From Conjecturing to Justification and Proof Using Geometric Explorations

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Outline

1. Why we think reasoning and proof is such an important part of instruction at all levels and should be a consistent part of classroom experiences
2. Research findings about proof and reasoning
3. How the structure of the tasks helps students learn about proof and reasoning
4. How we evaluate our students work during the activities
5. Five geometry explorations

Importance of Reasoning and Proof

1. "First, the notion of *mathematical understanding* is meaningless without a serious emphasis on reasoning" (Ball & Bass, 2003, p. 28)
2. Proof is an essential component of the work of doing mathematics.
3. Common Core State Standards emphasize reasoning and proof.

Findings from Research on Proof and Reasoning

- Many students accept examples as verification
- Many students do <u>NOT</u> accept deductive proofs as verification
- Many students do <u>NOT</u> accept counterexamples as refutation
- Many students accept arguments on bases other than logical coherence

(Chazan, 1993; Knuth, Chopin, & Bieda, 2009; Martin & Harel, 1989; Reid & Knipping, 2010)

Structure of Tasks

- We ask for conjectures instead of giving them
- The scenarios have many possible conjectures – they are not strategically designed so only one conjecture is the right conjecture
- Students are expected to explain their findings – we don't stop once students notice the conjecture and assume it is a theorem
- The conjectures are not perceptually obvious as to whether they are true or false
- Students with different levels of reasoning can still participate

Purpose of Structure

- By not asking for conjectures students learn that it is not an expectation for them and do not consider conjecturing as a mathematical activity (Herbst & Brach, 2010)
- One way of helping students develop an understanding about the limitations of empirical arguments is to give them tasks in which generalizing from several cases does not lead to a correct generalization (Knuth, Chopin, & Bieda, 2009)
- The open-nature of the tasks meets students where they are and by seeing other arguments and experiencing the limitations of their naïve arguments they build more formal arguments.

Framework for Analyzing Student Work

Categories	Characteristics of Categories
Level 0	**No justification given**. When asked to justify, students either said they "just knew" or restated their answers
Level 1:	**Appeal to external authority or rote procedures**. While students at all levels like a teacher's validation, students at this level relied exclusively on external authority and memorized (mis-memorized) procedures. They did not understand, nor were they interested in, why the idea was true
Level 2:	**Naïve reasoning, usually with incorrect conclusions**. Although the students use some deduction, the arguments started with an analogy or with something the students remembered hearing, often incorrectly. As a result, students came to a mostly incorrect conclusion (not just a computational error). If students did reach a correct conclusion, it was for the wrong reasons.
Level 3-A:	**Inductive reasoning A (examples, experiments, or empirical demonstrations).** Students concluded that an assertion was valid on the basis of a pattern or on a small number of cases. While students at all levels used examples to understand, students at this level thought that the examples were a proof.
Level 3-B:	**Inductive reasoning B (investigating if and why a generalization held).** While still focusing on examples, the students began to generalize by looking for counterexamples, cases of examples, or extreme cases. They showed some doubt that a pattern would necessarily hold indefinitely.
Level 4:	**Transition to formal reasoning (elements of formal reasoning but without the precision)**. Students used an informal version of a Level 5 argument.
Level 5:	**Formal reasoning (acceptable to a mathematician).** Students' arguments were precise and acceptable to a mathematician. Examples include theoretical probability and acceptable proof techniques-- such as proof by cases or proof by mathematical induction.

Adapted from Quinn (2009)

1. Area of Quadrilaterals

- In this activity we will work with quadrilaterals.
- Make a quadrilateral and then draw one of the diagonals.
  - What conjectures can you create?
  - What relationships do you see?
  - Are there any properties or special features that stand out?

1. Area of Quadrilaterals

- We will explore the conjecture;

"In a quadrilateral at least one diagonal cuts the area in half"

- Is the conjecture true or false? If you think it is true explain why it is true. If you think it is false can you create an example that shows that it is false? How do you know your example shows it is false?

1. Area of Quadrilaterals

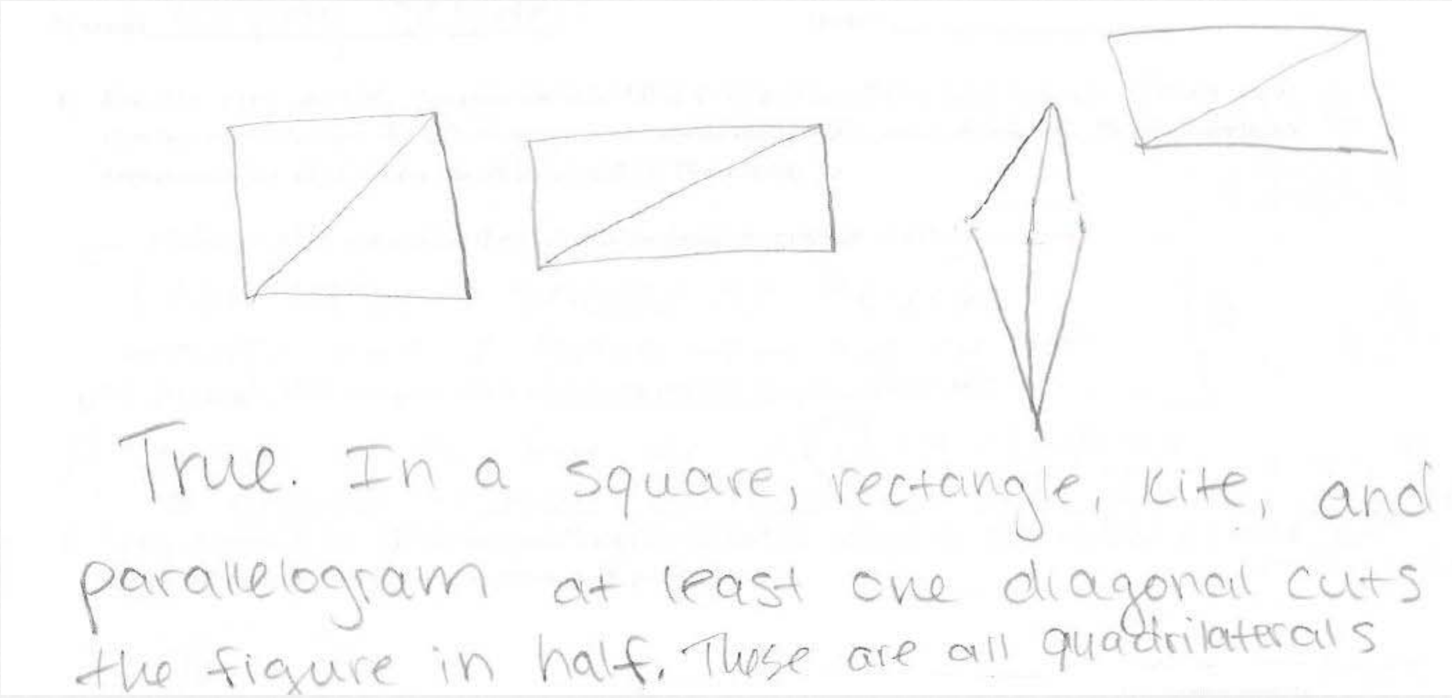
1. Consider a kite. Does this conjecture hold true for a kite ?
2. What if the quadrilateral is a rhombus?
3. What if the quadrilateral is a square?
4. Rectangle? Parallelogram?

Work on the exploration and write down points to take away.

Let us Share Solutions

Solutions We Have Seen

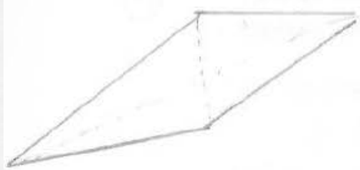
Solution 1.1



Solutions We Have Seen

Solution 1.2

False, for an irregular quadrilateral such as



These diagonals do not perfectly divide it in half.
you can see if you folded it over both diagonals,
there would be no line of symmetry either way,
meaning that at least one diagonal of
a quadrilateral does not have to cut the
quadrilateral in half, i can happen, but it is
not a rule.

Let Us Use the Framework to Examine some Solutions

- Solution 1.1: Level 3-A because they are reasoning based on examples.
- Solution 1.2: Level 3-B because they are reasoning based on a specific quadrilateral and their justification is true only for that quadrilateral. For example, a square has a diagonal that is also not a line of symmetry but does divide the area of the quadrilateral equally.

How to help students transition to higher levels

- The student who gave solution 1.1 will see that examples do not suffice as there could be an example that was not considered but should have been
- The student who gave solution 1.2 could be transitioned to formal reasoning by being asked to consider their argument in the case of a rectangle.

Area of Quadrilaterals

- Connections to CCSSM?
- CCSS.Math.Content.6.G.A.1: Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Powerful Pedagogical Aspects of the Activity

- Although examples are not sufficient to prove the truth of a statement, they do play an important role in making conjectures.
- Presenting students with a variety of tasks in which examples play different roles can help students develop an appreciation and understanding for their use as means of justification.

2. Reflecting a Triangle across its Sides

In this activity we will work with triangles. We will explore what shape is formed when a triangle is reflected across one of its sides.

What shapes are produced when we do this for different triangles?

2. Reflecting a Triangle across its Sides

1. Consider an obtuse or acute scalene triangle. Reflect it across its sides. What shape is formed?
2. What if the triangle is an isosceles?
3. What if the triangle is an equilateral?

Work on the exploration and write down points to take away.

2. Reflecting a Triangle across its Sides

We will explore this conjecture:

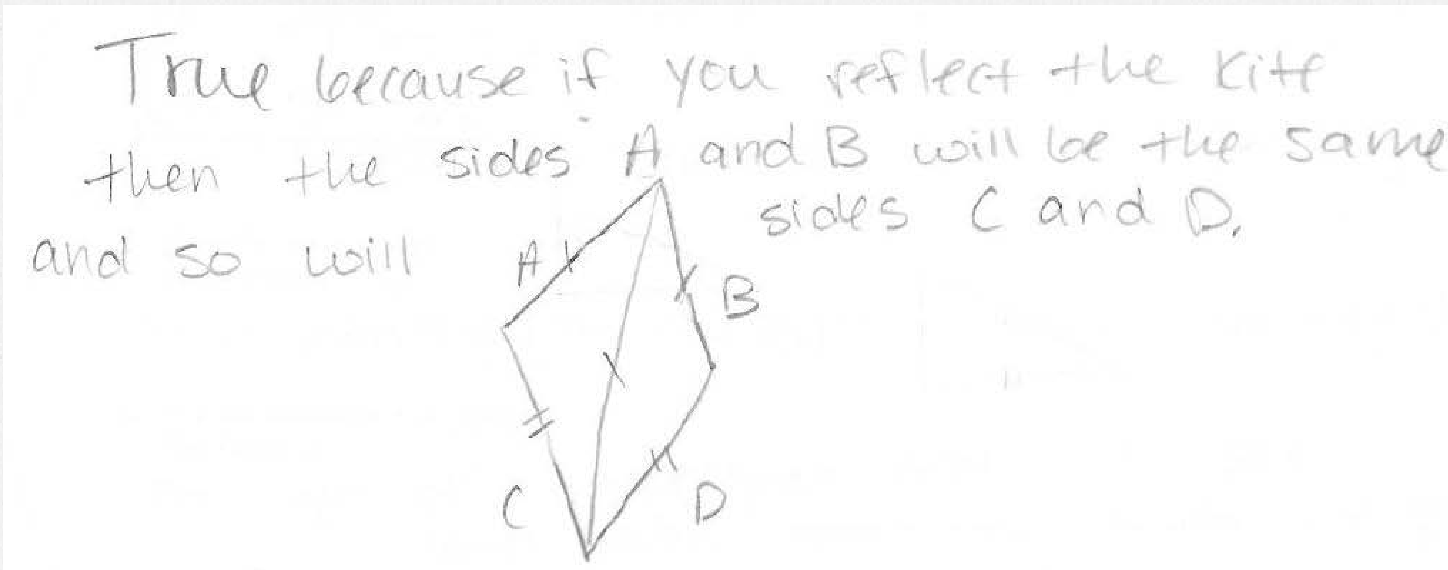
> *Consider a scalene triangle. Reflect it across one of its sides. What shape is formed?*

Work on the exploration and write down points to take away.

Let us Share Solutions

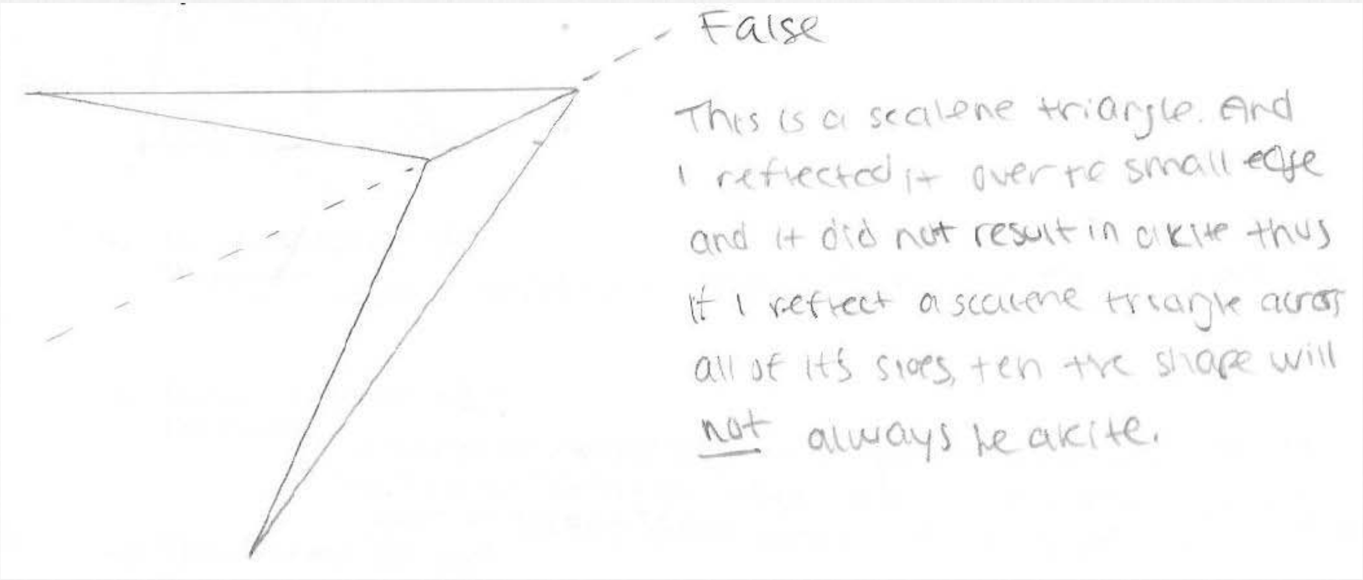
Solutions We Have Seen

Solution 2.1



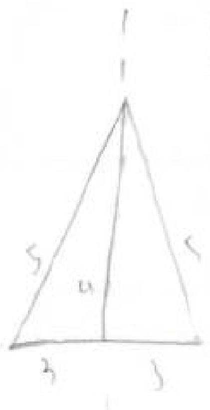
Solutions We Have Seen

Solution 2.2



Solutions We Have Seen

Solution 2.3



folse, right angles can forme a isuscles triangle

Let Us Use the Framework to Examine some Solutions

- Solution 2.1: Level 3B – The student has used the example to create a general argument. However they have not considered the extent to which that example could vary.
- Solution 2.2: Level 3B – The student has reasoned with an argument but their conception of kite is limited.
- Solution 2.3: Level 5 – correct use of counterexample to refute an argument.

How to help students transition to higher levels

- The students who gave solutions 2.1 and 2.2 will gain a similar benefit and that is a better understanding of the terms that were misused.
  - The student who gave solution 2.1 did not consider other types of triangles
  - The student who gave solution 2.2 does not consider the concave quadrilateral as a kite.
- Both students will experience the role that definitions and axioms play in reasoned arguments.

2. Reflecting a Triangle across its Sides

- Connections to CCSSM?
- CCSS.Math.Content.8.G.A.1 : Verify experimentally the properties of rotations, reflections, and translations.
- CCSS.Math.Content.8.G.A.3: Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

Powerful Pedagogical Aspects of the Activity

- Provides evidence not only about students ability to reason but also about their understanding of key geometric terms and concepts.
- Provides motivation for students to formalize the statements they use in arguments and attend to the full meaning of their statements in their arguments.

3. Connecting Consecutive Midpoints of the Sides of Quadrilaterals

In this activity we will work with quadrilaterals. We will make a shape by joining consecutive midpoints of the sides. We will explore what shape is formed.

What shapes are produced when we do this for different quadrilaterals?

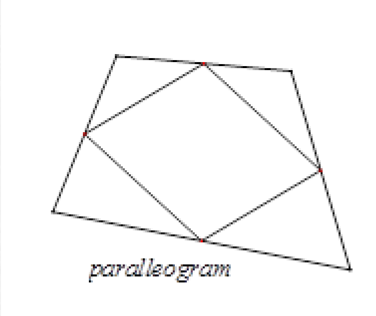
3. Connecting Consecutive Midpoints of the Sides of Quadrilaterals

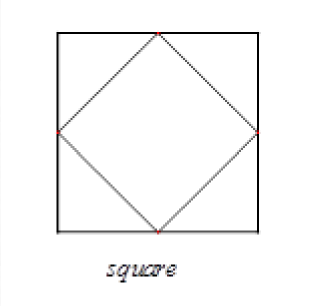
1. Consider a quadrilateral. Make a shape by joining consecutive midpoints of its sides. What shape is formed?

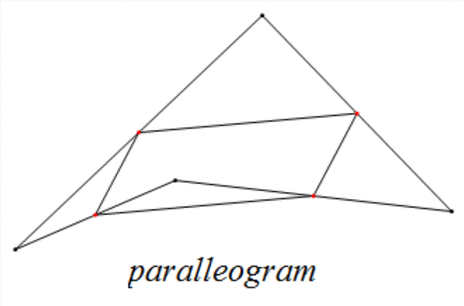
Do you believe the same type of quadrilateral will be formed if other quadrilaterals are considered?

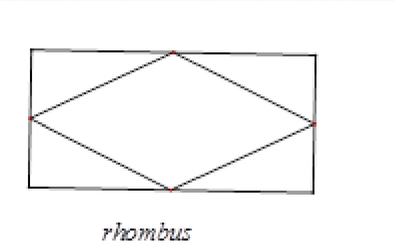
Work on the exploration and write down points to take away.

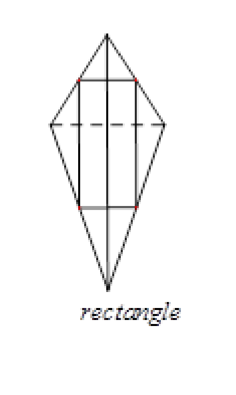
Possible Conjectures

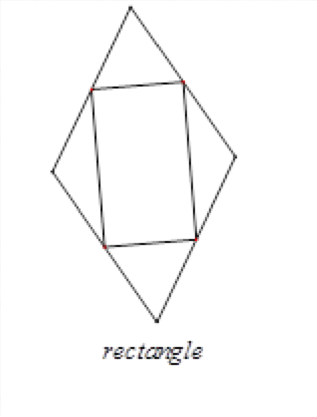












3. Connecting Consecutive Midpoints of the Sides of Quadrilaterals

- We will explore the conjecture;

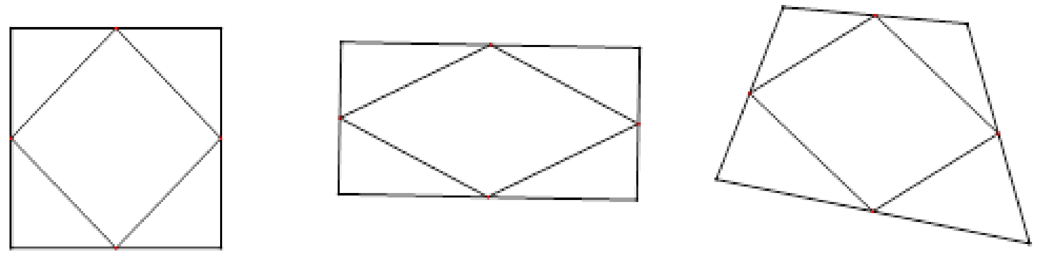
  "If you join consecutive midpoints of the sides of any quadrilateral, a parallelogram will be formed"

- Is the conjecture true or false?
- If you think it is true, justify why it is true.
- If you think it is false, can you create an example that shows that it is false?
- How do you know your students might justify this conjecture?

Let us Share Solutions

Some Solutions We Have Seen

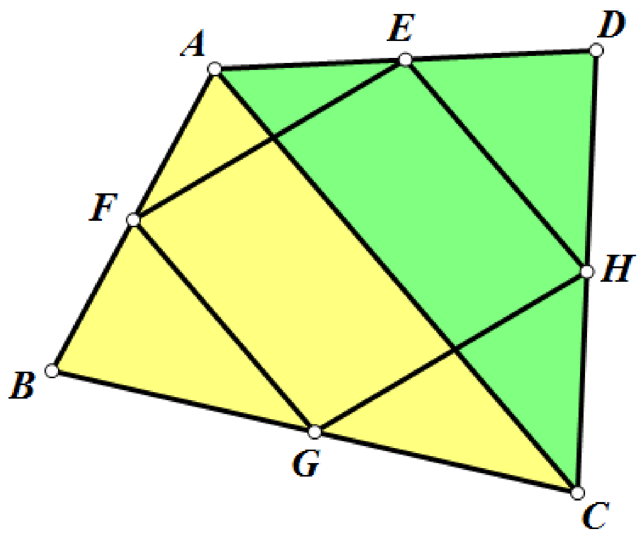
Solution 1.1



True! In a square, rectangle, and a generic quadrilateral, the consecutive mid points of sides formed a parallelogram.

Some Solutions We Have Seen

Solution 1.2



I drew one of the diagonals of the quadrilateral.Than I remembered that if the midpoints of the two sides of a triangle connected, this line segment will be parallel to the third side of the triangle. So in the ADC triangle the line segments EH and AC are parallel to each other. By the same way in the ABC triangle the line segments GF and AC are parallel to each other and so does EH and GF. When I did the same things for ABD andADC triangles that are obtained by constructing the second diagonal of the quadrilateral, I have found that the line segments EF and GH are also parallel to each other. Therefore, the EFGH is a parallelogram.

Let Us Use the Framework to Examine some Solutions

- Solution 1.1: Level 3-A because they are reasoning based on examples.
- Solution 1.2: Level 5 because they are reasoning based on logical reasoning and their justification is true not only for that quadrilateral but also for all quadrilaterals.

How to help students transition to higher levels

- The student who gave solution 1.1 will see that examples do not suffice as there could be an example that was not considered but should have been

3. Connecting Consecutive Midpoints of the Sides of Quadrilaterals

Connections to CCSSM?

- CCSS.Math.Content.HSG-SRT.B.4 Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
- CCSS.Math.Content.HSG-SRT.B.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
- CCSS.Math.Content.HSG-CO.B.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

3. Connecting Consecutive Midpoints of the Sides of Quadrilaterals

- Connections to CCSSM?
  - CCSS.Math.Content.7.G.A.2 Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions.
  - CCSS.Math.Content.8.G.A. Given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

3. Summary

4. Rotating Triangles

In this activity we will rotate shapes and deduce properties of the new shapes formed using knowledge that the images are congruent to the original triangles.

4. Rotating Triangles

Consider a right triangle. Rotate it 180 degrees through the midpoint of it's hypotenuse.

What shape will be formed?
Do you believe the same type of shape will be formed if other triangles are rotated?

Let us Share Solutions

Some Solutions We Have Seen

Let Us Use the Framework to Examine some Solutions

Rotating Triangles

- Connections to CCSSM? High School
- CCSS.Math.Content.HSG-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software.
- CCSS.Math.Content.HSG-CO.B.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure;

Rotating Triangles

- Connections to CCSSM?
- CCSS.Math.Content.8.G.A.1 Verify experimentally the properties of rotations, reflections, and translations:
  - CCSS.Math.Content.8.G.A.1a Lines are taken to lines, and line segments to line segments of the same length.
  - CCSS.Math.Content.8.G.A.1b Angles are taken to angles of the same measure.
  - CCSS.Math.Content.8.G.A.1c Parallel lines are taken to parallel lines.

5. Triangle Angle Sum

- In this activity we will work with triangles
- Make a triangle and then draw a line through one vertex that is parallel to the opposite side
  - What conjectures can you create?
  - What relationships do you see?
  - Are there any properties or special features that stand out?

5. Triangle Angle Sum

- We will explore the conjecture;

"The sum of the interior angles of a triangle is 180 degrees"

- Is the conjecture true or false?
- If you think it is true explain why it is true.
- If you think it is false can you create an example that shows that it is false?
- How do you know your example shows it is false?

Let us Share Solutions

Some Solutions We Have Seen

Let Us Use the Framework to Examine some Solutions

Questions and Discussion

Points to take away about:

- Reasoning and Proof
- Connections to CCSSM
- Standards for Mathematical Practices

References

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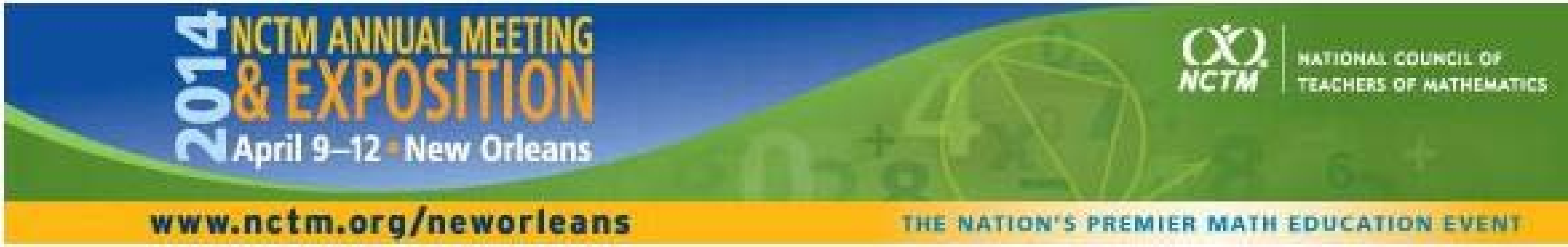

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