From Concrete to Abstract: Developing Students' Understanding of Functions through Geometric Growing Patterns

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Welcome!

- Introduction
- Supplies
- Questions

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Some preliminary exploration….

Task: Identify a function rule that relates the input values to the output values.

Input	Output
1	5
2	7
3	9
4	11
5	13
6	15

Did you identify the following rule?

Output = Input · 2 + 3

Reflect: What did you already need to know and be able to do in order to identify that rule?

Some preliminary exploration….

Task: Identify a function rule that relates the input values to the output values.

Input	Output
1	5
2	10
3	17
4	26
5	37
6	50

Did you identify the following rule?

$\text{Output} = \text{Input}^2 + 2 \cdot \text{Input} + 2$

Reflect: What did you already need to know and be able to do in order to identify that rule?

Familiar Strategies

- Guess and Check
- Procedures:
  - Identifying the constant rate of change / constant difference
  - Substituting these values, and the y-intercept, into known equations, such as $y = mx + b$
- A combination of guess and check and known procedures?

Pitfalls for Students

- Potential for meaningless procedures
- Therefore, procedures easily forgotten or confused
- Limited access for students – especially with guess and check

Long Term Implications

- Mathematics as meaningless procedures
- Missing the opportunity to use mathematics to model situations
- Lack of access to algebra for ALL

For now, let's put those strategies aside….

How can students, prior to a formal course in algebra, develop functional thinking?

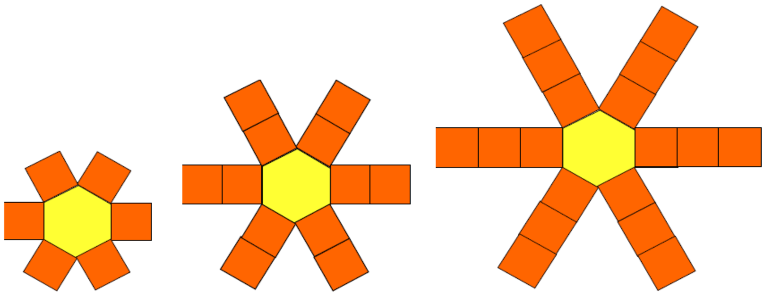
How can we, as educators, construct tasks and experiences to best support this development of functional thinking?

My Research

- *Geometric growing patterns*
  - Prior to a formal course in algebra
  - As early as third grade
- Why these patterns?
  - Engaging
  - Concrete, hands on
  - Ideal for bridging patterns with functions
- Emphasis on *figural reasoning*

What is a geometric growing pattern?

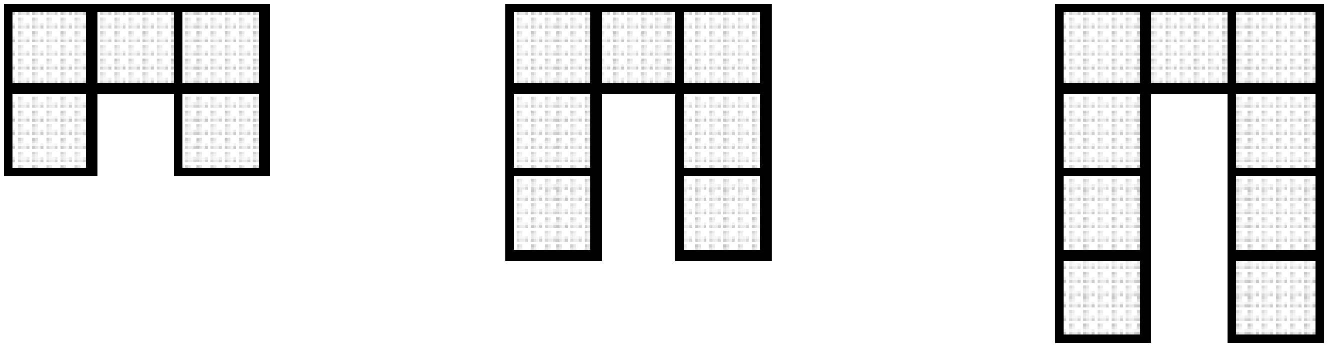
- Comprised of shapes & grows in a predictable manner
- For example:



What is figural reasoning?

Figural reasoning “relies on relationships that could be drawn visually from a given set of particular instances” (Rivera & Becker, 2005, p. 199).

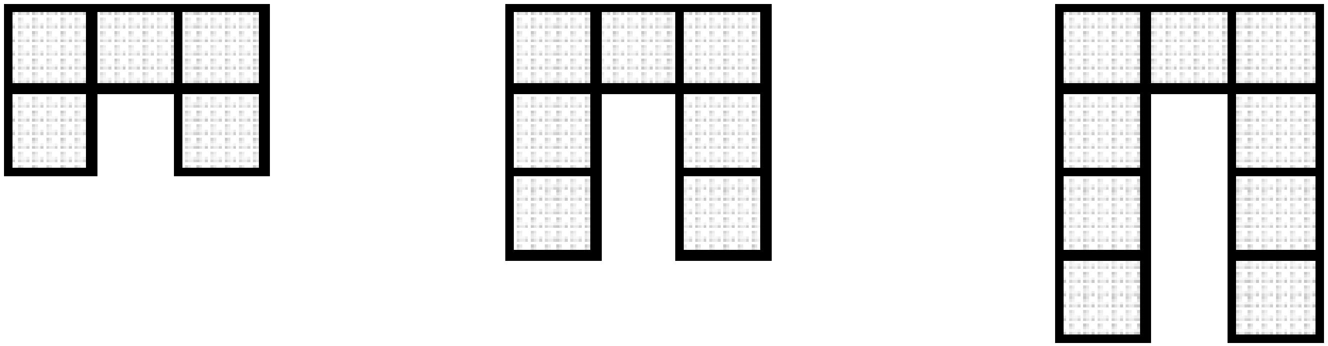
The Tunnel



Consider the pattern above.

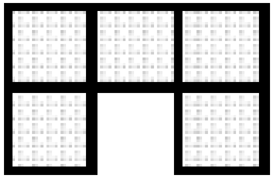
- What do you notice?
- Write down 2-3 observations you have.

The Tunnel

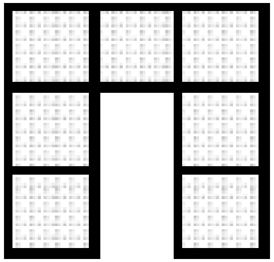


What observations do you make?

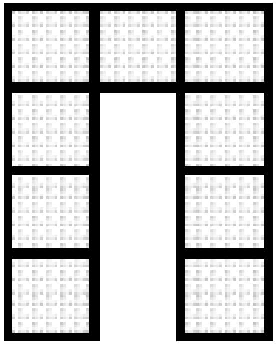
The Tunnel



Tunnel 1



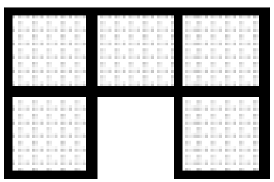
Tunnel 2



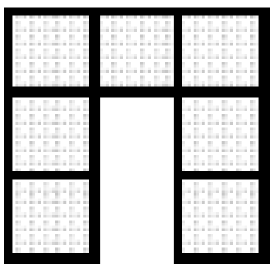
Tunnel 3

The importance of labeling….

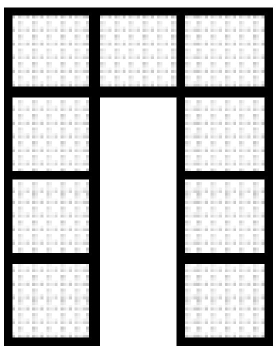
The Tunnel



Tunnel 1



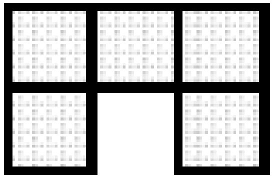
Tunnel 2



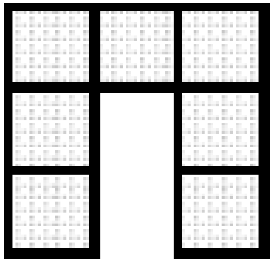
Tunnel 3

How many of you started counting square tiles?

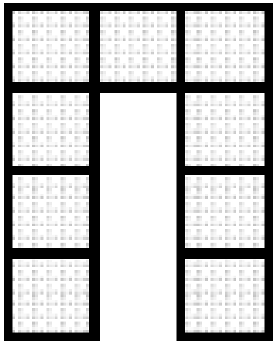
The Tunnel



Tunnel 1



Tunnel 2



Tunnel 3

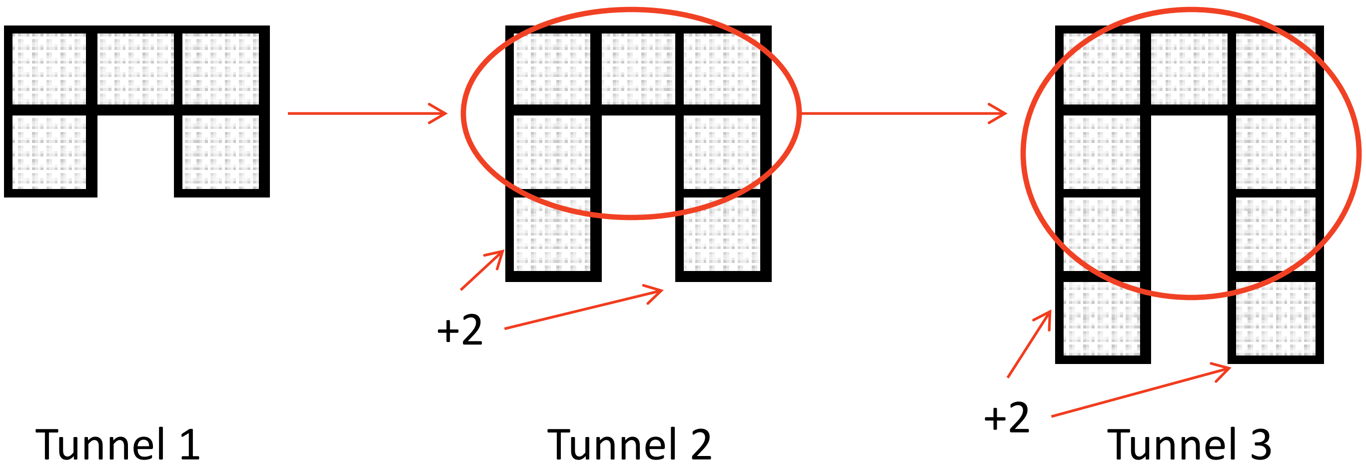
Quick Task:

- Build (or draw) Tunnel 4.
- Build (or draw) Tunnel 10.
- Explain how you would build or draw Tunnel 37.

Figural Reasoning

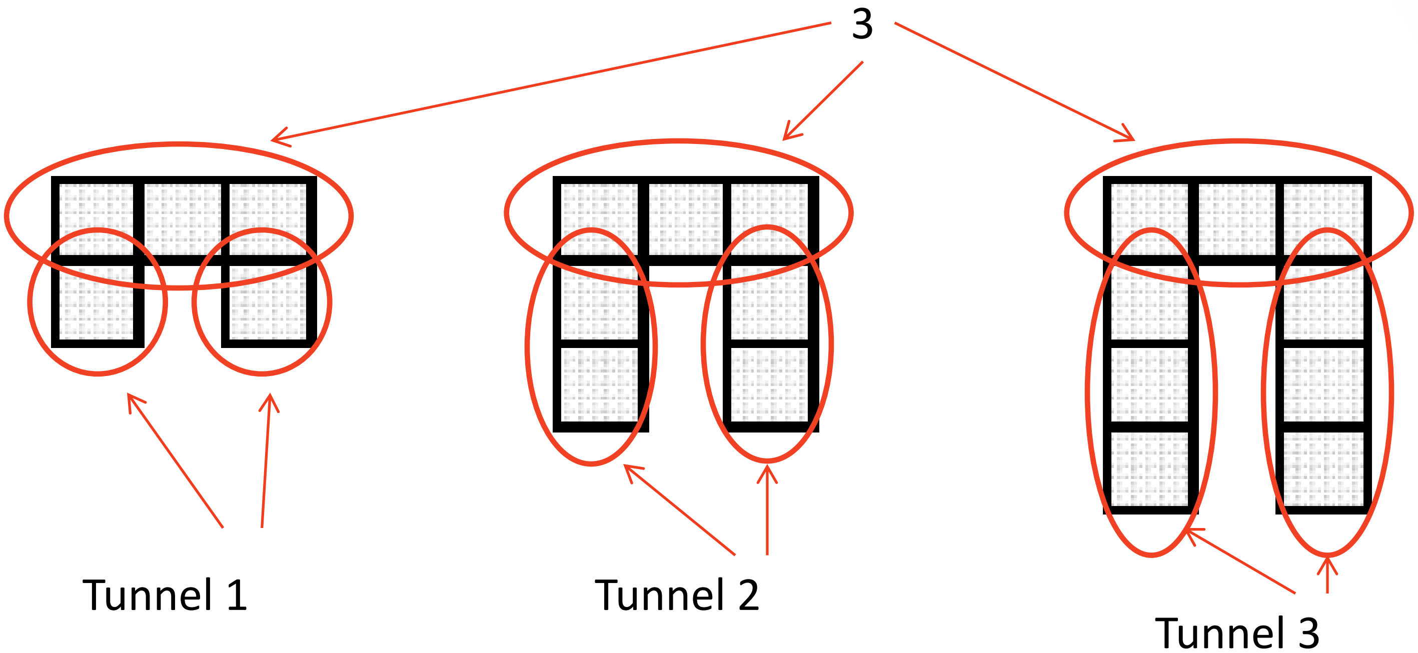
What ways of seeing might there be for the Tunnel pattern?

The Tunnel



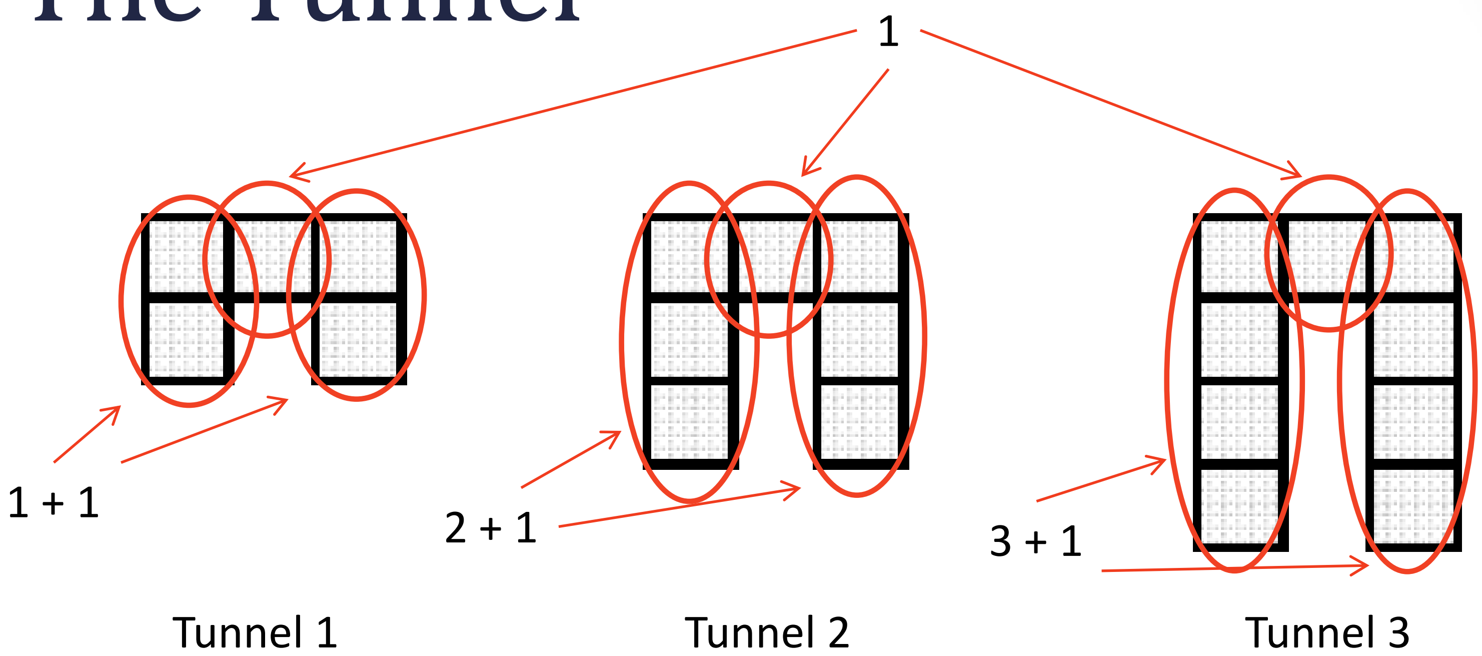
One way of seeing might be to recognize that two more square tiles are added to the previous tunnel. This is a *recursive* way of seeing. Although accurate, it is not easily applicable to later stages.

The Tunnel



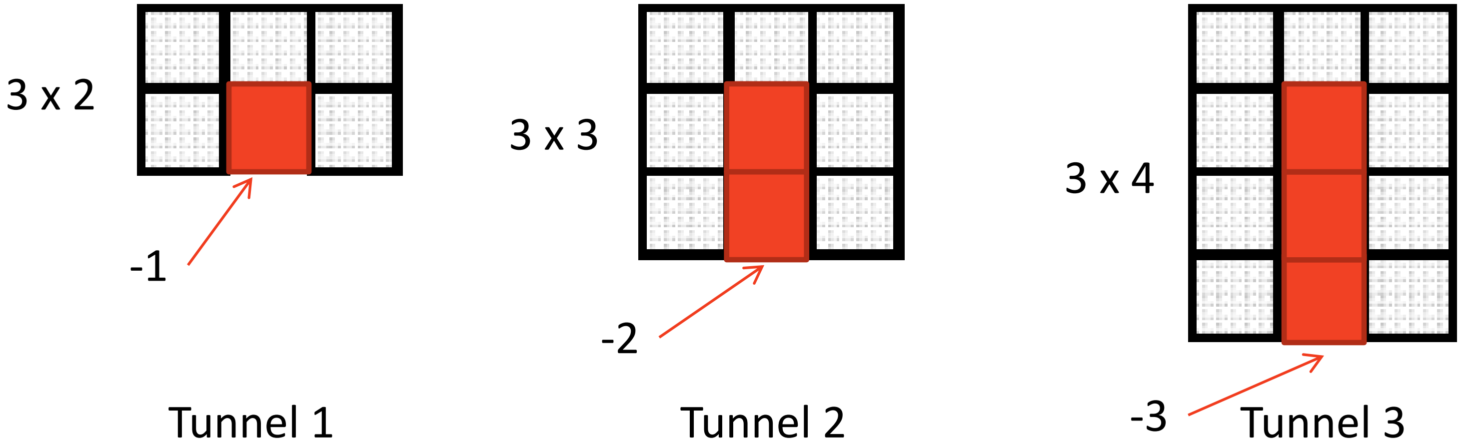
Another way of seeing might be to recognize that there are always three square tiles for the top row. The number of square tiles in each side of the tunnel corresponds to the tunnel number.

The Tunnel



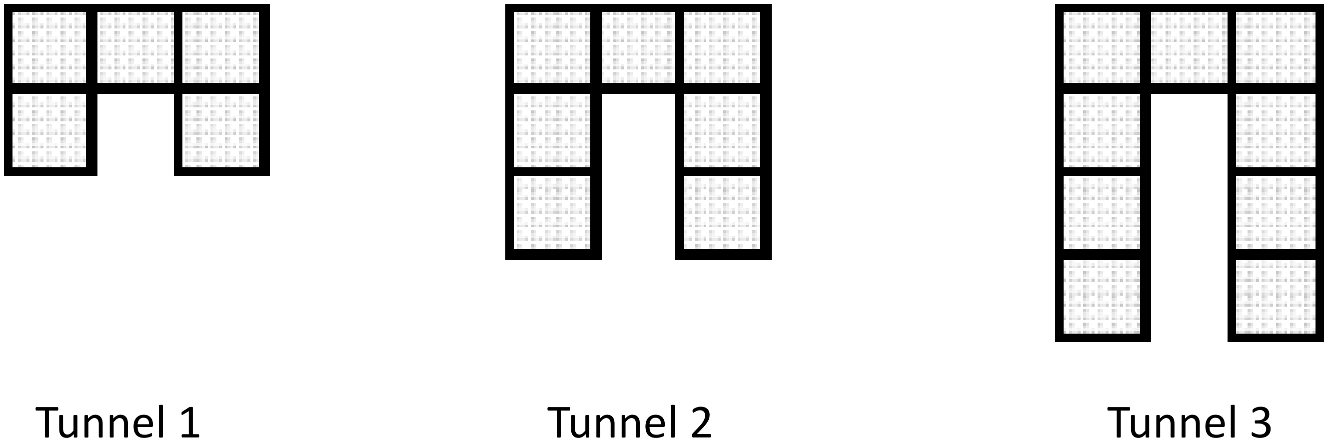
Another way of seeing might be to recognize that each of the two columns consists of one more square tile than the tunnel number. The two columns are connected with one additional square tile.

The Tunnel



A final way of seeing might be to recognize each tunnel as part of a larger rectangle, with dimensions 3 by 1 more than the tunnel number. Missing from this rectangle is the tunnel number of square tiles in the center column.

The Tunnel



NOW…. Let's think about *how many* square tiles we see.

A Typical Approach

- Often, teachers and students use patterns such as these to fill in quantities in a two-column table:

Tunnel Number	Total Number of Square Tiles
1	5
2	7
3	9
4	11

- Then, they try guess and check or procedures identified earlier to generate a rule for the functional relationship.
- The structure of the geometric growing pattern is never consulted again….

Instead....

- Instead, we can use an alternative tool that affords students the opportunity to translate their figural reasoning to a numerical reasoning.

Tunnel Number	Total Number of Square Tiles
1	5
2	7
3	9
4	11

The Three-Column Table

- Consider how a three-column table might serve this purpose:

The Three-Column Table

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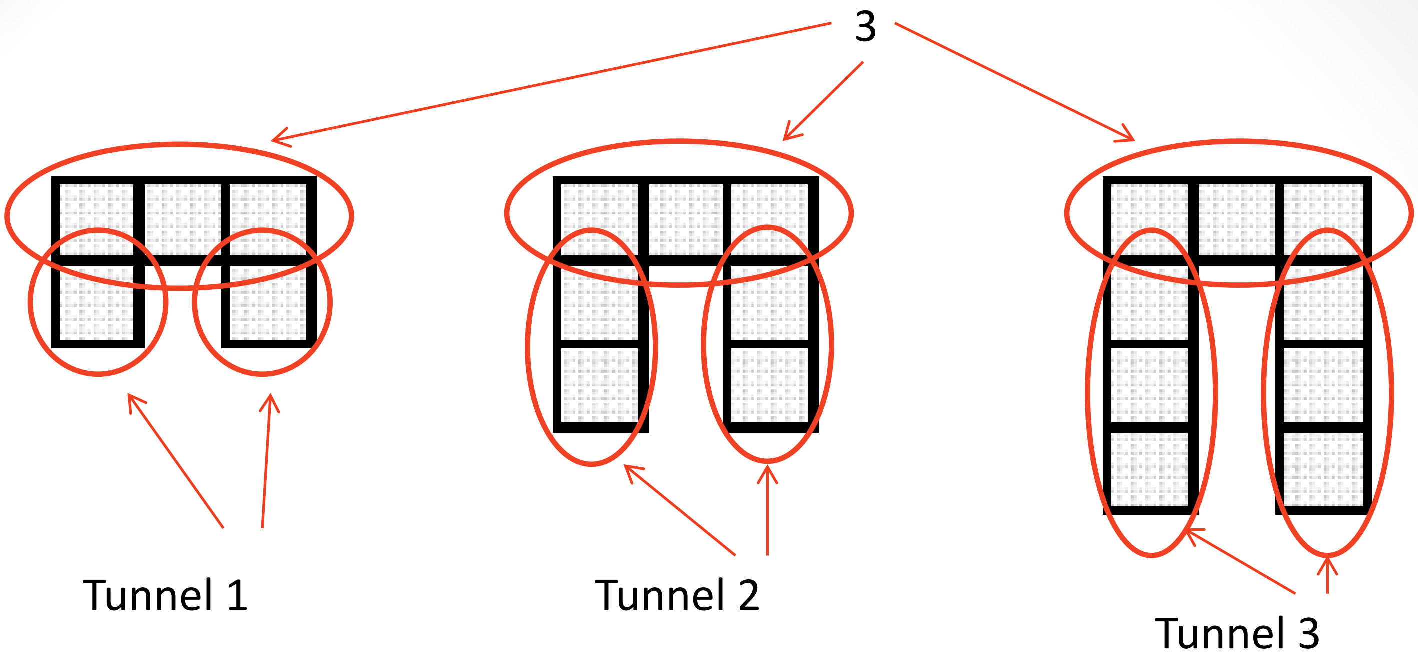
Tunnel Number		Total Number of Square Tiles

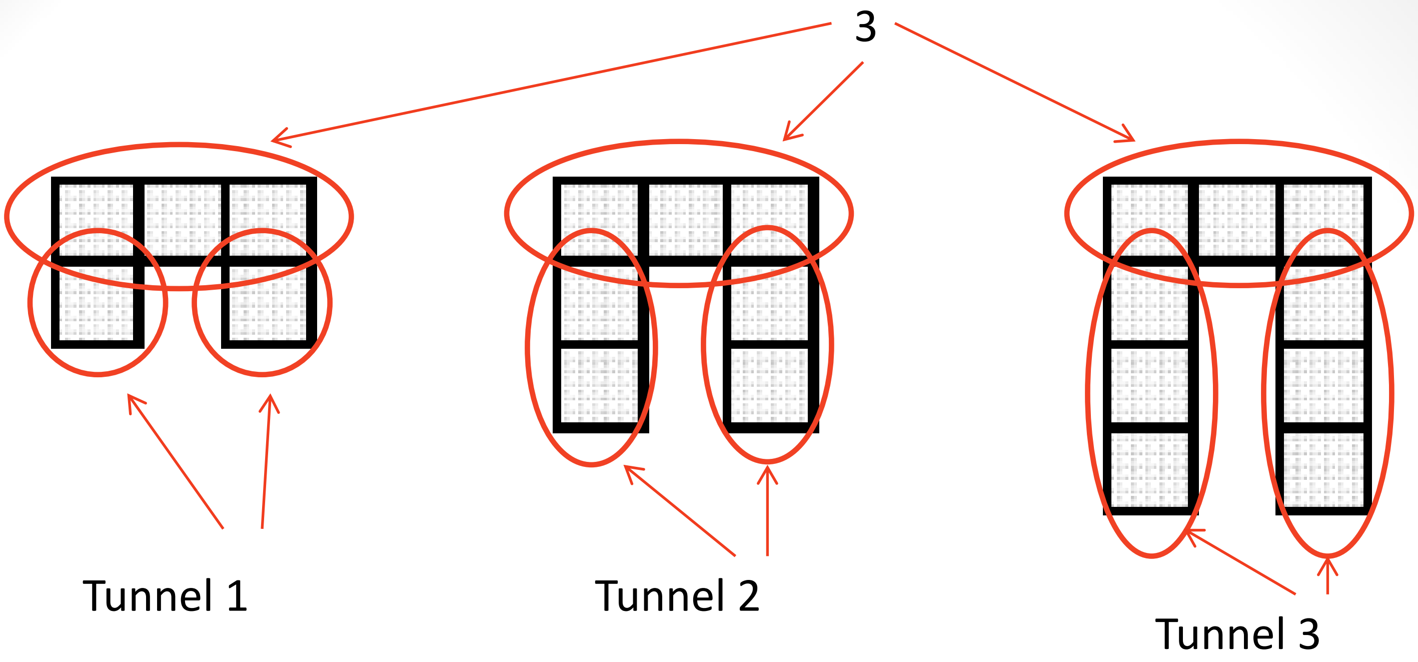
The Three-Column Table

- Consider how a three-column table might serve this purpose:

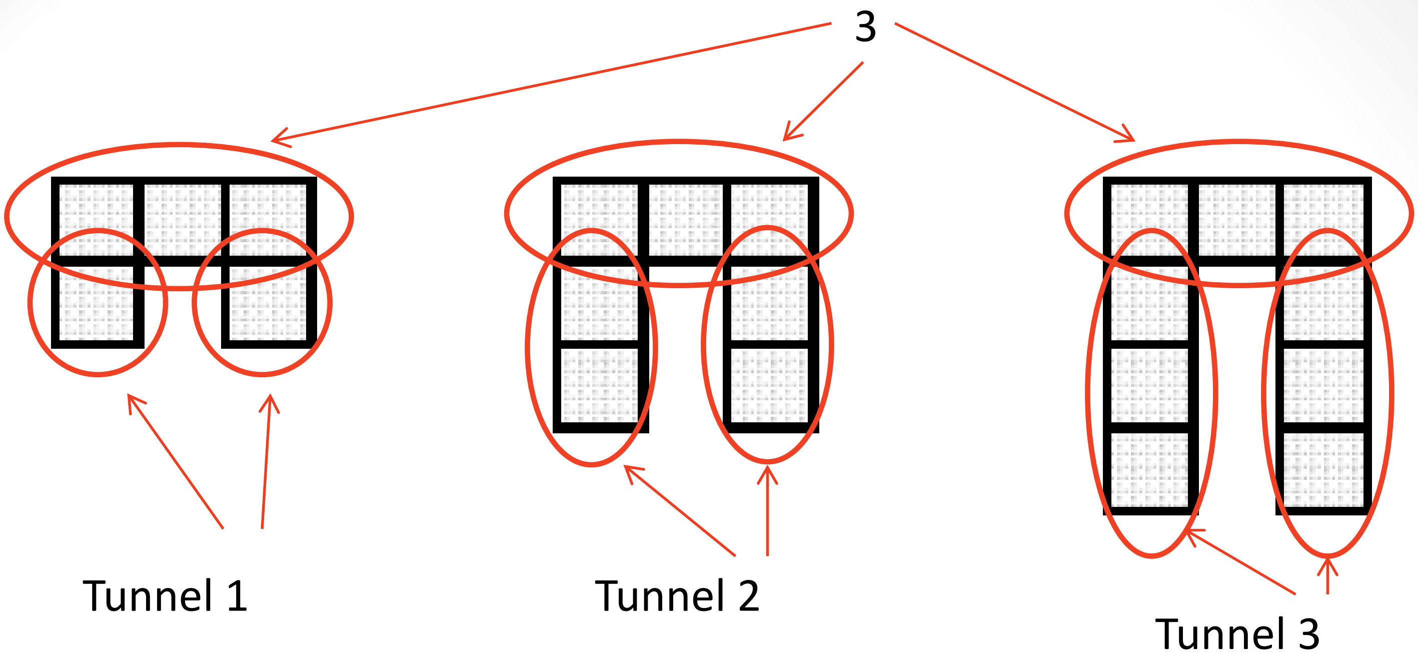
Tunnel Number	My Thinking	Total Number of Square Tiles



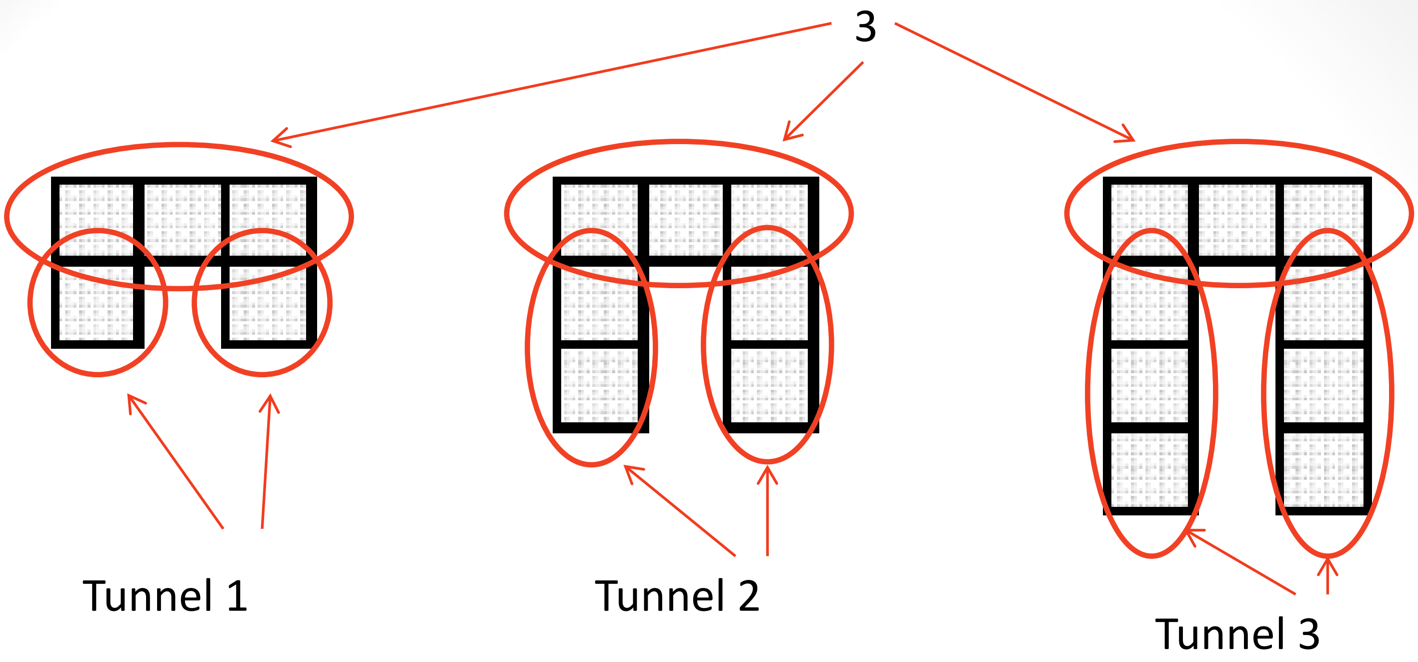
Tunnel Number	My Thinking	Total Number of Square Tiles
1		
2		
3		



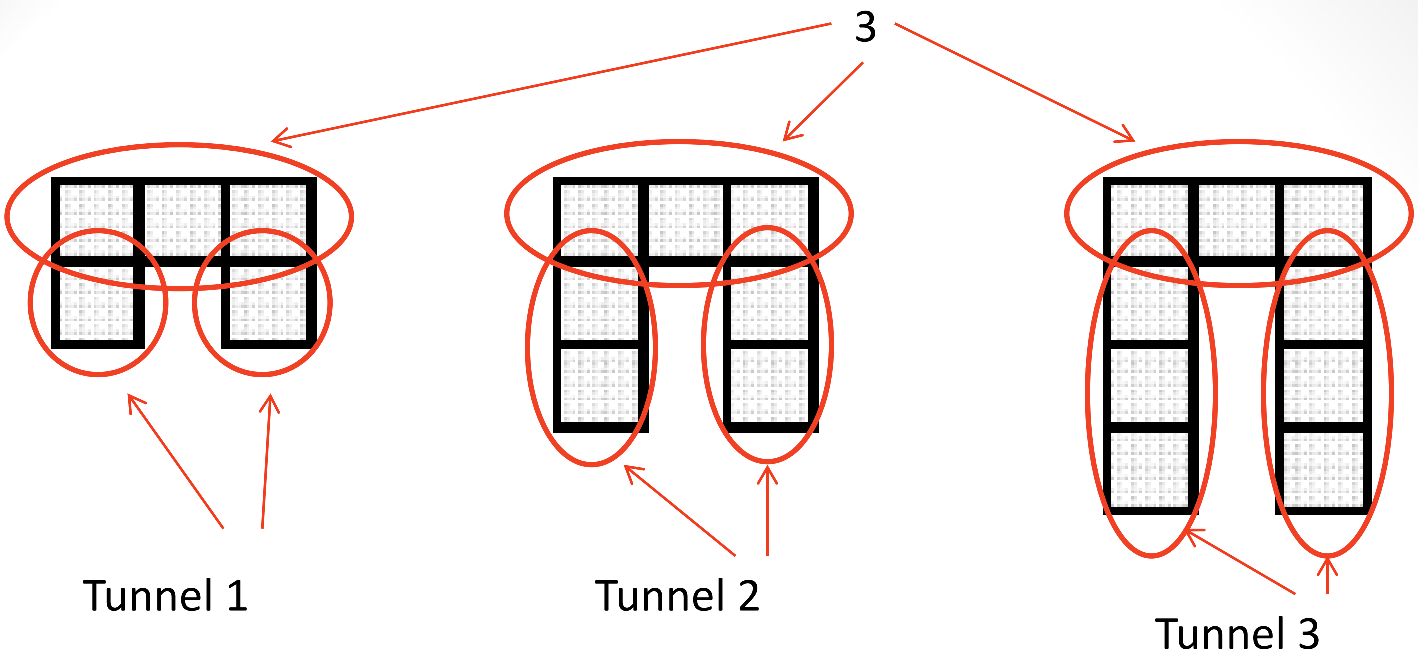
Tunnel Number	My Thinking	Total Number of Square Tiles
1		5
2		7
3		9



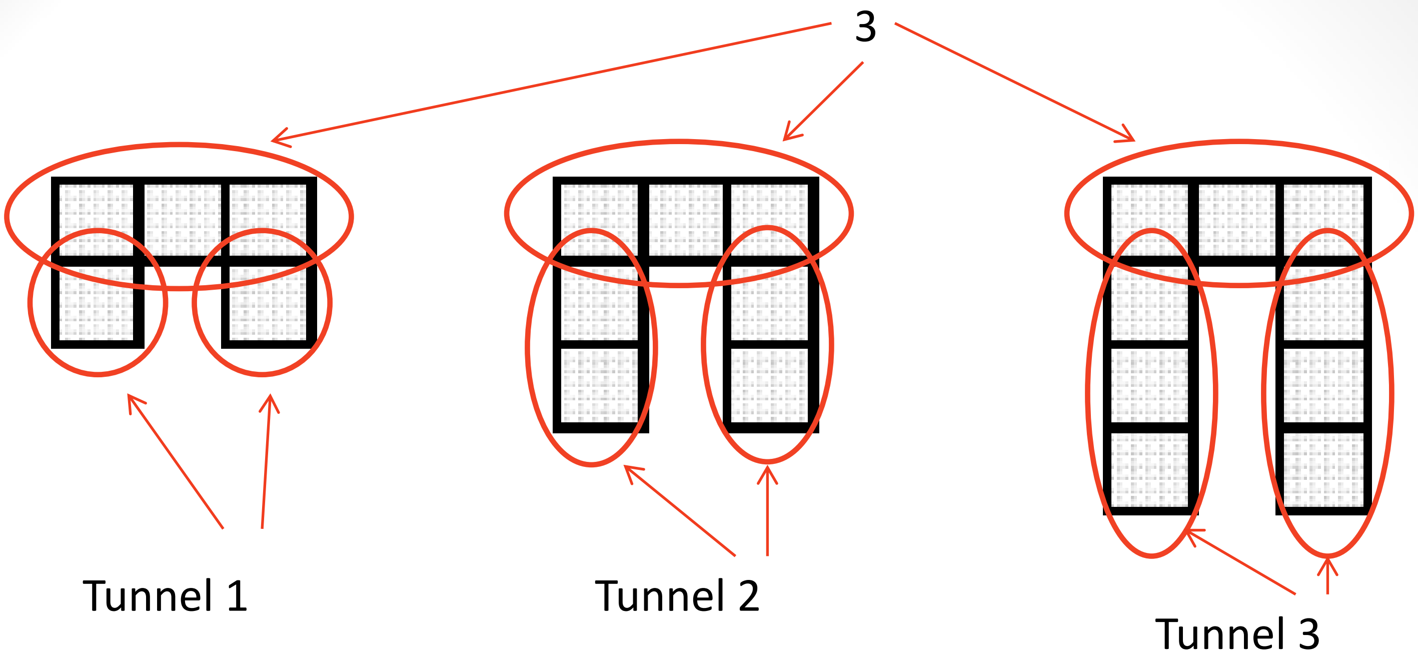
Tunnel Number	My Thinking	Total Number of Square Tiles
1		
2		
3		



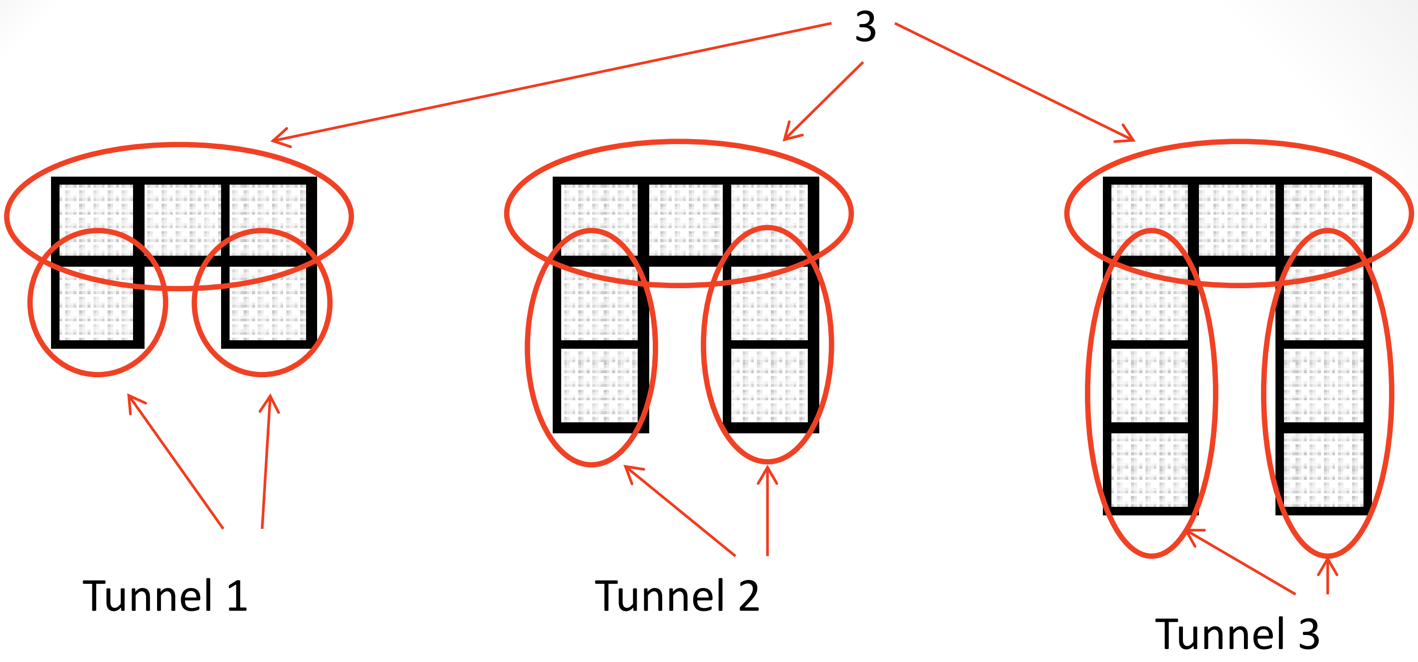
Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2		
3		



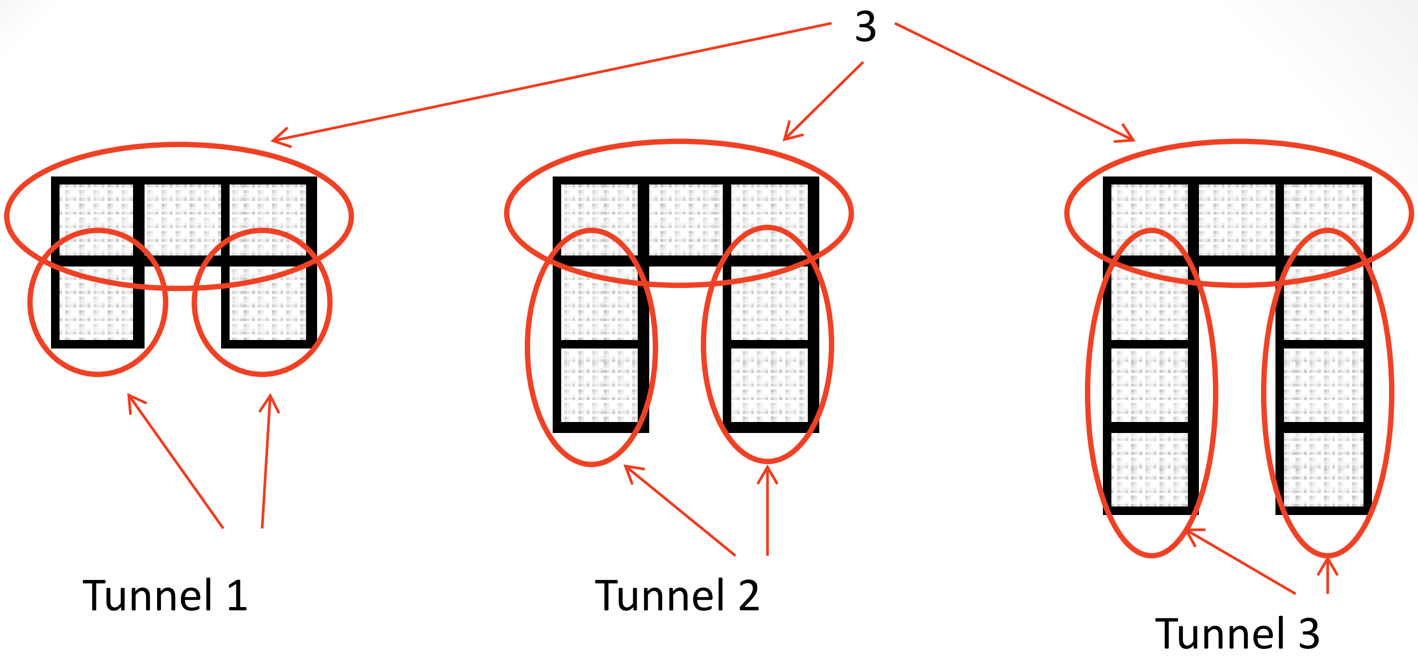
Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3		



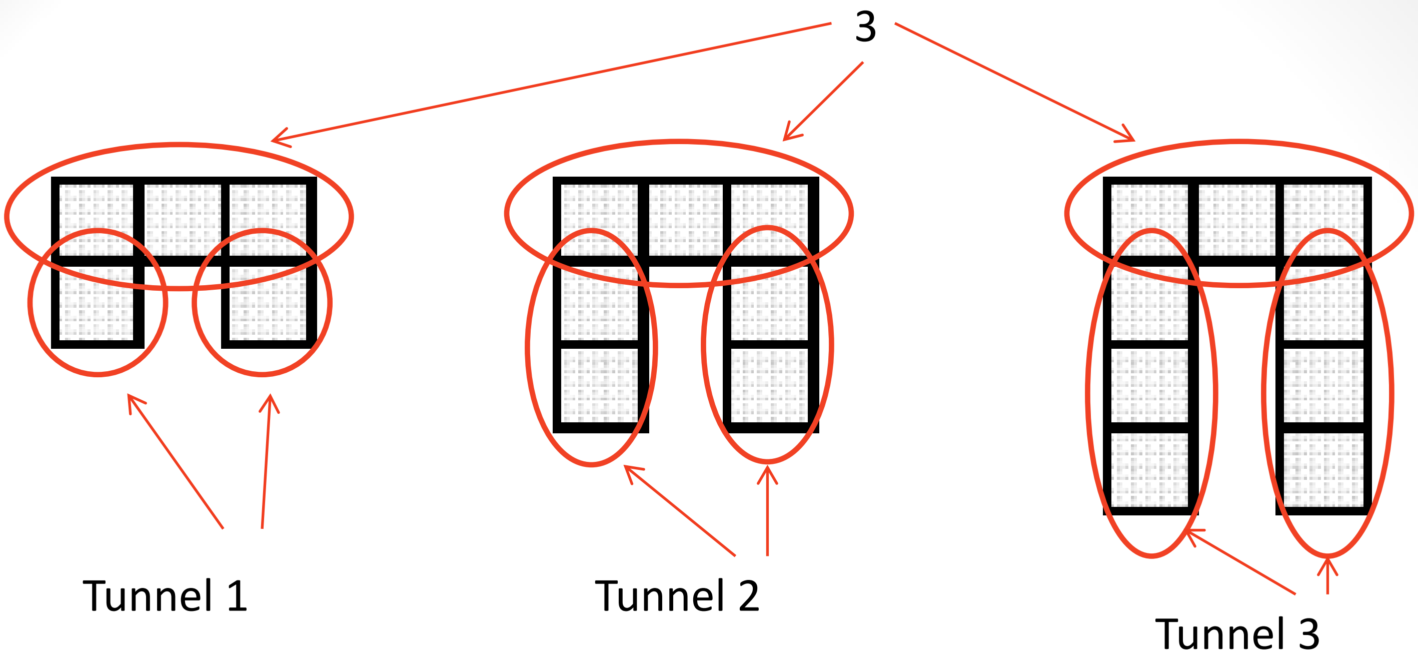
Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3	3 + 3 + 3	9



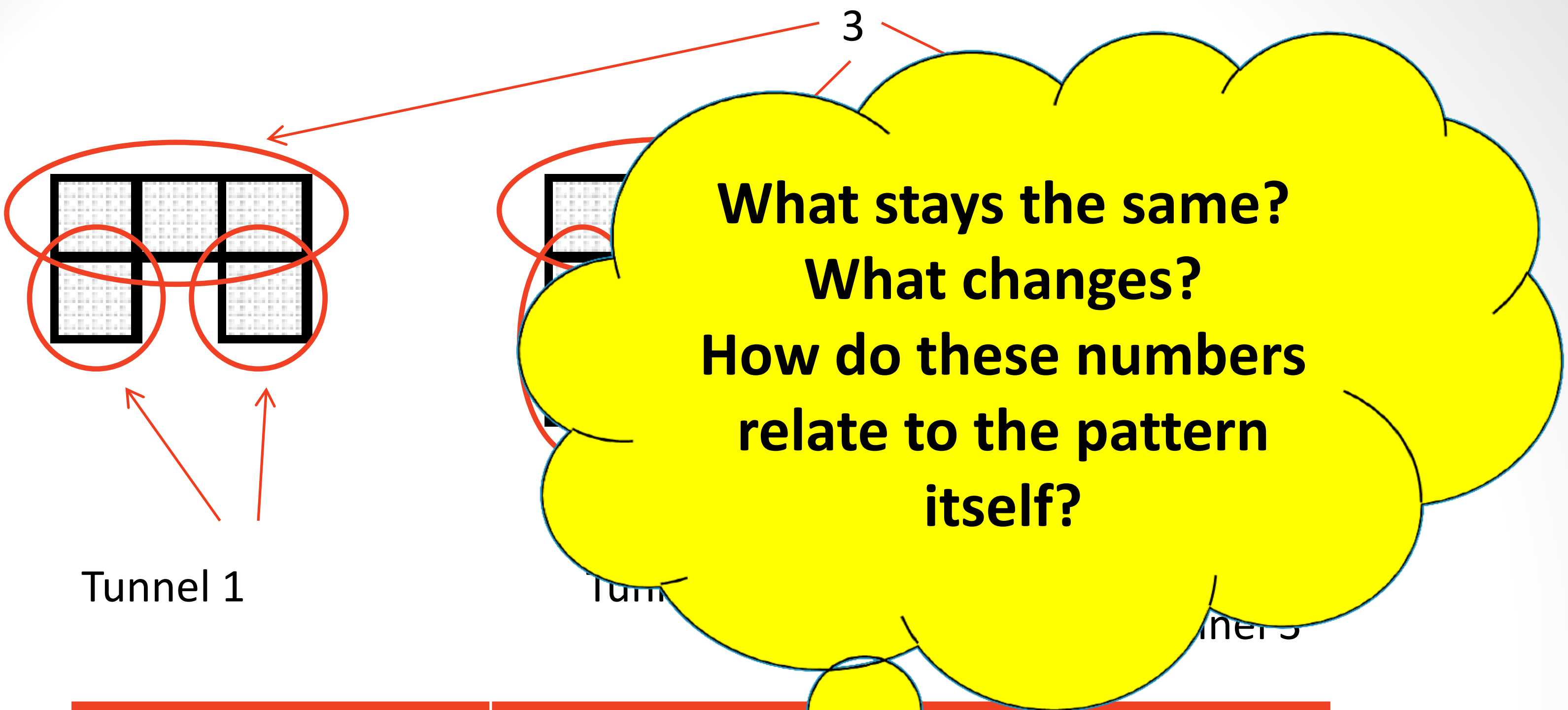
Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3	3 + 3 + 3	9
4	3 + 4 + 4	11



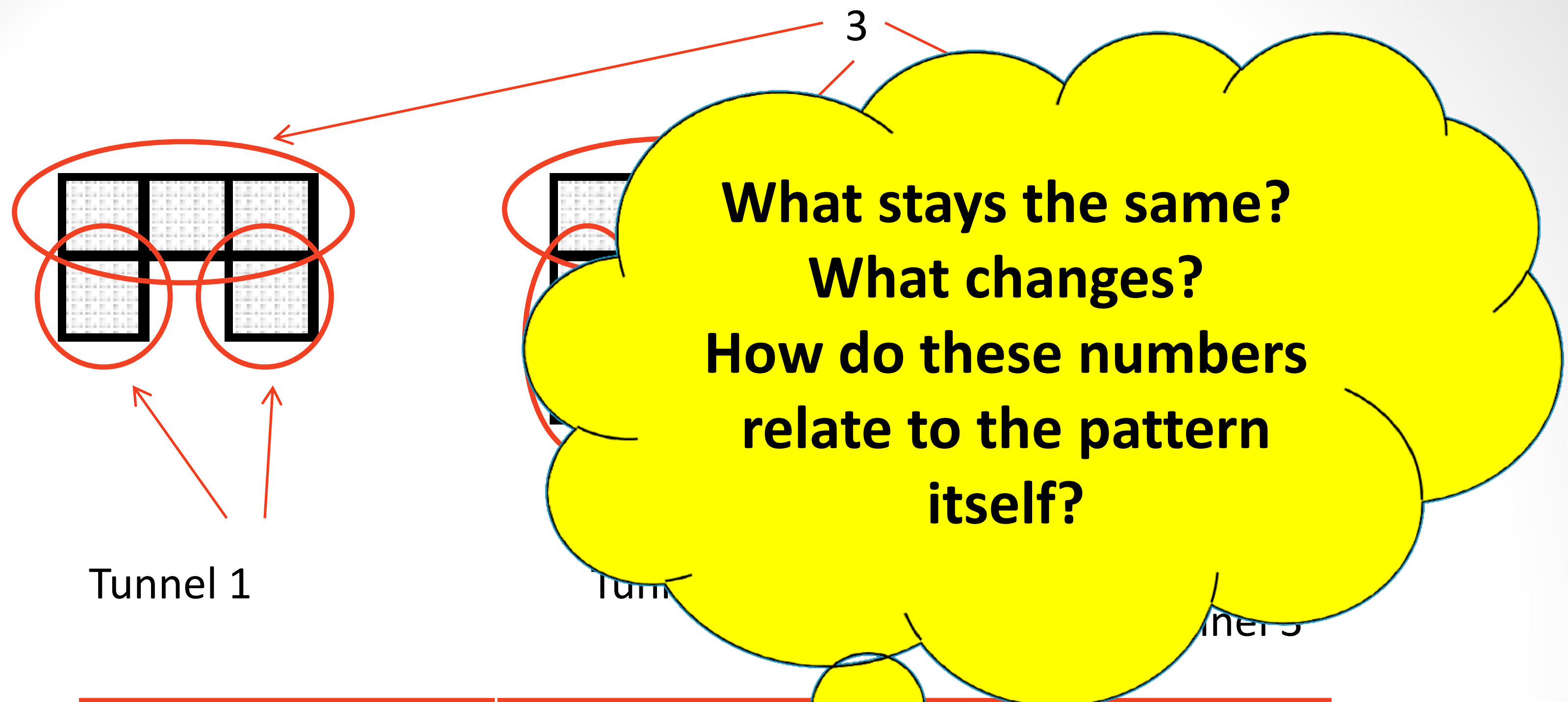
Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3	3 + 3 + 3	9
4	3 + 4 + 4	11
10	3 + 10 + 10	23



Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3	3 + 3 + 3	9
4	3 + 4 + 4	11
10	3 + 10 + 10	23
37	3 + 37 + 37	77



Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3	3 + 3 + 3	9
4	3 + 4 + 4	11
10	3 + 10 + 10	23
37	3 + 37 + 37	77
n	?	



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2	3 + 2 + 2	7
3	3 + 3 + 3	9
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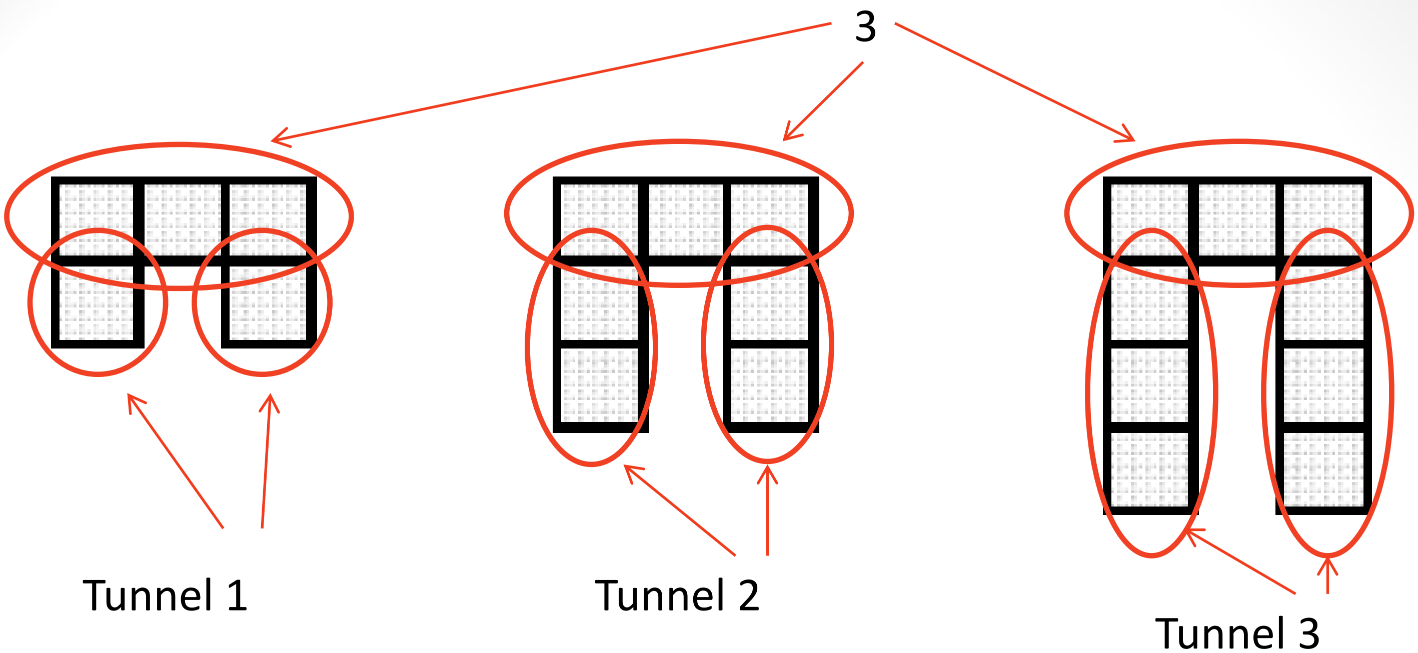
n	?	



Tunnel Number	My Thinking	Total Number of Square Tiles
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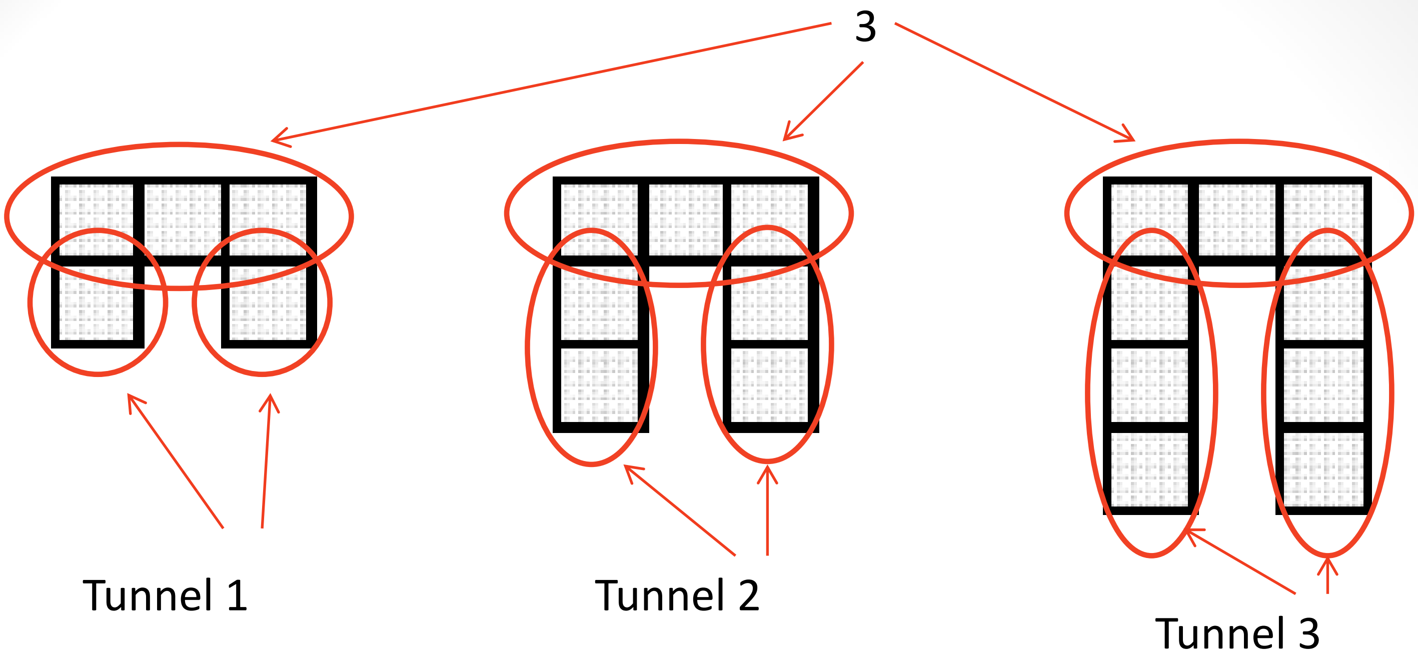
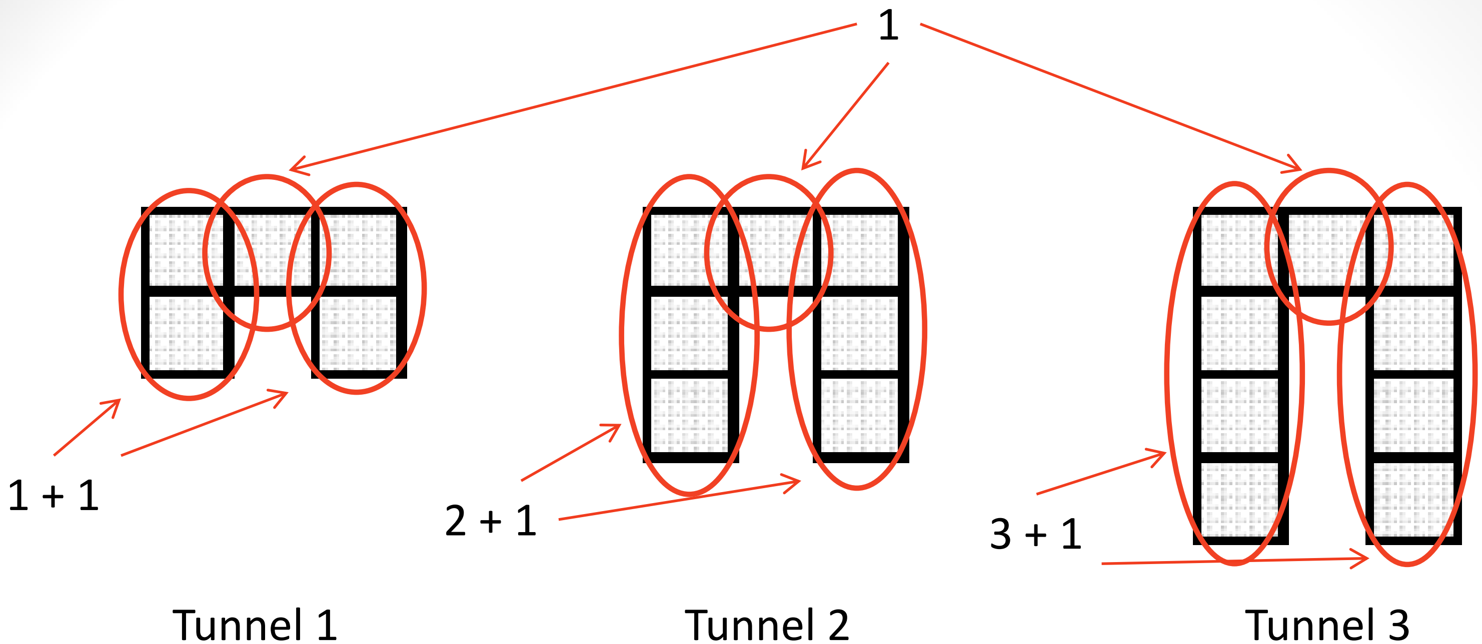
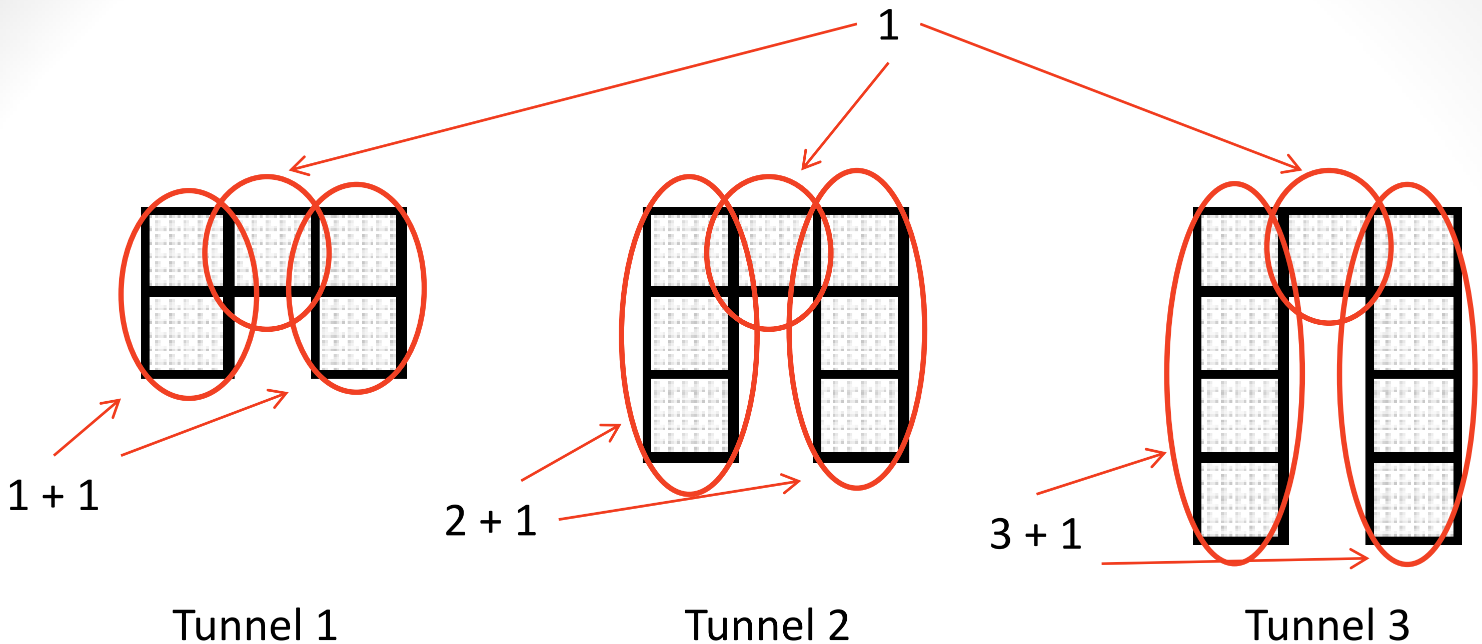
2	3 + 2 + 2	7
3	3 + 3 + 3	9
4	3 + 4 + 4	11
10	3 + 10 + 10	23
37	3 + 37 + 37	77
n	3	



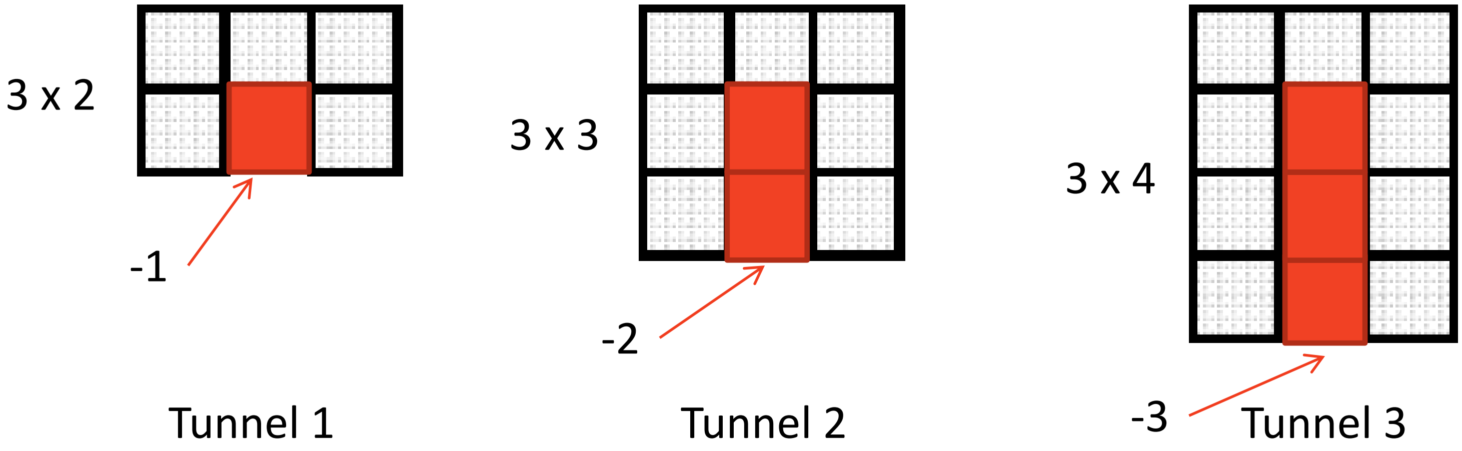
Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 + 1 + 1	5
2	3 + 2 + 2	7
3	3 + 3 + 3	9
4	3 + 4 + 4	11
10	3 + 10 + 10	23
37	3 + 37 + 37	77
n	$3 + n + n$	



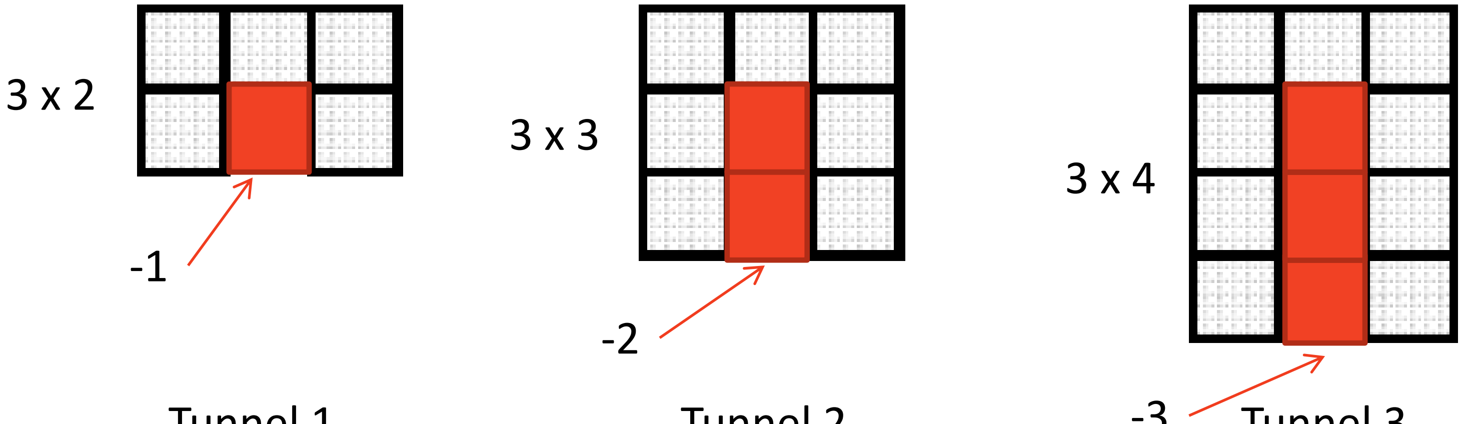
Tunnel Number	My Thinking	Total Number of Square Tiles
1	2 + 2 + 1	5
2	3 + 3 + 1	7
3	4 + 4 + 1	9
4	5 + 5 + 1	11
10	11 + 11 + 1	23
37	38 + 38 + 1	77
n		



Tunnel Number	My Thinking	Total Number of Square Tiles
1	2 + 2 + 1	5
2	3 + 3 + 1	7
3	4 + 4 + 1	9
4	5 + 5 + 1	11
10	11 + 11 + 1	23
37	38 + 38 + 1	77
n	$(n + 1) + (n + 1) + 1$	



Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 x 2 – 1	5
2	3 x 3 – 2	7
3	3 x 4 – 3	9
4	3 x 5 – 4	11
10	3 x 11 – 10	23
37	3 x 38 – 37	77
n		



Tunnel 1 Tunnel 2 Tunnel 3

Tunnel Number	My Thinking	Total Number of Square Tiles
1	3 x 2 – 1	5
2	3 x 3 – 2	7
3	3 x 4 – 3	9
4	3 x 5 – 4	11
10	3 x 11 – 10	23
37	3 x 38 – 37	77
n	3 x (n + 1) – n	

How can students, prior to a formal course in algebra, develop functional thinking?

How can we, as educators, construct tasks and experiences to best support this development of functional thinking?

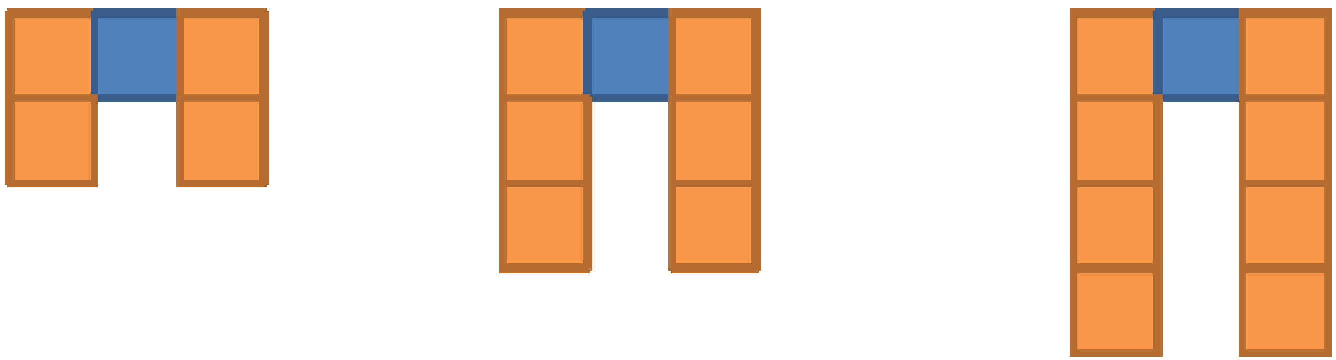
How can students, prior to a formal course in algebra, develop functional thinking?

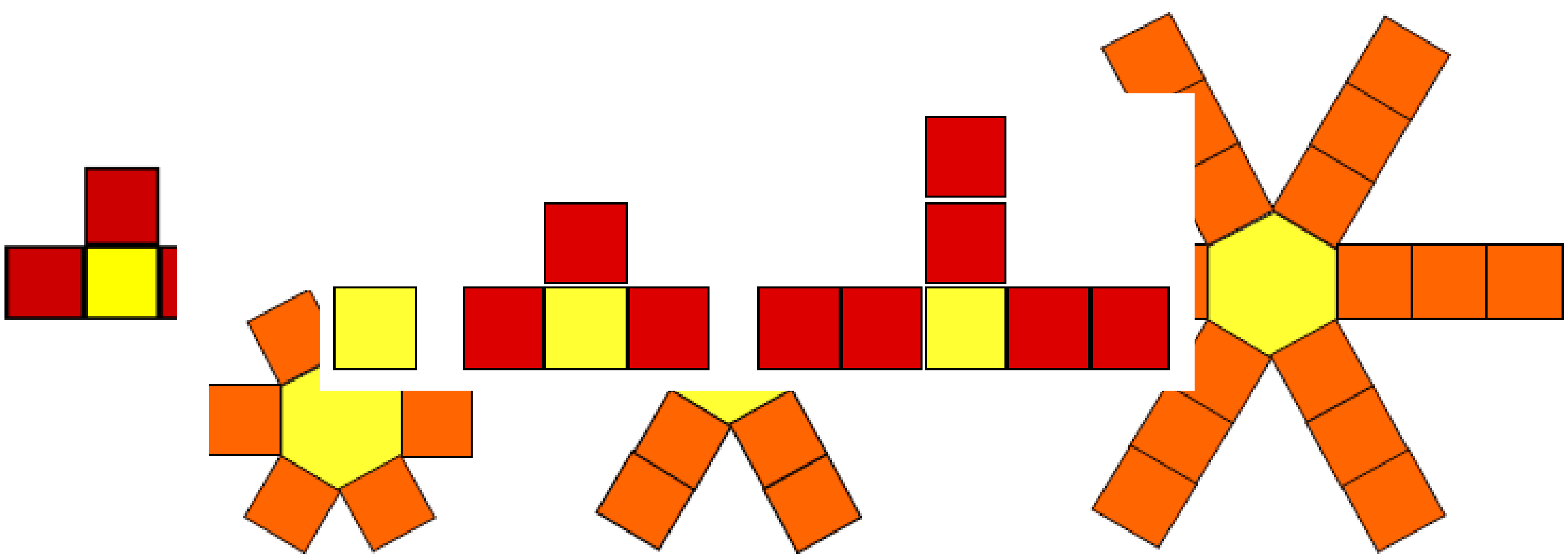
- Concrete experiences
- Accessible tasks – multiple ways of seeing
- An opportunity to make meaning of the numbers, variables, and symbols that are used in the algebraic expression or equation

How can we, as educators, construct tasks and experiences to best support this development of functional thinking?

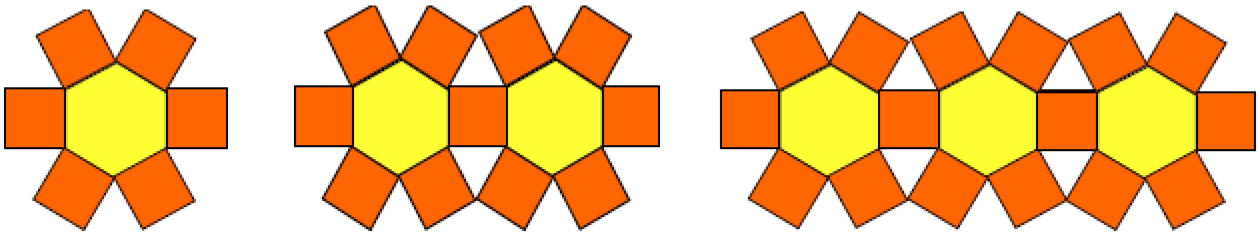
- Focus on labeling of the independent variable
- Problem solving process that encourages (and then capitalizes upon) figural reasoning
- Three-column table
- Accessible patterns....

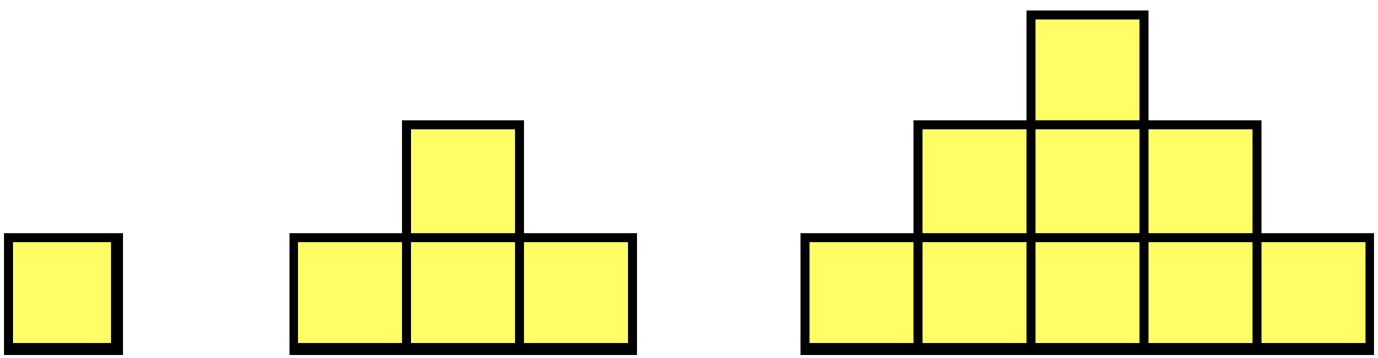
Pattern Complexity





Pattern Complexity

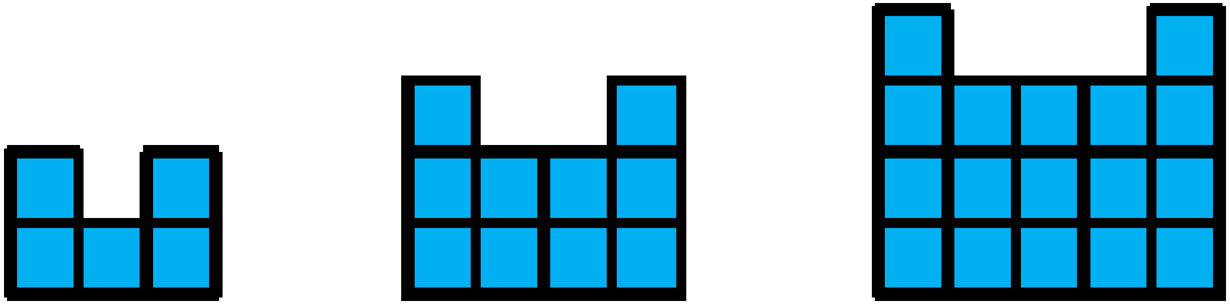




Beyond Linear Relationships

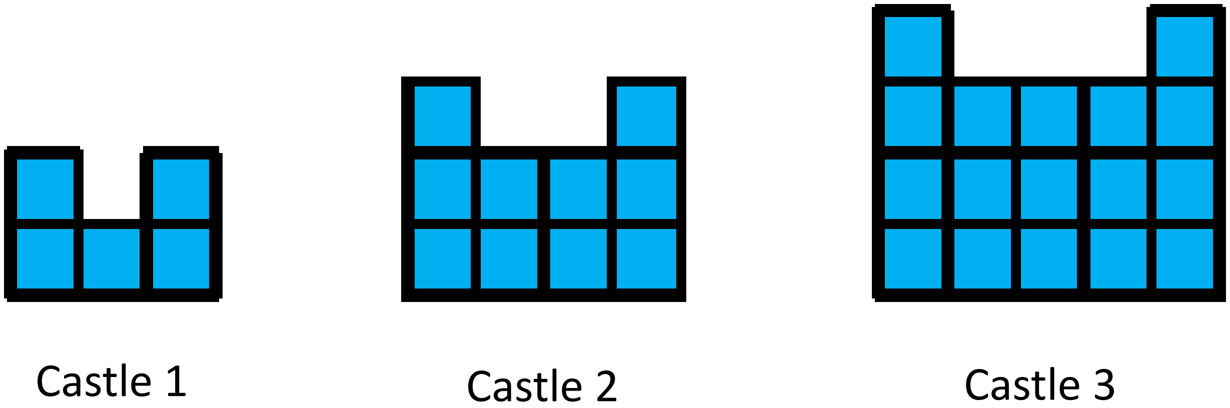
What happens when we work through a more complicated pattern task using these design features?

Consider the pattern below. What do you notice?



Castle 1   Castle 2   Castle 3

The Castle



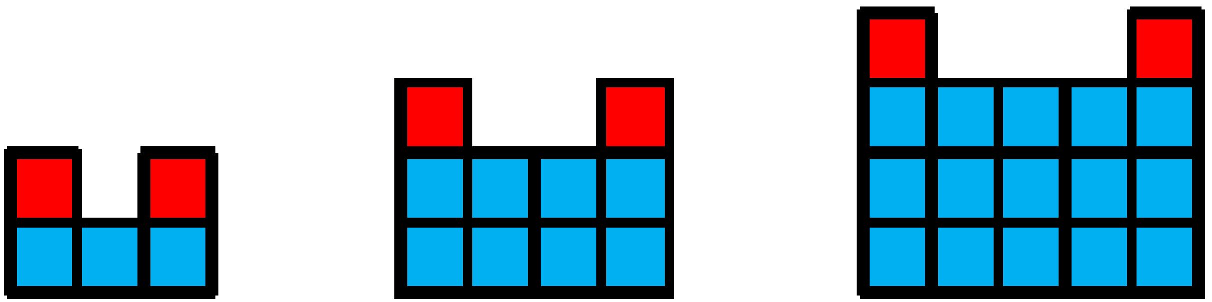
Focus on figural reasoning:

- Build or draw Castle 4.
- Build or draw Castle 10.
- Explain how you would build or draw Castle 41.

If time:

- Construct a three-column table for Castles 1, 2, 3, 4, 10, 41, and *n*.

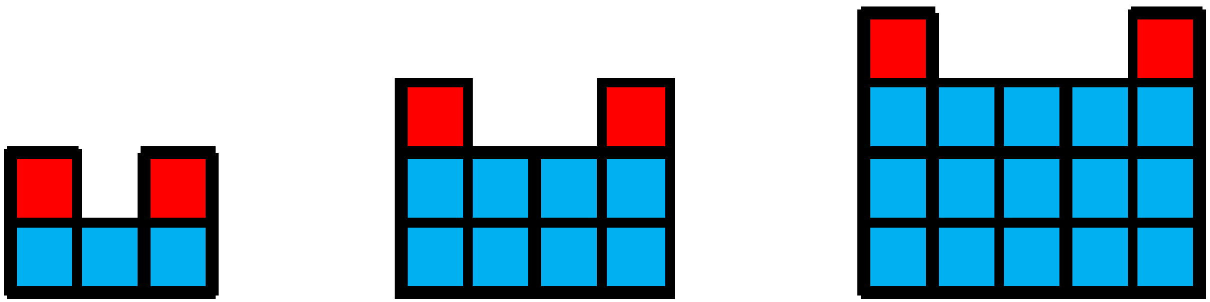
The Castle



Castle 1 Castle 2 Castle 3

Castle Number	My Thinking	Total Number of Square Tiles
1	1 x 3 + 2	5
2	2 x 4 + 2	10
3	3 x 5 + 2	17
4	4 x 6 + 2	26
10	10 x 12 + 2	122
41	41 x 43 + 2	1765
n		

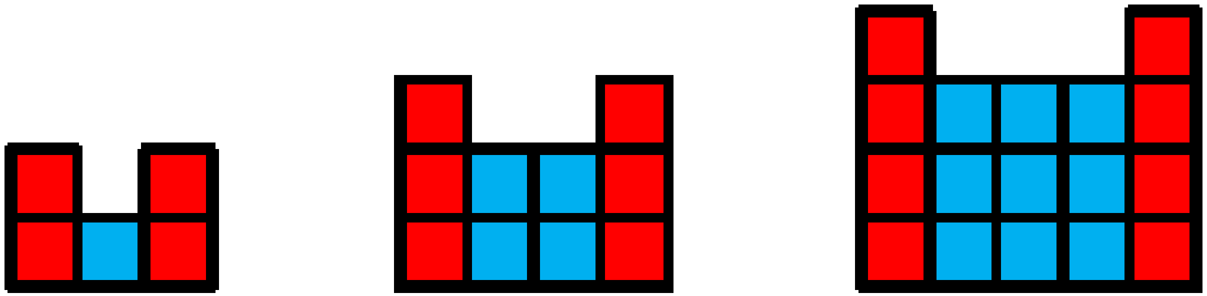
The Castle



Castle 1 Castle 2 Castle 3

Castle Number	My Thinking	Total Number of Square Tiles
1	1 x 3 + 2	5
2	2 x 4 + 2	10
3	3 x 5 + 2	17
4	4 x 6 + 2	26
10	10 x 12 + 2	122
41	41 x 43 + 2	1765
n	n x (n + 2) + 2	

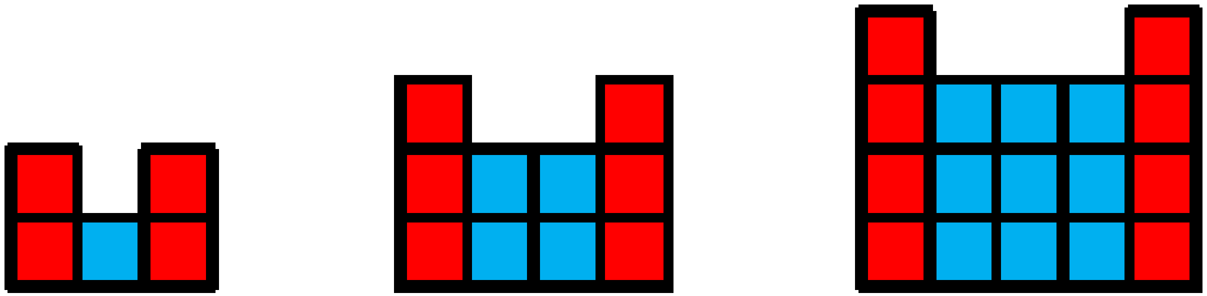
The Castle



Castle 1 Castle 2 Castle 3

Castle Number	My Thinking	Total Number of Square Tiles
1	1 x 1 + 2(1 + 1)	5
2	2 x 2 + 2(2 + 1)	10
3	3 x 3 + 2(3 + 1)	17
4	4 x 4 + 2(4 + 1)	26
10	10 x 10 + 2(10 + 1)	122
41	41 x 41 + 2(41 + 1)	1765
n		

The Castle



Castle 1 Castle 2 Castle 3

Castle Number	My Thinking	Total Number of Square Tiles
1	1 x 1 + 2(1 + 1)	5
2	2 x 2 + 2(2 + 1)	10
3	3 x 3 + 2(3 + 1)	17
4	4 x 4 + 2(4 + 1)	26
10	10 x 10 + 2(10 + 1)	122
41	41 x 41 + 2(41 + 1)	1765
n	*n* x *n* + 2(*n* + 1)	

The Castle

Were you able to generate these outputs fairly successfully?

- n x (n + 2) + 2
- n x n + 2(n + 1)

Do you think students might be able to generate a rule of a more complex, quadratic function such as this one? How does the geometric growing pattern help?

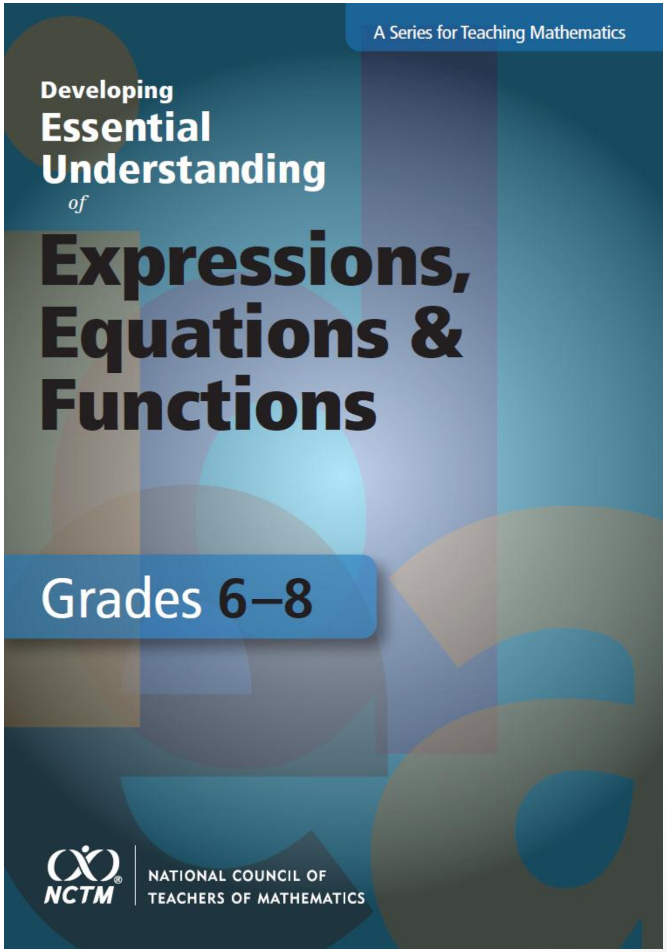
Look Back….

Input	Output
1	5
2	7
3	9
4	11
5	13
6	15

Input	Output
1	5
2	10
3	17
4	26
5	37
6	50

Big Idea #1: Expressions as Building Blocks

"Expressions are foundational for algebra; they serve as building blocks for work with equations and functions" (Lloyd, Herbel-Eisenmann, & Star, 2011, p. 8).

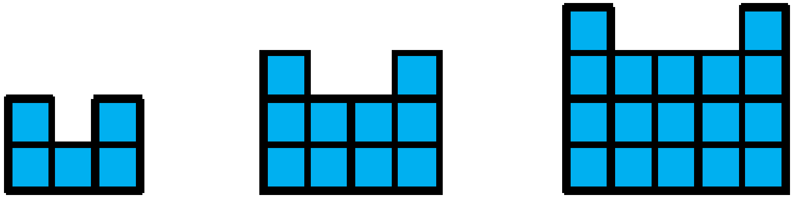


Essential Understanding 1a

"Expressions are powerful tools for exploring, reasoning about, and representing situations" (Lloyd, Herbel-Eisenmann, & Star, 2011, p. 9).

- How have we used expressions to represent these pattern situations?

  - $n \times (n + 2) + 2$
  - $n \times n + 2(n + 1)$

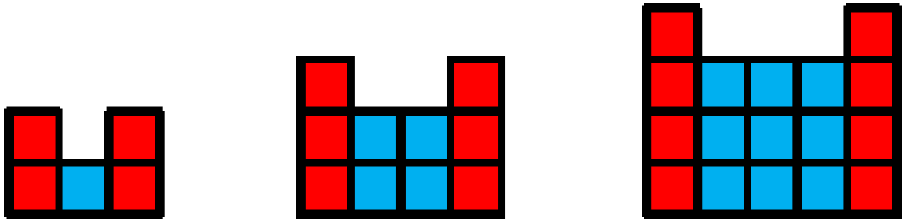


Essential Understanding 1b

"Two or more expressions may be equivalent, even when their symbolic forms differ" (Lloyd, Herbel-Eisenmann, & Star, 2011, p. 9).

- How do different ways of seeing these patterns generate different, yet equivalent, expressions?

  - $n \times (n + 2) + 2$
  - $n \times n + 2(n + 1)$



Essential Understanding 1c

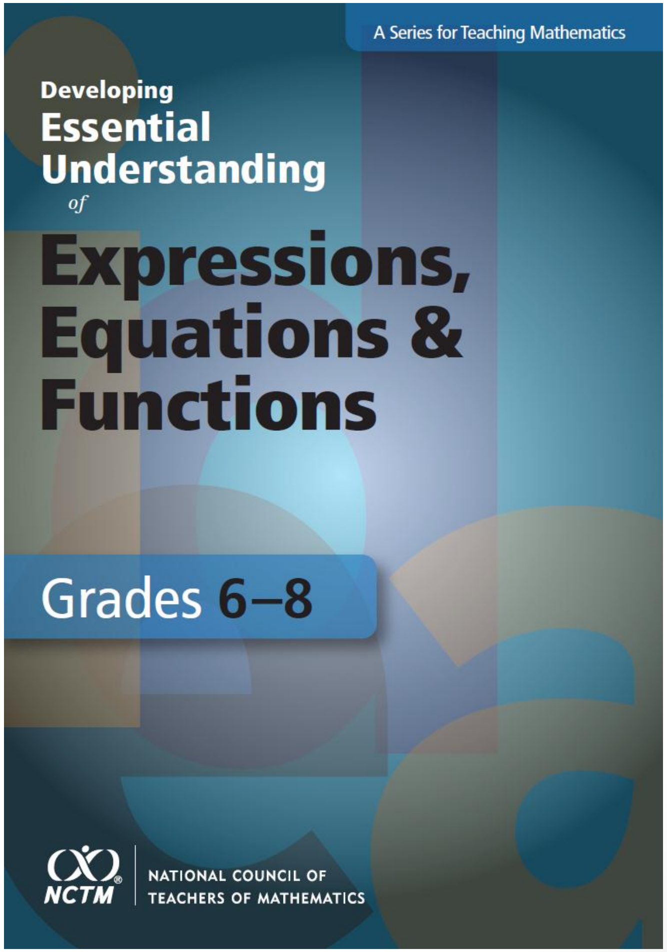
"A relatively small number of symbolic transformations can be applied to expressions to yield equivalent expressions" (Lloyd, Herbel-Eisenmann, & Star, 2011, p. 9).

- How can we demonstrate that are expressions are indeed equivalent?

  - $n \times (n + 2) + 2 \rightarrow n^2 + 2n + 2$
  - $n \times n + 2(n + 1) \rightarrow n^2 + 2n + 2$

Other Connections

There are certainly other connections to big ideas….



Standards for Mathematical Practice of the CCSSM

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

CCSSM Content Standards

Expressions and Equations

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason abou and inequalit
- Represent ar relationships independent

Expressions and Equations

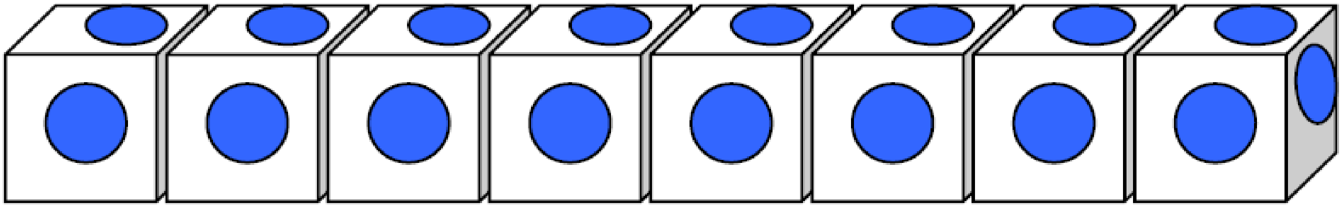
- Use properties of operations to generate equivalent expressions.
- Solve real- using num equations.

Expressions and Equations

- Work with radicals and integer exponents.
- Understand the connections between proportional relationships, lines, and linear equations.
- Analyze and solve linear equations and pairs of simultaneous linear equations.

Challenges and Limitations

- Challenge for students to find a useful *way of seeing*
  - Focus on a way of counting, without counting one by one
  - "Jumping in" with later stages
  - "Efficient counting techniques"



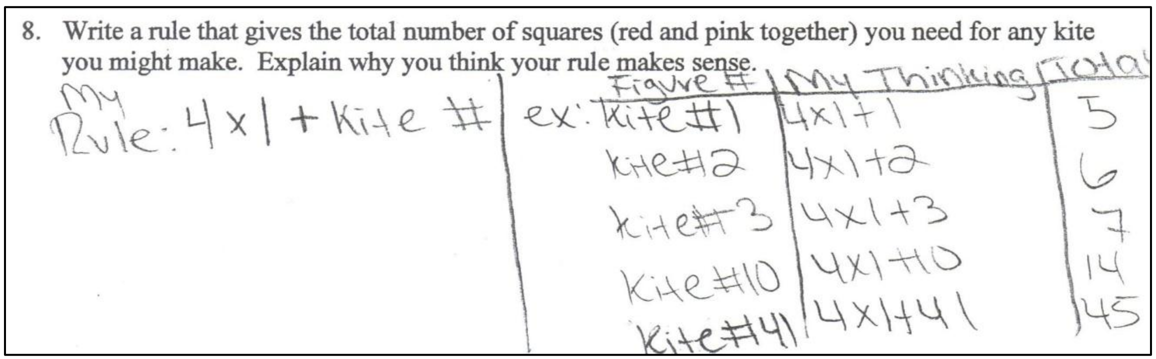
Challenges and Limitations

- Domain limitations
  - Negative numbers
  - Non-whole numbers
- Norms of variable use

Figure Number	Our Thinking	Total Number of Square Tiles
1	$(2 \cdot 1) + 2$	4
2	$(3 \cdot 2) + 2$	8
3	$(4 \cdot 3) + 2$	14
f	$(?? \cdot f) + 2$	

Challenges and Limitations

- Getting students to use the three-column table on their own as a tool



8. Write a rule that gives the total number of squares (red and pink together) you need for any kite you might make. Explain why you think your rule makes sense.

My Rule: 4x1 + Kite #

Figure #	My Thinking	Total
ex: Kite#1	4x1+1	5
Kite#2	4x1+2	6
Kite#3	4x1+3	7
Kite#10	4x1+10	14
Kite#41	4x1+41	45

- Slowing students down to analyze the patterns and identify their own *ways of seeing*

Wrap Up

How can students, prior to a formal course in algebra, develop functional thinking?

- Concrete experiences
- Accessible tasks – multiple ways of seeing
- An opportunity to make meaning of the numbers, variables, and symbols that are used in the algebraic expression or equation

How can we, as educators, construct tasks and experiences to best support this development of functional thinking?

- Focus on labeling of the independent variable
- Problem solving process that encourages (and then capitalizes upon) figural reasoning
- Three-column table
- Accessible patterns….

Questions?

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