# Developing Algebraic Reasoning from Quantitative Reasoning

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# Goals of this Presentation

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To explore how the development of algebraic thinking through quantitative reasoning:

- Rate problems as a tool: They are not all created equally
- How to create these problems
  - Spoiler alert: DON'T create … find and tweak
  - Again, context matters…
- How they are used with HoM, WoT, SMP's in mind
- What student thinking can look like

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# From Macro to Micro (or Theory and Practice)

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Influential Sources:

- Usiskin's Definition of Algebra
- Pat Thompson's Theory of Quantitative Reasoning (2011)
- Harel's Necessity Principle (Harel, 2013a and 2013b)
- Fostering Algebraic Thinking (Driscoll et al)
- CCSS-M Appendix A, p. 17.

# Harel (Research Perspective for CCSS)

Quantitative reasoning is a way of thinking by which one reasons with quantities and about relations among quantities. It entails the habits of creating a coherent image of the problem at hand; considering the units involved; continually attending to the meaning of quantities, in addition to how to compute them; and having multiple images of a concept and being flexible in transitioning among them.

# Reason Abstractly and Quantitatively

CCSS MP2

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

# Some Principles

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1. "Friendly numbers" can be used to help create a coherent image of the problem.
2. Making the number "less friendly" can help focus attention on structure and meaning.
3. A context with units can be used to help attend to meaning.
4. The ability to change perspectives (or images) of a problem is essential.
5. Writing different expressions representing the same quantity can necessitate algebraic manipulation and help students attend to the meaning of the equal sign.
6. *An approach to introducing algebraic reasoning:*
    1. Noticing a pattern from repeated observations with attention to causality.
    2. Change the way you ask your question… "At any in point in time".
    3. Build scaffolds to support repeated reasoning before asking to solve…

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# Gen's Sequence/Version of Harel's (2013) Motion Problems

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1) From two towns, A and B, two cars left at the same time toward each other – one from Town A and the other from Town B. The speed of the first car is 70 mph, and the speed of the second car is 80 mph. If the distance between the two towns is 300 miles, how long does it take for the two cars to meet?

2) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 12 mph, and the motor cycler at 40 mph. In how many hours will the distance between them be more than 105 miles?

3) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 15 mph, and the motor cycler at 55 mph. In how many hours will the distance between them be more than 170 miles?

# Building Algebraic Thinking through Quantitative Reasoning: Applying Harel's Necessity Principle

Sequence for Distance Problems:

1) From two towns, A and B, two cars left at the same time toward each other – one from Town A and the other from Town B. The speed of the first car is 70 mph, and the speed of the second car is 80 mph. How long does it take for the two cars to meet, if the distance between the two towns is 300 miles? 150 miles? 525 miles? 273 miles?

2) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 12 mph, and the motor cycler at 40 mph. In how many hours will the distance between them be more than 56 miles? 140 miles? 42 miles? 105 miles?

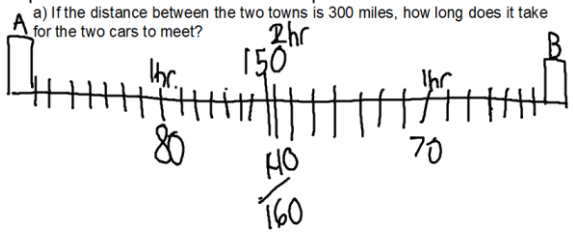
2) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 12 mph, and the motor cycler at 40 mph.

c) In how many hours will the distance between them be more than 105 miles?

1hr 40-12=28 30min 20-6=14
2hr 80-24=56 10-3=7
3hr 120-36=84 15min
160-48=112 14
98
+7
105
3hr 45min

From two towns, A and B, two cars left at the same time toward each other – one from Town A and the other from Town B. The speed of the first car is 70 mph, and the speed of the second car is 80 mph.

a) If the distance between the two towns is 300 miles, how long does it take for the two cars to meet?




A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 15 mph, and the motor cycler at 55 mph.

c) In how many hours will the distance between them be more than 170 miles?

55-15=40
55(2)-15(2)=80
55(3)-15(3)=120
55(4)-15(4)=160

1(55-15)=40
2(55-15)=80x
40x/40 = 170/40
60.00
.25
15
4.25
40|170
4 hrs. 15 mins.

- "Friendly" solutions first.
- Opportunities to use knowledge of motion.
- Changing to less "friendly" solutions allows repeated reasoning about the meaning of quantities until a pattern was noticed
- An equation based on the quantity that is varying was written.

# Difference between Representation and Problem Solving Approaches

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- Form versus substance
  - Representations are useful, but not the point
- Student work shows:
  - Double number line
  - Tables and pseudo-tables
  - Equations
- Closed versus open computations
  - Open form emphasizes structure

# Flooded Basement Problems (FBP's)

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Flooded Basement Problem: My basement flooded and there are 2.5 inches of water in it. Last time when it flooded there was 3/8 inch of water, and it took my pump 45 minutes to pump it out.

(1) How long will it take this time?

(2) When I started my pump, I realized that the pump has enough gasoline for one hour. According to the pump manual, the capacity of the gasoline tank is 0.5 gallon, which is sufficient for 5 hours work. What is the least amount of gasoline I need to add to my pump to ensure that it can pump all of the water out of my basement?

(3) Unfortunately, I have only 1/5 gallon of gasoline to put in my pump. My neighbor has a portable pump which holds the same amount of gasoline as my pump and pumps water out at the same rate. It has ¼ gallon of gasoline. Using all of the gasoline available, would the two pumps be able to pump 100% of the water out of my basement?

(4) If there were enough gasoline and if the two pumps were working together, how long would it take them to pump 100% of the water out of my basement?

(5) My neighbor isn't home, so I have to borrow a pump from my brother. My brother's pump pumps out water at a rate 4/5 as fast as my pump. If the two pumps were working together, how many inches of water would be pumped out of the basement at any given moment from the start of their working together?

(6) If my and my brother's pumps were working together, how long would it take them to pump all of the water out of the basement?

# Flooded Basement Problems (FBP's) Focus Problems

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Flooded Basement Problem: My basement flooded and there are 2.5 inches of water in it. Last time when it flooded there was 3/8 inch of water, and it took my pump 45 minutes to pump it out.

(1) How long will it take this time?

(5) My neighbor isn't home, so I have to borrow a pump from my brother. My brother's pump pumps out water at a rate 4/5 as fast as my pump. If the two pumps were working together, how many inches of water would be pumped out of the basement at any given moment from the start of their working together?

(6) If my and my brother's pumps were working together, how long would it take them to pump all of the water out of the basement?

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# Possible approaches/lessons to FBP – Repeated Addition Approach

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## Tabulate with purpose and organization, change forms, switch perspectives, persist

| Number of Min's | Number of inches | Goal $2.5\,\text{inches}$ | | | |
|---|---|---|---|---|---|
| 45 | $\frac{3}{8}$ inch | | | | |
| 90 | $\frac{6}{8}$ inch | | | | |
| 135 | $1\frac{1}{8}$ inch or $\frac{9}{8}$ inch | | | | |
| 180 | $1\frac{4}{8}$ inch or $\frac{12}{8}$ inch | | | | |
| 225 | $1\frac{7}{8}$ inch or $\frac{15}{8}$ inch | $2.5\,\text{inch}$ | $2\frac{1}{2}$ inch | $2\frac{4}{8}$ inch | $\frac{20}{8}$ inch |
| 270 | $2\frac{2}{8}$ inch or $\frac{24}{8}$ inch | $2.5\,\text{inch}$ | $2\frac{1}{2}$ inch | $2\frac{4}{8}$ inch | $\frac{20}{8}$ inch |
| 315 | $2\frac{5}{8}$ inch or $\frac{21}{8}$ inch | $2.5\,\text{inch}$ | $2\frac{1}{2}$ inch | $2\frac{4}{8}$ inch | $\frac{20}{8}$ inch |
| ***15*** | $\frac{1}{8}$ inch | | | | |

# Possible approaches/lessons to FBP – Repeated Subtraction Approach

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## Repeated subtraction with decimal approximation:

$$2.5 \text{ inch} - \tfrac{3}{8} \text{ inch} = 2.5 - 0.375 \text{ inch} = 2.125 \text{ inch}$$

$$2.125 \text{ inch} - \tfrac{3}{8} \text{ inch} = 2.125 - 0.375 \text{ inch} = 1.75 \text{ inch}$$

$$1.75 \text{ inch} - \tfrac{3}{8} \text{ inch} = 1.75 - 0.375 \text{ inch} = 1.375 \text{ inch}$$

$$1.375 - \tfrac{3}{8} \text{ inch} = 1.375 - 0.375 \text{ inch} = 1 \text{ inch}$$

$$1 - \tfrac{3}{8} \text{ inch} = 1 - 0.375 \text{ inch} = .625 \text{ inch}$$

$$.625 - \tfrac{3}{8} \text{ inch} = .625 - 0.375 \text{ inch} = .25 \text{ inch}$$

$$.25 - \tfrac{3}{8} \text{ inch} = .25 - 0.375 \text{ inch} = ??$$

– It takes longer than 6(45) minutes… How much?

# Possible approaches/lessons to FBP – Proportional Reasoning

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- Immediately reason that $^{3}/_{8}$ inch every 45 min means $^{1}/_{8}$ inch every 15 min.

- Realizing that 2.5 inch = $^{20}/_{8}$ inch and thinking of $^{1}/_{8}$ as an object rather than a process.

- Note: Some MfA teachers observed this as the most popular approach taken by incoming freshmen.

# Referential Versus Non-referential Symbolic Reasoning

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- Compare these two approaches … both correct:

Approach 1:

Let x be the number of minutes it takes to remove 2.5 inches of water.

$$\frac{\frac{3}{8}\text{ inch}}{45\text{ min}}(x\text{ min}) = \frac{20}{8}\text{ inch}$$

$$x = \frac{20}{3}(45\text{ min}) = 300\text{ min}$$

Approach 2:

$$\frac{\frac{3}{8}}{45} = \frac{\frac{20}{8}}{x}$$

$$x = \frac{20}{3}(45) = 300$$

# FBP's #1

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Some lesson so far:

- Tabulation as a tool helps communicate and organize what I think is happening in this problem
- Representations indicate mental images
- Proportional reasoning can be represented in several ways

# FB's #5

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(5) My neighbor isn't home, so I have to borrow a pump from my brother. My brother's pump pumps out water at a rate 4/5 as fast as my pump. If the two pumps were working together, how many inches of water would be pumped out of the basement at any given moment from the start of their working together?

# What does “4/5 as fast as” mean?

| | Image #1<br>Hold time steady and remove 4/5 as many inches… | Image #2<br>Hold inches steady and take 5/4’s as much time… |
|---|---|---|
| My pump | $\frac{3/8 \text{ inch}}{45 \text{ min}} = \frac{1/8 \text{ inch}}{15 \text{ min}} = \frac{1/2 \text{ inch}}{1 \text{ hour}} = \frac{1 \text{ inch}}{2 \text{ hours}}$ | $\frac{3/8 \text{ inch}}{45 \text{ min}} = \frac{1/2 \text{ inch}}{1 \text{ hour}}$ |
| My brother’s pump | $\frac{4/5 \text{ inch}}{2 \text{ hours}} = \frac{2/5 \text{ inch}}{1 \text{ hour}} = \frac{1 \text{ inch}}{2\,1/2 \text{ hours}}$ | $\frac{1/2 \text{ inch}}{5/4 \text{ hour}} = \frac{2 \text{ inches}}{5 \text{ hours}} = \frac{2/5 \text{ inch}}{1 \text{ hour}} = \frac{1 \text{ inch}}{2\,1/2 \text{ hours}}$ |

Image #2 came about through a student error. Student work looked more like this:

4/5 as fast → my brother’s pump takes 4/5 as long to pump 3/8 inch → $\frac{3/8 \text{ inch}}{4/5 \text{ of } 45 \text{ min}}$

Main Point: We had to discuss the second image in order to address students’ images.

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# To students an answer is …

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- The domain is whole number of hours… Therefore the list is the answers.
- Consistent with image of function as a set of points - Several ways to handle this pedagogically.
- In problem 6 there will be a need to refine this table b/c $^{2}/_{10}$ inch too much water has been pumped out.
- Usefulness of the question, "What about between the hours?" is in problem #6.

| Time (hrs) | Mine (in.) | Bro's (in.) | Both (in.) |
|---|---|---|---|
| 1 | $\frac{1}{2}$ inch= $\frac{5}{10}$ inch | $\frac{2}{5}$ inch $= \frac{4}{10}$ inch | $\frac{9}{10}$ inch |
| 2 | $\frac{10}{10}$ inch | $\frac{8}{10}$ inch | $\frac{18}{10}$ inch |
| 3 | $\frac{15}{10}$ inch | $\frac{12}{10}$ inch | $\frac{27}{10}$ inch |

Problem 6:

- 3 hours → $\frac{2}{10}$ inch too much was removed
- Every hour we remove $\frac{9}{10}$ inch
- In 6 min we remove $\frac{1}{10}$ inch → 12 min removes $\frac{2}{10}$ inch
- So, it takes us 2 hours and 48 min together

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# FB's #5 and 6

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More lessons

- Massage wording: Phenomenon BEFORE label … "At any point in time".
- Student conceptions of function … list vs. process.
- Teacher's role: Attention to domain
- Teacher's role: Point out values

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- For more information on Harel's DNR Theoretical, visit http://www.math.ucsd.edu/~harel
- For more information on Thompson's Theory of Quantitative Reasoning, visit http://pat-thompson.net/
- Flooded Basement (and other holistic problems) http://www.math.ucsd.edu/~harel/projects/Downloadable/Holistic%20Problems.pdf

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