Area and Perimeter

How are the area and perimeter of a rectangle related? You probably know the formulas by heart:

$$Area = Length \cdot Width$$

$$Perimeter = 2(Length + Width)$$

But if you look at data for many different rectangles, would you notice any patterns? Are there relationships between area and perimeter? You'll use Fathom to explore these questions.

EXPERIMENT

The first step is to experiment with many different rectangles and gather data. You could get data by actually drawing or cutting out rectangles and measuring their lengths and widths, but that would get tiring. Instead, you'll use Fathom to generate random numbers for the lengths and widths. Using technology to randomly generate data for an experiment is one method of doing a *simulation*.

1. Open a new Fathom document and create a new case table.

2. Make two new attributes, *Length* and *Width.*

3. With the case table selected, choose **New Cases** from the **Collection** menu. Type `200` in the dialog box and click **OK.** The cases appear in your table, although they have no values.

For this activity, you'll assume that the length or width of a rectangle can be between 0 and 10 units.

4. Select the attribute *Length* and choose **Edit Formula** from the **Edit** menu. Type `random(10)` in the formula editor and click **OK.** The *Length* column fills with randomly generated numbers. If you want, choose **Rerandomize** from the **Collection** menu to see a different set of random numbers.

Q1 Look at the values of *Length.* What does the formula *random*(10) do? What would *random*(100) do? How about *random*()? (*Hint:* If you aren't sure, read about the *random* function in Fathom Help.)

5. Use the same formula to define *Width.*

Graph

6. Drag a new empty graph from the object shelf. Drag *Length* from the case table to the horizontal axis. You'll see a dot plot; you can change the graph to a histogram if you like. Make a dot plot or histogram of *Width,* too.
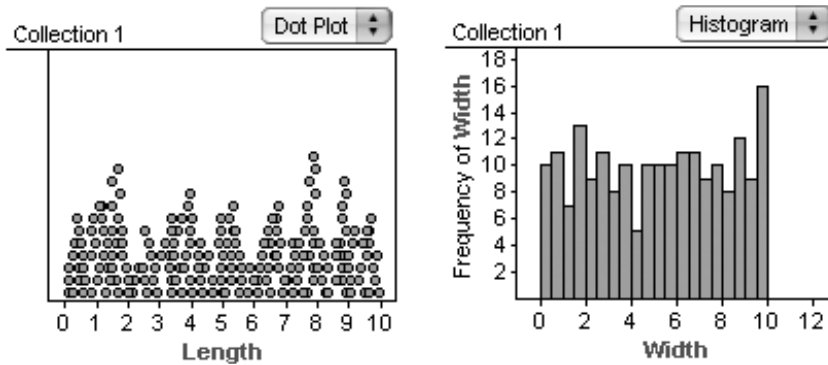
Q2 Describe the general "shape" of the graphs of *Length* and *Width.* If you want, rerandomize the data several times to get a better feel for the general shape. Why do these graphs have this shape?

7. Make another new graph. This time make a scatter plot with *Length* on one axis and *Width* on the other axis. Rerandomize the data a few times.

Q3 Describe any patterns in the scatter plot. Why does the scatter plot look the way it does?

Now that you have data that represent random rectangles, you need to find the perimeter and area of each. After all, you want to look for relationships between area and perimeter.

8. Make two new attributes, *Area* and *Perimeter.* Define each with a formula that uses *Length* and *Width.*

Q4 Copy on paper all of the values for the first two cases in your case table. Use by-hand calculations to verify that your formulas for *Area* and *Perimeter* are correct.

MAKE A CONJECTURE

So far, you've simulated many rectangles by randomly generating length and width, and you've calculated the area and perimeter. You've looked at graphs that help you see how length and width are related. Now, let's see how area and perimeter are related. But first, predict what you expect to find.

Q5 If you made a scatter plot of *Area* versus *Perimeter,* what do you think it would look like? Make a sketch of what you would expect to see or describe it thoroughly. Explain why you think it would look that way.

INVESTIGATE

9. In Fathom, actually make the scatter plot of *Area* versus *Perimeter.* Put *Perimeter* on the horizontal axis and *Area* on the vertical axis. You might want to rerandomize a few times to make sure you understand the general relationship.

Q6 Make a sketch of the actual graph or describe it thoroughly. Did the actual graph match your prediction? Is there anything about the graph that surprises you? Explain.

Q7 If everything went well, there should be no points in the upper-left part of the graph—a region bounded by a curve. Explain why there are no points in that region.

Now your goal is to figure out the function for that curved boundary. Because calculating area involves squaring, you might reason that the function is quadratic: *Area = Perimeter²*.

10. Select the graph and choose **Plot Function** from the **Graph** menu. In the formula editor, type `Perimeter²` and click **OK.**

11. Unfortunately, the curve is too steep. Double-click the function equation at the bottom of the graph. Change the formula to divide *Perimeter²* by some number, such as *Perimeter²*/3.

Q8 What equation did you try? What happened to the curve—did it get steeper or shallower? Do you think the divisor needs to be bigger or smaller than the number you used?

You can use a slider to try lots of divisors quickly.

12. Drag a new slider from the object shelf. The slider that appears looks like a number line with a pointer on it. It's probably called *V1* and has a value of 5.
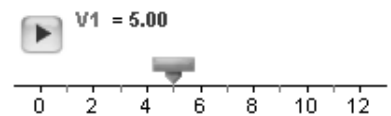
13. Double-click the function equation at the bottom of the graph and change it to use *V1* as the divisor. Now you can control the value of the divisor and change the function by dragging the slider. You will probably need to change the scale of the slider. The easiest way is to drag the numbers on the number line.

Q9 What value of *V1* makes the curve fit the boundary the best?

14. Double-click the collection (the box of gold balls) to show the collection inspector. Then go to the scatter plot of *Area* versus *Perimeter* and click any of the points nearest to the curved boundary. When you do, that case appears in the inspector. Do this for several of the points and notice the *Length* and *Width* of each point.

Q10 What do the cases near the boundary have in common? Why do all those rectangles have that relationship? What type of rectangles are they?

Q11 Use your observation from Q10 and the formulas for area and perimeter to find the boundary curve's function algebraically. Show that it is the same formula you got by using the slider to fit the curve.

EXPLORE MORE

1. There is another empty zone in the lower-right part of the scatter plot of *Area* versus *Perimeter*. Why are there no points in that region? What is the equation of that boundary? Why? Use this and your answer to Q11 to completely describe the relationships between area and perimeter.

2. Look at scatter plots of *Area* versus *Length* and *Perimeter* versus *Length*. Are there maximum and minimum relationships between these attributes? If so, find functions for the boundaries and explain why they exist.

AREA AND PERIMETER (PAGE 29)

Activity Time: 40–55 minutes

Fathom Prerequisites: Students should be able to

- Create a collection by entering data into a case table
- Define attributes with formulas
- Make two-variable graphs (scatter plots)

Fathom Skills: Students will learn how to

- Add cases to a case table without adding data
- Use random numbers to simulate data
- Rerandomize data
- Make one-variable graphs (dot plots, histograms)
- Plot functions
- Create a slider and use its value as a parameter

Mathematics Prerequisites: Students should be able to calculate the area and perimeter of a rectangle; make a scatter plot of points; and graph linear and nonlinear functions.

Mathematics Skills: Students will learn how to simulate an experiment using technology; describe the distribution of one-variable data; fit curves (specifically, a parabola) to data; use algebra to mathematically explain a geometric situation; and see boundaries (limits) from a graph.

General Notes: In this activity, students simulate random rectangles and look for relationships between length, width, area, and perimeter. The activity takes students significantly deeper than the customary formulas $A = lw$ and $P = 2(l + w)$. It exposes students to statistics, algebraic modeling, and limits, and shows how these diverse areas of mathematics can be used to support each other. The activity also touches on the concept of optimization—How do you maximize area without maximizing perimeter?—which has many real-world applications, such as packaging design and manufacturing. That is, it illuminates the geometric concept that a square maximizes area for any perimeter.

As presented on the student worksheet, students use random numbers to simulate creating random rectangles. Before jumping into the Fathom activity, you may prefer to

have students draw rectangles by hand, cut them out, and measure the lengths and widths. A quick way to do this is to give each student a piece of 8.5-by-11-inch paper and ask them to make three straight cuts to form three random rectangles. If you make a lot of rectangles as a class (say, 100 or more), you could enter your real data into Fathom and analyze the data in addition to or as a replacement for the computer simulation.

When students make the scatter plot of *Area* versus *Perimeter* in step 9, make sure that they understand what the position of each point represents in terms of a physical rectangle. Ask questions such as "What do rectangles in the lower-left look like?" (small in both dimensions) or "What do rectangles in the lower-right look like?" (long and skinny). Although the student worksheet eventually asks, "What type of rectangles are [along the curved boundary]?," it is worth asking this at step 9 as well, to make sure that students see them as squares.

A challenging discussion question is "Why do you suppose so many of the points are close to the curved boundary?" Large-area rectangles must be close to being squares; otherwise, they'd have a dimension larger than 10. For smaller-area rectangles, you can have a surprisingly large length-to-width ratio before the rectangle moves a long way from the border. Statistically speaking, you should get more squarish rectangles because the distribution of the quotient of two uniform random variables, larger to smaller, is greatly skewed toward 1. In this case, that means the majority of values for *Length* ÷ *Width* will be close to 1; so, the majority of rectangles will have about the same length as width, or be close to a square.

Although the context of this activity is geometric, it is also an excellent activity for algebra, probability, and statistics. Algebra and Precalculus students can practice creating scatter plots, fitting linear and quadratic functions, identifying limits, and algebraically proving why a particular function fits a set of data. Statistics students can practice using random numbers, performing simulations, comparing experimental versus theoretical probability, generating frequency distributions, and describing (and justifying) the shape and spread of distributions.

EXPERIMENT

Q1 The formula *random*(10) creates random numbers between 0 and 10; *random*(100) would create random numbers between 0 and 100; *random*( ) would create random numbers between 0 and 1. In general, the function *random* creates a random number between 0 and 1, which is then multiplied by the number in parentheses.

Q2 Technically these distributions are called *rectangular*. Because each random number between 0 and 10 has the same probability, stacks or bins of about the same height are created. (*Note:* Rather than making two separate graphs, students familiar with Fathom's ability to put multiple attributes on one axis might combine the dot plots or histograms, as shown in the third graph.)
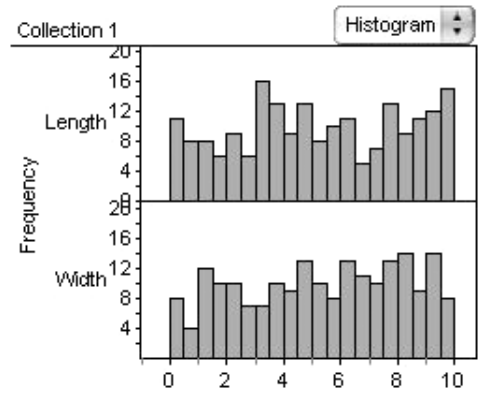
Q3 Students should somehow mention that the scatter plot is a random arrangement of points between (0, 0) and (10, 10).
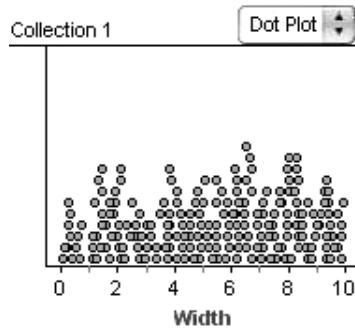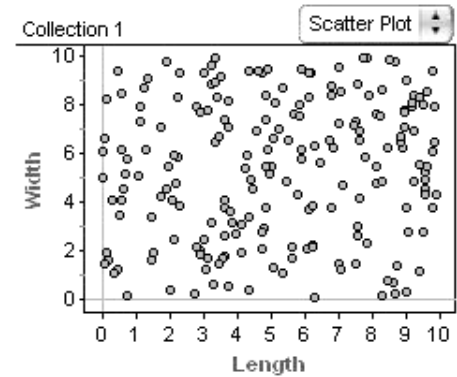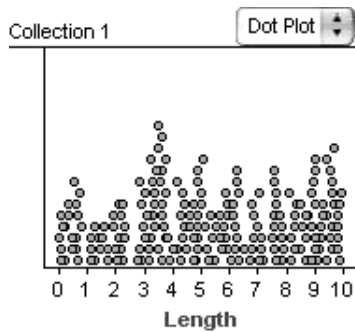
Q4 Students should use the formulas from the introduction to the worksheet: *Area = Length • Width* and *Perimeter = 2(Length + Width)*. Students should verify the formulas by showing hand calculations. For the first case in the table below, 9.42286 • 5.50248 = 51.8491 and 2(9.42286 + 5.50248) = 29.8507.

|	Length	Width	Area	Perimeter
1	9.42286	5.50248	51.8491	29.8507

MAKE A CONJECTURE

Q5 Answers will vary.

INVESTIGATE

Q6 The actual scatter plot has a curved upper boundary and a linear lower boundary, as shown on the next page. This probably is not what students expected,

especially after having seen *Width* versus *Length,* which had no discernible shape. Students will probably be surprised to see that the points form such clear boundaries. Even seasoned geometry students who know that squares maximize area and who might expect the parabolic curve may be surprised by the linear lower boundary.
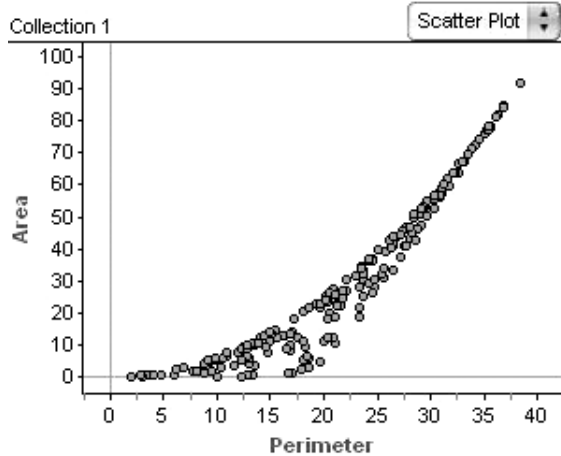
Q7 There are several ways to explain why there are no points in the upper-left region. An intuitive approach is to say that such a rectangle would have a big area and a small perimeter, which is impossible. For a given perimeter there's only so much area you can enclose (and some students may already know that a square maximizes area). For example, with a perimeter of 20 units, the most area you can enclose is a 5-by-5 square, or 25 square units. Therefore, the region above that limit is empty. Or, for a given area there's a minimum perimeter that encloses it. For example, with an area of 25 square units, 20 units is the smallest perimeter possible.

Q8 If students try a number between 0 and 1, the curve will get steeper, so they need to try larger numbers. If students try a number between 1 and 15, the curve will get shallower, but they'll still need to try a larger number. A few students might try 16, which will fit perfectly. If students try numbers greater than 16, they'll overshoot the boundary and will need to try smaller numbers. Students probably won't think to try negative numbers, but if they do, the graph will be reflected across the horizontal axis.

Q9 *V*1 = 16 makes a perfect fit.
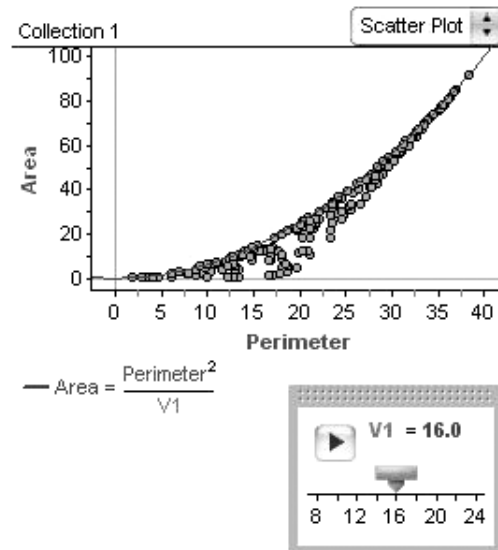
Q10 The rectangles near the boundary have (approximately) the same length and width, which means they are squares or almost squares. Squares form the upper boundary because squares maximize area for any given perimeter.

Note: You'll probably want to discuss how Q10 is directly related to Q7.

Q11 The area of a square is *Area* = *Length*^{2}. The perimeter of a square is *Perimeter* = 4 · *Length,* or *Length* = $\frac{Perimeter}{4}$. Substituting for *Length* gives *Area* = $\left(\frac{Perimeter}{4}\right)^2$, or *Area* = $\frac{Perimeter^2}{16}$.

EXPLORE MORE

1. The empty region below the points exists because the maximum values of *Length* and *Width* limit the sizes of the rectangles. For example, if *Perimeter* were 30 units, then *Area* could not be 0 square units because then either *Length* or *Width* would have to be 15 units, yet the maximum for either is 10 units. This means there is a minimum area for any given perimeter. To minimize *Area* for a given *Perimeter,* two of the sides (say, *Length*) have to be as long as possible, 10 units each and 20 units combined, and the other two sides (say, *Width*) have to be as short as possible, or *Width* = $\frac{Perimeter - 20}{2}$. Then the

minimum area is *Area = Length • Width = 10 •* $\frac{Perimeter - 20}{2} = 5 \cdot Perimeter - 100$. Students could also find this linear equation by using a movable line:

Collection 1 — Scatter Plot
Area = 5Perimeter − 100
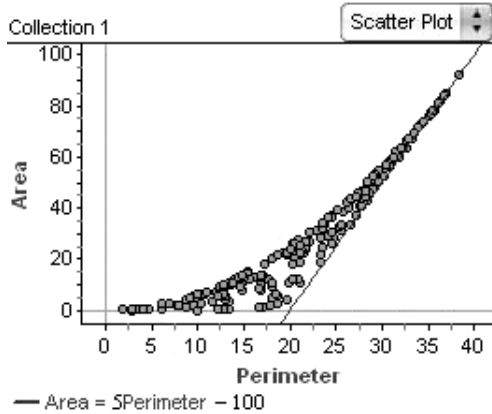
2. *Area* versus *Length* has one linear upper boundary: *Area* = 10 • *Length*. This maximum occurs when *Width* is as large as possible, or 10 units.


Collection 1 — Scatter Plot
Area = 10Length
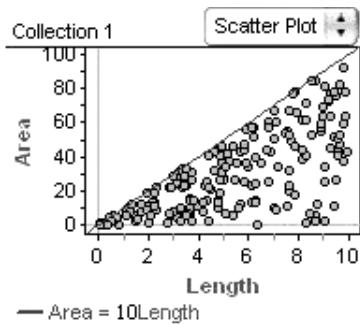
Perimeter versus *Length* has a linear upper boundary and a linear lower boundary: *Perimeter* = 2 • *Length* + 20 and *Perimeter* = 2 • *Length*. The maximum again occurs when *Width* is as large as possible, or 10 units. The minimum occurs when *Width* is as small as possible and can still make a rectangle, or very close to 0 units.


Collection 1 — Scatter Plot
Perimeter = 2Length + 20
Perimeter = 2Length
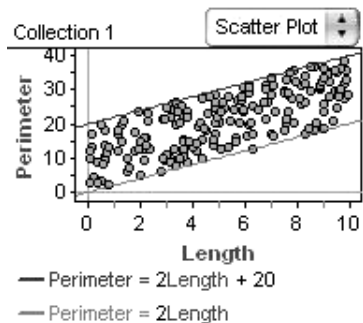
EXTENSIONS

1. Have students look at histograms of *Area* and *Perimeter,* describe the shape of these graphs, and explain why these shapes make sense. This is a particularly good extension for Statistics students.

   The graph of *Area* is skewed right, which means that smaller areas are more likely than larger areas. This makes sense because area is a product, and to get a large product you need two large factors; but because the length and width are randomly generated, it is less likely to get two large numbers paired together. As a simple example, imagine that length and width could each be 1 or 10. Then 1 • 1 = 1 (small), 1 • 10 = 10 (small), 10 • 1 = 10 (small), and 10 • 10 = 100 (large).

Collection 1 — Histogram
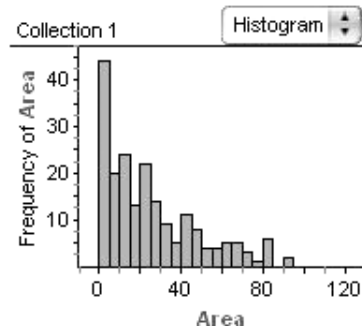
   The graph of *Perimeter* is triangular, which means that perimeters in the middle (between 15 and 25 units) are more likely than small or large perimeters. This makes sense because perimeter is a sum, and to get a small or large sum you need two small or large addends. Imagine the same simple example: 2(1 + 1) = 4 (small), 2(1 + 10) = 22 (medium), 2(10 + 1) = 22 (medium), and 2(10 + 10) = 40 (large).

Collection 1 — Histogram
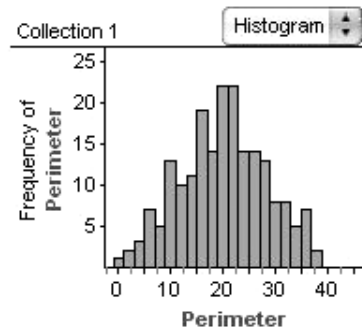
After students have created histograms of *Area* and *Perimeter,* have them select any one bin in either histogram. Have them observe where the points are in the scatter plots of *Length* versus *Width* and *Area* versus *Perimeter.* (If they did Explore More 2, also look at the location of points in *Area* versus *Length* and *Perimeter* versus *Length.*) Ask students to describe the patterns that they see as they gradually select consecutive bins in the histogram. Challenge them to explain why these patterns exist.
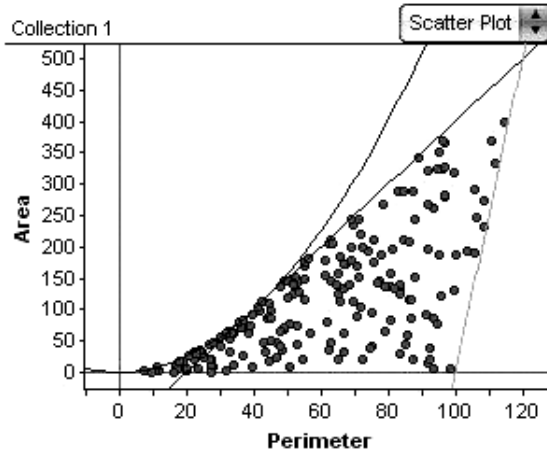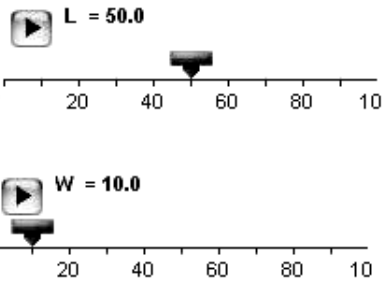
2. Have students explore what happens when they change the formulas for *Length* and *Width* to a different random number, say, r*andom*(5) or r*andom*(100), or what happens when they are different formulas, say, *Length = random*(10) and *Width = random*(5). Ask students to first predict how the scatter plot of *Area* versus *Perimeter* and its boundaries would change, then actually try it in Fathom. Challenge students to generalize the results to describe what happens for *Length = random*(*L*) and *Width = random*(*W*). This is a particularly good extension for Algebra students, because it involves generalizing functions for different parameters (see below).
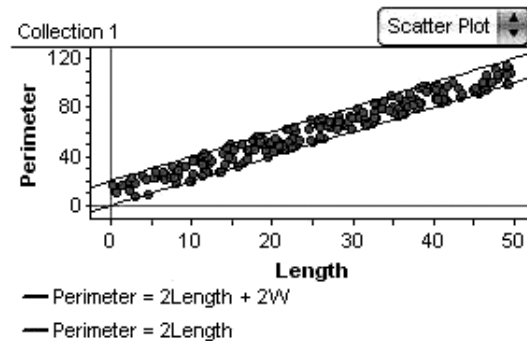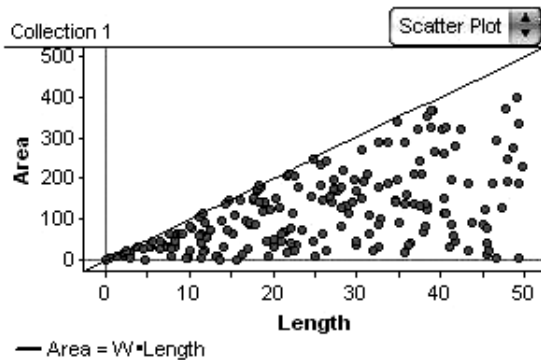
Area and Perimeter, Extension 2

Collection 1

	Length	Width	Area	Perimeter
=	random (L)	random (W)	Length•Width	2 (Length + Width)
1	19.7112	6.27622	123.712	51.9748
2	45.0115	1.90632	85.8064	93.8356
3	37.0252	1.76616	66.0810	78.3837

L = 50.0

W = 10.0

$$\text{Area} = \frac{\text{Perimeter}^2}{16}$$

$$\text{Area} = \frac{if\ (L > W)\begin{cases}W\\L\end{cases}}{2}\ \text{Perimeter} - if\ (L > W)\begin{cases}W^2\\L\end{cases}$$

$$\text{Area} = \frac{if\ (L > W)\begin{cases}L\\W\end{cases}}{2}\ \text{Perimeter} - if\ (L > W)\begin{cases}L^2\\W\end{cases}$$

Area = W•Length

Perimeter = 2Length + 2W

Perimeter = 2Length

If *Length* = *random*(*L*) and *Width* = *random*(*W*), the results are similar to the results throughout this activity, except that many of the boundary functions are scaled accordingly. For the graph of *Area* versus *Perimeter,* part of the upper boundary is always *Area* = $\frac{Perimeter^2}{16}$ (and this is the only upper boundary if *L* = *W*). But if *L* ≠ *W*, a second upper boundary comes into play: *Area* = $\left(\frac{\text{lesser of } L \text{ or } W}{2}\right)$ • *Perimeter* − (lesser of *L* or *W*)^2. The lower boundary of *Area* versus *Perimeter* always depends on the greater of *L* or *W*:
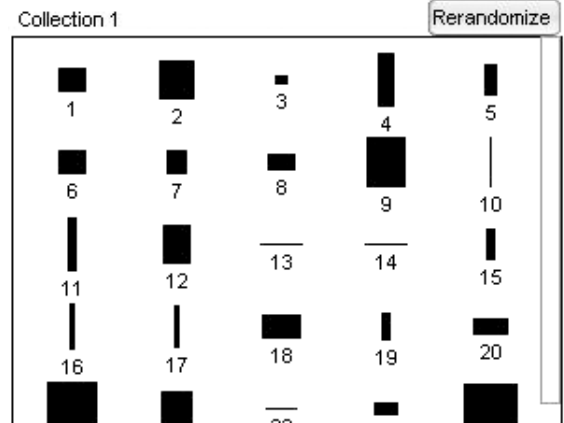
$$Area = \frac{\text{greater of } L \text{ or } W}{2} \cdot Perimeter -$$
$$(\text{greater of } L \text{ or } W)^2$$

Students might also notice that the upper boundary of the scatter plot of *Area* versus *Length* is always *Area* = *W* • *Length.* For *Perimeter* versus *Length,* the boundaries are always *Perimeter* = 2*Length* + 2*W* and *Perimeter* = 2*Length*.

Students can create sliders for *L* and *W* and confirm that their boundaries always work.

3. Statistics students can explore sample size by adding and deleting cases to see how the histograms and scatter plots compare for 10, 20, 50, 100, 200, 500, and 1000 cases. At what sample size can they begin to discern important patterns in the data? Are there pros and cons to using larger sample sizes? Students will probably find that they need at least 50 or 100 cases before they can describe patterns in the data. Even more cases, 500 or 1000, help "fill in" the patterns in the scatter plots, but don't help them see the pattern any better than smaller numbers.

4. For fun, show students how they can use the collection as a geometric representation of the rectangles. Drag the bottom corner of the collection to expand it. You'll see a gold ball for each rectangle. Double-click the collection to show the inspector and use the settings shown at right on the **Display** panel. The gold balls will change to rectangles.

If you asked the question "Why do you suppose so many of the points are close to the curved boundary?," students can now see that many of the rectangles are indeed close to being squares.