least one number for each one that Carolyn removes and there are only 14 numbers.
Since Paul always removes numbers that are divisors of the number just removed by Carolyn, then Paul can never remove a number larger than $\frac{1}{2}n$. (For him to do so, Carolyn would have to have removed a number larger than $2 \times \frac{1}{2}n = n$. This is impossible.)
Therefore, if Carolyn actually removes 7 numbers, then she must remove the 7 numbers larger than $\frac{1}{2}n$ (that is, 8, 9, 10, 11, 12, 13, 14).
Whichever number Carolyn removes first, Paul will remove 1 on his first turn, as it is a positive divisor of every positive integer. (Paul might remove other numbers too.)
At this stage, at least one of 11 and 13 is left in the list (depending on whether Carolyn removed one of these on her first turn).
For the sake of argument, assume that 11 is still left in the list. (The argument is the same if 13 is left.)
Carolyn cannot now remove 11 from the list. This is because 11 is a prime number and its only positive divisors are 1 and 11, so 11 does not have a positive divisor other than itself left in the list, so by the last rule, Carolyn cannot remove 11.
Thus, Carolyn cannot remove all of the numbers from 8 to 14, so cannot remove 7 numbers from the list.

12. (a) Since 3, then 1, then 4 toothpicks have been removed from the initial pile of 11 toothpicks, there are now 3 toothpicks remaining.
Since players have removed 1, 3 and 4 toothpicks on turns already, then Chris can only remove 2 or 5 toothpicks now on his turn, because of rules 2 and 3.
Since there are only 3 toothpicks remaining, Chris must remove 2 toothpicks.
This leaves 1 toothpick in the pile, and the only possible move that Gwen can now make is to remove 5 toothpicks, which is impossible.
Therefore, Gwen cannot make her turn.
Since Chris was the last player able to move, then Chris wins.

(b) After Gwen has removed 5 toothpicks, there are 5 remaining and Chris can remove 1, 2, 3, or 4 on his turn.
If Chris removes 1, there are 4 remaining and Gwen can remove all of them (since no one has yet removed 4 toothpicks on a turn). This empties the pile, so Gwen wins.
If Chris removes 2, there are 3 remaining and Gwen can remove all of them (since no one has yet removed 3 toothpicks on a turn). This empties the pile, so Gwen wins.
If Chris removes 3, there are 2 remaining and Gwen can remove all of them (since no one has yet removed 2 toothpicks on a turn). This empties the pile, so Gwen wins.
If Chris removes 4, there is 1 remaining and Gwen can remove all of them (since no one has yet removed 1 toothpick on a turn). This empties the pile, so Gwen wins.
Therefore, no matter what Chris removes, Gwen can always win the game.

(c) After Gwen has removed 2 toothpicks, there are 7 toothpicks remaining, and Chris can take 1, 3, 4, or 5 on his turn.
If Chris removes 5 toothpicks, there are 2 remaining and Gwen can remove 1, 3 or 4. Thus, Gwen must remove 1, leaving 1 toothpick and Chris can remove 3 or 4. He is unable to make his turn, so Gwen wins. So Chris should not remove 5 toothpicks.

If Chris removes 3 or 4 toothpicks, there are 4 or 3 toothpicks remaining, and Gwen can remove all of them (since in either case that number of toothpicks hasn't yet been removed on a turn), so Gwen wins. So Chris should not remove 3 or 4 toothpicks. (If Gwen removed 1 toothpick instead of 4 or 3 toothpicks, she would still be guaranteed to win, since Chris would be unable to go again. Why?)

If Chris removes 1 toothpick, there are 6 remaining and Gwen can remove 3, 4 or 5. If Gwen now removes 5 toothpicks, there is 1 remaining, and Chris is unable to make his move, since he can now only remove 3 or 4 toothtpicks. So Gwen wins. Similarly, if Gwen had removed 4 toothpicks, there would be 2 remaining and Chris cannot remove 1 or 2 since these numbers have already been used, so Gwen wins. If Gwen had removed 3 toothpicks, there would be 3 remaining and Chris cannot remove 1, 2 or 3, since these numbers have already been used, so Gwen wins.

Thus, regardless of what Chris does on his turn, Gwen will win.

13. (a) *Solution 1*
If Xavier goes first and calls 4, then on her turn Yolanda can call any number from 5 to 14, since her number has to be from 1 to 10 greater than Xavier's.
But if Yolanda calls a number from 5 to 14, then Xavier can call 15 on his next turn, since 15 is from 1 to 10 bigger than any of the possible numbers that Yolanda can call.
So Xavier can call 15 on his second turn no matter what Yolanda calls, and is thus always guaranteed to win.

Solution 2
If Xavier goes first and calls 4, then Yolanda will call a number of the form $4 + n$ where n is a whole number between 1 and 10.
On his second turn, Xavier can call 15 (and thus win) if the difference between 15 and $4+n$ is between 1 and 10. But $15(4+n) = 11n$ and since n is between 1 and 10, then $11n$ is also between 1 and 10, so Xavier can call 15.
Therefore, Xavier's winning strategy is to call 15 on his second turn.

(b) In (a), we saw that if Xavier calls 4, then he can guarantee that he can call 15.
Using the same argument, shifting all of the numbers up, to guarantee that he can call 50, he should call 39 on his previous turn.
(In this case, Yolanda can call any whole number from 40 to 49, and in any of these cases Xavier can call 50, since 50 is no more than 10 greater than any of these numbers.)
In a similar way, to guarantee that he can call 39, he should call 28 on his previous turn, which he can do for the same reasons as above.
To guarantee that he can call 28, he should call 17 on his previous turn.
To guarantee that he can call 17, he should call 6 on his previous turn, which could be his first turn.
Therefore, Xavier's winning strategy is to call 6 on his first turn, 17 on his second turn, 28 on his third turn, 39 on his fourth turn, and 50 on his last turn.
At each step, we are using the fact that Xavier can guarantee that his number on one turn is 11 greater than his number on his previous turn. This is because Yolanda adds 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 to his previous number, and he can then correspondingly add 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1 to her number, for a total of 11 in each case.

(c) In (b), we discovered that Xavier can always guarantee that the difference between his numbers on two successive turns is 11.
In fact, Yolanda can do the same thing, using exactly the same strategy as Xavier did.
If the target number is between 1 and 9, then Xavier will win on his very first turn by calling that number.
If the target number is then 11 greater than a number between 1 and 9, Xavier will win as in (b). Thus, Xavier wins for 12 through 20.
What about 10 and 11? In each of these cases, Yolanda can win by choosing 10 or 11 on her first turn, which she can do for any initial choice of Xavier's, since he chooses a number between 1 and 9.
Therefore, Yolanda will also win for 21 and 22, and so also for 32 and 33, and so on.
Since either Yolanda or Xavier can repeat their strategy as many times as they want, then Xavier can ensure that he wins if the target number is a multiple of 11 more than one of 1 through 9.

14. (a) *Solution 1*
First we ask the question: When does a player have a winning move? Since to win a player must remove the last coin, then a player has a winning move when he or she is choosing from a position with coins in only one pile (and a second empty pile).

So for Yolanda to win, she wants to ensure that she will always be passed an empty pile and a non-empty pile at some point. How can she force Xavier to pass her an empty pile? Xavier can only be forced to empty a pile if he receives two piles both of which have 1 coin (otherwise, he could reduce, but not empty, one of the piles). So if Yolanda chooses from piles with 1 and 3 coins or 1 and 2 coins, then she can pass back piles with 1 and 1 coins, and be sure to win.

Thus, Xavier does not want to initially remove 2 coins from one pile, otherwise Yolanda can follow her strategy above. Also, Xavier does not want to remove 3 coins from one pile, or Yolanda can immediately win by removing the other 3 coins.

So Xavier should start by removing 1 coin, and passing piles with 2 and 3 coins to Yolanda. She does not want to pass an empty pile or a pile with 1 coin in it to Xavier (or he can use her strategy from above), so she removes 1 coin from the larger pile, and passes back 2 and 2 coins. Xavier is then forced to empty one pile, or reduce one pile to 1 coin, and so Yolanda can then guarantee that she wins.

Therefore, in all cases, Yolanda can guarantee that she wins.

Solution 2
Yolanda will always win the game if she can guarantee that at some point when it is her turn to choose that she is selecting coins from just one pile. If she is selecting coins from just one pile, she will win the game by removing all of the coins from that pile.

She can guarantee that this will happen by duplicating Xavier's move only in the other pile. Thus, if Xavier takes 1, 2 or 3 coins, then Yolanda will take the same number of coins from the other pile. This strategy, on Yolanda's part, will mean that Xavier will always empty one pile first, and thus guarantee that Yolanda will win.

Solution 3
If Yolanda can ensure that she passes two equal piles to Xavier, then Xavier can never win, because he can never empty the last pile (he'll always have two non-empty piles).
So if Xavier reduces to 2 and 3 coins, Yolanda passes back 2 and 2 coins.
If Xavier reduces to 1 and 3 coins, Yolanda passes back 1 and 1 coins.
If Xavier reduces to 0 and 3 coins, Yolanda can immediately win by removing the last 3 coins.
From 1 and 1 coins, Xavier must reduce to 1 and 0 coins, and so Yolanda wins.
From 2 and 2 coins, Xavier must reduce to 1 and 2 coins (allowing Yolanda to pass back 1 and 1 coins) or to 0 and 2 coins, allowing Yolanda to win immediately.
Therefore, Yolanda can always win by following an equalizing strategy

(b) In part (a), we saw that Yolanda always won the game if she could guarantee that Xavier was choosing when there were two piles with an equal number of coins in each pile.
Starting with piles of 1, 2 and 3 coins, Yolanda can always win, because she can always after her first turn give two equal piles (and an empty third pile) back to Xavier. We see this by examining the possibilities:

Xavier's First Move	Yolanda's First Move
0,2,3	0,2,2
1,1,3	1,1,0
1,0,3	1,0,1
1,2,2	0,2,2
1,2,1	1,0,1
1,2,0	1,1,0

In any of these cases, Yolanda can be sure to win by following her equalizing strategy from part (a).
So Yolanda's strategy is to create two equal piles (and a third empty pile) after her first turn, and so force Xavier to lose, using her strategy from (a).

(c) In part (b), we saw that if Xavier chooses first from three piles with 1, 2 and 3 coins, then Yolanda can always win.
In part (a), we saw that if Xavier chooses first from two piles with an equal number of coins, then Yolanda can again always win.
So on his first move, Xavier does not want to create two equal piles (eg. 2,4,4 or 2,2,5 etc.), otherwise Yolanda would remove the third unequal pile and Xavier would then be choosing on his second turn from two equal piles.
Similarly, Xavier does not want to create a situation where Yolanda can reduce immediately to 1,2,3, otherwise Yolanda will win by following the strategy from (b).

So we consider the possible first moves for Xavier:

Xavier's First Move	Yolanda's First Move	Winner
2,4,4	0,4,4	Yolanda
2,4,3	2,1,3	Yolanda
2,4,2	2,0,2	Yolanda
2,4,1	2,3,1	Yolanda
2,4,0	2,2,0	Yolanda
2,3,5	2,3,1	Yolanda
2,2,5	2,2,0	Yolanda
2,1,5	2,1,3	Yolanda
2,0,5	2,0,2	Yolanda
1,4,5	??	??
0,4,5	0,4,4	Yolanda

So if Xavier makes any move other than to 1,4,5, Yolanda will win by following the correct strategy.

What if Xavier moves to 1,4,5? There are then 10 possible moves for Yolanda. As above, if Yolanda makes her first move to 0,4,5 or 1,1,5 or 1,0,5 or 1,4,4 or 1,4,1 or 1,4,0, then Xavier can reduce to two equal piles.
If Yolanda reduces to 1,3,5 or 1,2,5 or 1,4,3 or 1,4,2, then Xavier can reduce to some ordering of 1,2,3, and so Xavier can win.
Therefore, Xavier can win by reducing first to 1,4,5, and then to either two equal piles or some ordering of 1,2,3, and then following Yolanda's strategy from (a) or (b).

15. (a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.
Since Bob can place a 3 in any the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.

(b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

Bob's First Turn	Total So Far	Avril's Second Turn
1	6	9
2	7	8
3	8	7
4	9	6
5	10	5
6	11	4
7	12	3
8	13	2
9	14	1

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn

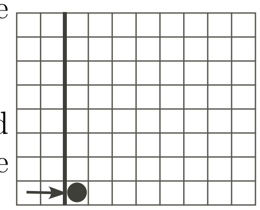
(c) Bob can place any of the numbers 1, 2, 4, 5, 7, 8 in any of the six empty circles. We can pair these numbers up so that the sum of the two numbers in the pair plus 6 is equal to 15:

1 and 8; 2 and 7; 4 and 5

So when Bob uses one of these numbers, Avril can use the other number from the pair, place it directly opposite the one that Bob entered, and the total of the three numbers on this line through the centre will be 15, so Avril will win the game

16. Here we can make use of symmetry to find a winning strategy by 'equalizing' the piles, or making the piles have the same number of matchsticks.
This strategy leads to a win for the player who makes the first move.
We will call the player who goes first player A.
Player A begins by taking 4 matchsticks, leaving 13 in each pile.
This strategy wins since, if player A equalizes the piles, then player B must make them unequal. Suppose B takes 3 matchsticks leaving piles of 10 and 13. Then A equalizes the piles again leaving 10 matchsticks in each pile. No matter how many matchsticks player B takes from one of the piles, player B will remove enough matchsticks from the other pile so that the two piles once again have the same number of matchsticks.
Since a player can only remove matchsticks from a single pile, eventually player B will be forced to take all the matchsticks from one of the piles and then player A will win on the next move by taking all the sticks in the second pile.
Thus, the winning strategy is for player A to return a symmetrical position to her opponent on every move.

17. Again, the use of symmetry solves this game.
Whatever move Al makes, Bert simply mirrors the move to return the piece to the main diagonal.
For example, if Al moves 3 squares across, Bert moves 3 squares up.
Since Al must always move off the diagonal, Bert can always respond by moving back to the diagonal. Since the winning square is on the diagonal, Bert wins.



In the second version, we have an 8 by 10 board.
In this case, Al is able to turn the tables on Bert by simply moving 2 squares to the right to start since this converts the game to an 8 by 8 board with Bert going first.

18. The winning strategy in this game is, on your turn, to make the gap between A and B the same number of squares (possibly 0) as the gap between C and D. Therefore, in the position shown, you should move A two squares to the right, or move D two squares to the right. Your opponent must now move one of the counters so that the two gaps will be different and on your subsequent turns it will always be possible to make the two gaps the same. As the game continues, D will, sooner or later, be moved to the extreme right square and, on a subsequent move, C will be moved to the last square it can occupy i.e. the second square from the right. If your opponent moves C to this position, then you will move A to the square immediately to the left of B so that both gaps are now zero. Alternatively, you may move C to its final position yourself if D occupies the last square and your opponent places A on the square adjacent to B. In either case, your opponent is faced with a situation in which B must be moved at least one square. On your turn, you move A the same number of squares to once again reduce the gap to zero. Eventually, your opponent must move B to the square adjacent to C and you then win the game by moving A to its final position.