Sequences and Series before Formulas

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Students must distinguish between:

Sequence:	3,	5,	7,	9,	11,	...	(Terms)
	1	2	3	4	5	...	(Term numbers)

Series:	3 +	5 +	7 +	9 +	11 +	...	(Terms)
	1	2	3	4	5	...	(Term numbers)

Some definitions:

- Sequence: A function with domain {positive integers}
- Series: The indicated sum of the terms of a sequence

Background: In algebra, students learn about *functions*.

$y = 3x + 5$ — Linear function

$y = -5x^2 + 7x + 11$ — Quadratic function

$y = 6 \times 2^x$ — Exponential function

Objective for functions: Given a function:

- Find y if x is given.
- Find x if y is given.

Important, because x and y could represent related quantities in the real (or mathematical) world.

e.g., an arrow y meters above the ground after x seconds

Objective for sequences: Given a sequence,

- Find a term if the term number is given.
- Find the term number if the term is given.

e.g. t_n:	1,	1,	2,	3,	5,	8,	13,	...
n:	1	2	3	4	5	6	7	...

Pattern (iteratively): Add the two preceding terms ... .

Important b/c n and t_n could represent related quantities in the real (or math.) world where the independent variable changes *discretely,* rather than continuously.

e.g. t_n pairs of rabbits in a population after n generations

Special Kinds of Sequence:

Arithmetic sequence (defined *iteratively*):

t_n:	3,	7,	11,	15,	19,	23,	27,	...
n:	1	2	3	4	5	6	7	...
Pattern:	+4	+4	+4	+4	+4	+4		...

Iteratively: *Add* a constant to get the next term.
How many 4s do you add to the first term, 3, to get t_7?
Answer: *Six* of them. [Look at the pattern.]

$t_7 = 3 + 6(4) = \underline{\underline{27}}$, which agrees!

- For t_{100}, add 99 4s to 3.

$t_{100} = 3 + 99(4) = \underline{\underline{399}}$ [A procedure. No formula yet.]

• For this arithmetic sequence, find n if $t_n = 13071$

Now, write and use an *explicit formula.*

Words: "3 + enough 4s."

$t_n = 3 + (n-1)(4)$ ["Enough" 4s is 1 less than n of them.]

$\therefore\ 13071 = 3 + (n-1)(4)$ [Substitute 13071 for t_n.]

[Solve by "peeling away" the numbers around the n.]

$13068 = (n-1)(4)$ [Peel the most exposed no. *first*.]

$3267 = (n-1)$ [Peel the next-most-exposed no.]

$\underline{3268} = n$ [Peel away the last no., leaving n by itself.]

<u>Geometric</u> sequence: (defined *iteratively*):

t_n:	3,	6,	12,	24,	48,	96,	192,	...
n:	1	2	3	4	5	6	7	...
Pattern:	×2	×2	×2	×2	×2	×2		...

Iteratively: *Multiply* by a constant to get the next term.

How many 2s do you multiply the first term by to get t_7?

Answer: *Six* of them. [Look at the pattern!]

$t_7 = 3 \times 2^6 = \underline{192}$, which agrees!

• For t_{20}, multiply 3 by 19 2s.

$t_{20} = 3 \times 2^{19} = \underline{1572864}$ [Procedure. No formula yet.]

• For this geometric sequence, find n if $t_n = 402653184$.

Now write and use an *explicit formula.*

Words: "3 × enough 2s." ["Enough" is 1 less than n.]

$t_n = 3 \times 2^{n-1}$

$402653184 = 3 \times 2^{n-1}$ [Substitute for t_n.]

$134217728 = 2^{n-1}$ [Peel away the most exposed no.]

$\log 134217728 = \log (2^{n-1})$ [Log both sides. Easy!]

$\log 134217728 = (n-1) \log 2$ [log of a power]

$\dfrac{\log 134217728}{\log 2} = (n-1)$ [Peel next-most-exposed no.]

$27 = (n-1) \Rightarrow \underline{28} = n$ [by calculator, then peel the 1]

Or: Use SOLVER to get $n = 28$ ["Indiana Jones" method]

<u>Series</u>

This is an **arithmetic series**: [Why?]

	3 +	7 +	11 +	15 +	19 +	23 +	27 +	...
n:	1	2	3	4	5	6	7	...

How is this *sequence* formed from the arithmetic series?

S_n:	3,	10,	21,	36,	55,	78,	105,	...
n:	1	2	3	4	5	6	7	...

The *terms* of this sequence are *partial sums* of the series.

e.g., $S_6 = 3 + 7 + 11 + 15 + 19 + 23 = \underline{78}$, which agrees.

• How about finding S_{100}? This is a *big* problem!

Notice a pattern:

$S_6 = 3 + 7 + 11 + 15 + 19 + 23$

$S_6 = (3 + 23) + (7 + 19) + (11 + 15)$ [Com. & and assoc.]

$S_6 = 26 + 26 + 26 = 3(26) = \underline{78}$, which agrees!

There are *three* pairs, each equal to the sum of the first and the last term of the partial sum. [not "series."]

From earlier, $t_{100} = 399$

So S_{100} will have 50 pairs, each equal to 3 + 399, or 402.

$S_{100} = 50(402) = \underline{20100}$

If in doubt, calculate all the terms and "Adam Upp!"

• Which partial sums for this series equals 149331?

$3 + 7 + 11 + 15 + 19 + 23 + 27 + \ldots$

n: 1 2 3 4 5 6 7 ...

$t_1 = 3,\ t_n = 3 + (n-1)(4)$ [t_1 + "enough" differences]

So each pair $= 3 + \big(3 + (n-1)(4)\big)$ or $\big(6 + (n-1)(4)\big)$

And there are $n/2$ pairs.

So $S_n = \dfrac{n}{2}\big(6 + (n-1)(4)\big)$

Check: $S_6 = \dfrac{6}{2}\big(6 + (6-1)(4)\big) = 3(26) = \underline{78}$, Agrees!

Substitute 149331 for S_n.

$149331 = \dfrac{n}{2}\big(6 + (n-1)(4)\big)$

$298662 = n\big(6 + (n-1)(4)\big)$ [Peel the 2, most exposed.]

$298662 = n(4n + 2)$ [Distribute, combine like terms.]

$0 = 4n^2 + 2n - 298662$ [Dist, make one side equal 0.]

$n = \underline{273}$ or -273.5 (out of domain) [Quadratic formula]

Check using the SERIES program.

$S_{273} = \underline{149331}$ [Reeee-markable!]

Geometric series

e.g. $100 + 90 + 81 + 72.9 + 65.61 + 59.049 + \ldots$

n: 1 2 3 4 5 6 ...

Pattern: ×0.9 ×0.9 ×0.9 ×0.9 ×0.9 ...

Numerically, the sixth partial sum is given by
$S_6 = 100 + 90 + 81 + 72.9 + 65.61 + 59.049$ [no "..."]

$S_6 = \underline{468.559}$

An explicit formula for partial sums of geometric series is trickier to find than for arithmetic series.

Here's an informative example. Use r for the ratio 0.9.

$$S_6 = 100 + 100r + 100 \cdot r^2 + 100 \cdot r^3 + 100 \cdot r^4 + 100 \cdot r^5$$

$$r \cdot S_6 = 100r + 100 \cdot r^2 + 100 \cdot r^3 + 100 \cdot r^4 + 100 \cdot r^5 + 100 \cdot r^6$$

Subtract, top equation – bottom equation.
The middle terms "telescope!"

$$S_6 - r \cdot S_6 = 100 - 100 \cdot r^6$$

$$S_6(1-r) = 100(1-r^6)$$

$$S_6 = 100 \cdot \frac{1-r^6}{1-r}$$ (first term) · (fraction involving r and 6)

Check: $S_6 = 100 \cdot \frac{1-0.9^6}{1-0.9} = 100 \cdot \frac{0.468559}{0.1} = \underline{\underline{468.559}}$

Medication Problem

Take 100 mg each 6 hr. 90% remains 6 hr later.

n doses	mg remaining
1	100
2	100(0.9) + 100
3	$100(0.9^2)$ + 100(0.9) + 100

After n doses, the number of milligrams remaining is the n^{th} partial sum of a geometric series!

- How much is left after 6 doses?

See calculations above. $S_6 = 468.559 \approx$ <u>469 mg</u>

You've taken 600 mg, of which 469 mg remain.

- Find number of mg remaining for large values of n.

$$S_n = 100 \cdot \frac{1-0.9^n}{1-0.9}$$ [Remember: first term *times* fraction]

n	S_n
10	651.3215...
50	994.8642...
100	999.9734...
200	999.999999294...

The values are *converging* to 1000!

$\lim_{n\to\infty} 0.9^n = 0$, so $\lim_{n\to\infty} S_n = 100 \cdot \frac{1}{1-0.9} =$ <u>1000 mg max.</u>

- Find smallest n so that $S_n > 800$ mg.

$100 \cdot \frac{1-0.9^n}{1-0.9} > 800$ [Substitute for S_n.]

$1 - 0.9^n > 0.8$ [Divide by 100, multiply by 0.1.]

$0.9^n < 0.2$ [Why "<" not ">"?]

$\log(0.9^n) < \log 0.2$ [log both sides. Easy!]

$n > \frac{\log(0.2)}{\log(0.9)}$ [Why back to ">"?]

$n > 15.2755... \Rightarrow n = \underline{16}$ [Why rounded *up*?]

Check using SERIES program.

In calculus, students learn about *power (Taylor) series.*

e.g. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \ldots$

n: 0 1 2 3 4 5 ...

Term *index,* vs. term number, starting at t_0 rather than t_1.

A "polynomial" with an infinite number of terms.

A formula for $t_n(x)$ is $\frac{(-1)^n}{(2n)!}x^{2n}$

• e.g., $\cos 0.6 \approx S_3(0.6) = 1 - \frac{1}{2!}0.6^2 + \frac{1}{4!}0.6^4 - \frac{1}{6!}0.6^6$

$= \underline{0.8253352}$, which is close to

$\cos 0.6 = 0.825335614\ldots$ [Reeee-markable!]

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Conclusions:

- "Sequence" vs. "Series"
- Remember patterns. Derive formulas if needed
- Find t_n if n is given, and vice versa
- Just examples of the function concept
- Math. models if indep. variable changes *discretely*
- Solving for t_n gives review of algebraic techniques.
- Program SERIES for partial sums numerically.
- The SOLVER for numerical solution of equation
- Partial sums of geometric series can *converge*.
- Series in algebra helps Taylor series in calculus.

Final Exam: Write the *one* most important thing you learned as a result of attending this session.

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Appendix: SERIES program for TI-84

Enter the formula for t_x in y_1, then run the program.

```
:Disp "HOW MANY TERMS?"
:Prompt N
:0→S
:For(X,1,N)
:S+Y1(X)→S
:Disp S
:End
:Disp "NO. OF TERMS ="
:Disp N
:Disp "PARTIAL SUM ="
:Disp S
```

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Name: ____________________ Group Members: ______________ ______________ ______________

Foerster Algebra II | Exploration 11-5a: Introduction to Series | Date: ____________

Objective: If a formula for t_n is known, find a partial sum of a series by adding up the terms.

1. Write out the indicated terms of this series:

$$\sum_{k=1}^{6} 3k + 5$$

2. Evaluate the partial sum in Problem 1.

3. Enter a program into your grapher to calculate the partial sum of a series. The formula for t_k should be entered as y_1. When the program runs, the grapher should ask you to input the number of terms. Then it should calculate and display each partial sum up to the one you asked for. Test your program by using it to find the partial sum in Problem 1. You may assume that the program is working correctly if it gives the answer you got in Problem 2.

4. Use your program from Problem 3 to evaluate S_{100} for the series in Problem 1. That is, find

$$\sum_{k=1}^{100} 3k + 5$$

5. Figure out a formula for t_k for the series below. Then use your program in Problem 3 to find the 5th partial sum, S_5, for the series. Confirm that the program gives you the correct answer by actually adding the terms shown here.

$2 + 5 + 10 + 17 + 26 + \ldots$

6. Use your program to calculate the 50th partial sum for the series in Problem 5.

7. Write out the first four terms of the geometric series with first term 1000 and common ratio 1.06. Calculate the fourth partial sum by adding up these terms.

8. Write a formula for the nth term of the geometric series in Problem 7. Confirm that your formula is correct by using the program of Problem 3 to find the fourth partial sum you calculated manually in Problem 7.

9. Use your program to find the 30th partial sum of the geometric series in Problem 7. (This number is the amount of money you would have at the end of 30 years if you invested $1000 a year in a savings account that pays 6% per year interest, APR.)

10. Calculate the following partial sums of the geometric series with first term 800 and common ratio 0.9.

S_{10} ______________________

S_{20} ______________________

S_{50} ______________________

S_{100} ______________________

S_{200} ______________________

11. The partial sums in Problem 10 are said to **converge to 8000**. What do you suppose this means?

12. What did you learn as a result of doing this Exploration that you did not know before? (Over.)