Expressing *Covariation* and *Correspondence* relationships in elementary schooling

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Abstract: *We focus on development of students' early expression of covariation and correspondence (functional) relationships through instructional tasks supporting generalization of pattern relationships. We present a teaching experiment conducted in a 5th grade classroom, and explore students' expressions of those relationships. Implications for CCSS-M implementation and research are also discussed.*

A focus on Early Algebra

Present in all the definitions of algebra and algebraic reasoning is the concept of *generalization*. Students' generalizing should become more sophisticated over time as their focus gradually shifts "from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables." (Carraher, Schliemann & Schwartz, 2007, p. 266). That is, students come increasingly to recognize the "general through the particular" and "the particular in the general" (Mason, Graham & Johnston-Wilder, 2005).

The aim of early algebra instruction in elementary schools include providing students the opportunity to establish the foundations for development of algebraic reasoning in middle and high school and building proficiency in constructing and interrelating different representations of mathematical situations, especially those involving unknown quantities. Longitudinal studies in classrooms have shown that elementary school students are capable of (a) representing unknown quantities using variables and solve equations with variables on both sides of the equality; (b) thinking about relations among sets of numbers and measures and describe and represent relations among variables; (c) solving problems using multiple representations such as tables, graphs, and equations and inter-relate different representations of functions (Brizuela & Earnest, 2008; Brizuela & Schliemann, 2004; Carraher & Schliemann, 2007; Carraher, Schliemann, & Schwartz, 2007; Carraher, Martinez & Schliemann, 2008; Schliemann, Carraher, & Brizuela, 2007; Schliemann et al., 2003).

The Common Core State Standards for Mathematics (CCSS-M, 2010) specify that 5th Grade students should be able to identify relationships between two patterns--what we may refer to as functional relationships (Standard 5.OA.B.3). Conventional functions curricula are dominated by defining relationships between two patterns via a rule, describing how to find y or $f(x)$ given a particular value for x (i.e. $y = 4x + 1$). This type of relationship is called a "*correspondence*" description of the relationship between two patterns (Confrey & Smith, 1995). Seeking a more balanced approach to the function concept, Confrey and Smith (1995) suggested that a second type of functional relationship, the "*covariation*" between quantities, should also be emphasized. A covariation description of the relationship between two patterns shows how a quantity in one pattern changes at the same time as a quantity in the other pattern changes: how x_1 changes to x_2 and how y_1 changes to y_2 simultaneously. Explaining how the values of the two patterns change simultaneously is foundational for the development of the correspondence rule (Blanton & Kaput, 2011; Confrey & Smith, 1991, 1994, 1995).

Elementary students are capable of reasoning about covariation and correspondence relationships (Blanton, & Kaput, 2004; Martinez & Brizuela, 2006; Stephens et al., 2012), so a challenge is to synthesize research on early algebra to inform the design of instructional tasks that promote the development of both types of relationships. The aim of the study described here is to satisfy teachers' requests for "creating a 'story' that illustrates how the ideas are likely to evolve in the minds of students when they are provided appropriate curriculum tasks, instruction, and opportunities for discourse" (Confrey 2012, p. 4).

The design study

Following this theoretical framework, we designed a sequence of tasks that would stimulate students to reason about and distinguish between covariation and correspondence relationships (Figure 1). We intended to examine how students' thinking progresses with regard to these relationships, using a sequence of easier to more sophisticated tasks, each represented in a different contextual problem.

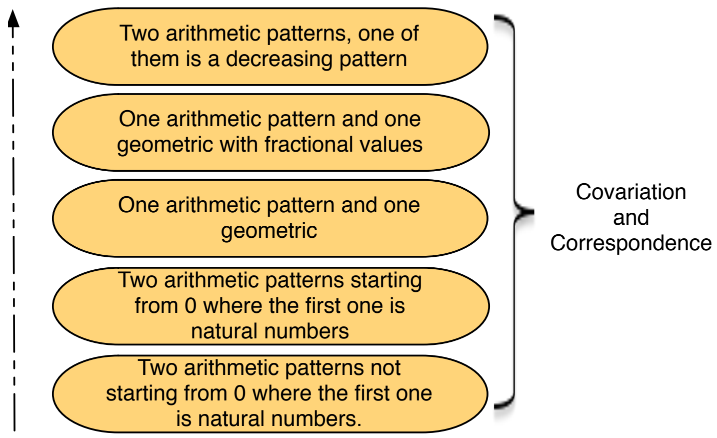

Figure 1: Sequence of task types

The purpose of this structure was for students to gain experience exploring covariation and correspondence relationships, using a variety of arithmetic and geometric sequences, all of which lay a strong foundation for the development of ratio and functions in middle school (Carraher & Schliemann, 2007; Zazkis & Liljedahl, 2002). Each contextual task included opportunities for students to describe relationships both within and between numerical patterns, and to establish distinctions between covariation and correspondence relationships, using models, such as diagrams and tables, verbally, symbolically (using letters as variables), and using graphical representations.

The curriculum used was based on the Early Equations and Expressions learning trajectory presented in the TurnOnCCMath.net project (Confrey et al., 2011), which unpacks the contents of the CCSS-M standard 5.OA.B.3. The descriptor introduces the distinction between covariation and correspondence relationships through contextual problems, aiming to "situate and deepen the learning of mathematics and generalizations and the use of multiple representations" (Carraher, Martinez, & Schliemann, 2008, p. 6).

Using a design research approach (Cobb, Confrey, Lehrer, & Schauble, 2003), we conducted a 6-day teaching experiment in a classroom of eighteen 5th graders in an elementary school in North Carolina. The students had already had some experience with arithmetic and geometric patterns, little experience in plotting ordered pairs on coordinate plane, and no formal instruction on expressing relationships between patterns according to their regular classroom teacher. Data from this work were comprised of video recordings and observation notes from classroom interactions during instruction, and student written work.

In this paper we present different strategies, justifications, names and mathematical relationships that students used as they engaged with one of the contextual problems, one involving two arithmetic patterns beginning from 0 (Figure 2).

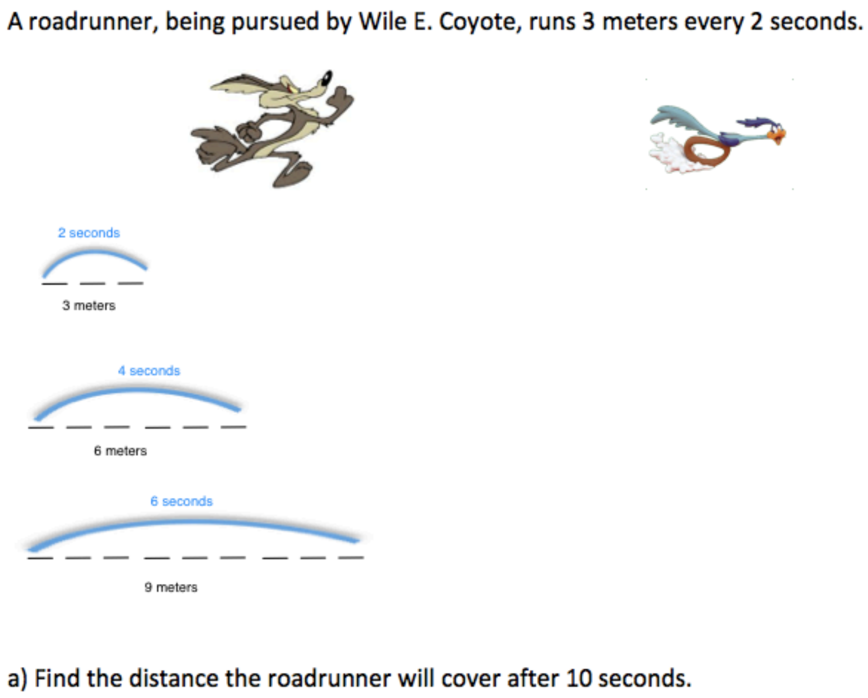

Figure 2: The Roadrunner Problem

The Roadrunner Problem was the third contextual problem presented to students. All contextual problems followed the same structure. Students were told they had to develop a strategy for solving the problem, and no instructional guidance was initially provided with regard to solution strategies (see question *'a'* above). The aim of the introductory question was to examine students' prior knowledge (which now included their work on the preceding two tasks) and to learn what strategies they would begin with if no further guidance was provided. Students typically extended the shape/drawing pattern given to them, used calculations, or drew tables. Subsequent questions were designed for students to gain experience in reasoning about the relationships with multiple representations: students were asked to complete a table, express the relationships in words, apply those relationships to find a different value of distance, and represent those relationships on a dynagraph and a coordinate graph.

Expressing Covariation and Correspondence

In the following paragraphs we present examples of students' expressions of the covariation and correspondence relationships as they described those using tables, using words (written language), using variables, and comparing representations (dynagraphs and graphs):

<u>Functional relationships in tables</u>: For each task, "relationships" tables were offered as a tool for solving the problems. These were incomplete tables that included column breaks similar to Schliemann et al.'s function tables (2001). We also added a middle column, the "Rule", for students to describe the correspondence relationship between the two patterns and we introduced the two types of relationships by showing them visually on the tables (Figure 3).

Correspondence

Covariation

Time in *s* (seconds)	Rule	Distance covered in *m* (meters)
0		
2		3
4		
6		
		12
10		
		21
30		
100		
t		

Covariation

Figure 3: "Relationships" tables.

Students first identified the “rule” of each pattern independently (e.g. the first column’s values increase by 2, the second’s increase by 3). This involved their recognizing that the patterns change simultaneously, suggesting a *covariation* relationship (i.e. the time quantity increases by 2 as the distance increases by 3). This strategy sufficed for completing the table down to a column break. The breaks in the columns, where values of the patterns increased by more than a single increment, were intended to draw attention to the relationships *between* the two patterns (i.e. to find the distances corresponding to the times 30s and 100s (Figure 3) it might be easier and, in the long run, more efficient to generate a more general strategy to relate the two patterns directly, instead of identifying every value in both patterns). Students were able to find the distance corresponding to 30s using covariation but this strategy proved inefficient for finding the distance when time was 100s. Some students continued with a covariation strategy, but most tried to identify a rule that related the two patterns, as a shortcut. Some students considered a recursive rule (e.g. in the example above +1, +2, +3, etc.), but noticed that it was not helpful for finding higher values, because this rule is not the “same” (invariant) for each corresponding pair of numbers.

Functional relationships in words: The table served as a starter for class discussion about expressing relationships between the two patterns in words. Students generally found the covariation approach easier and more intuitive. Many students expressed both relationships verbally, which helped them to draw distinctions between the two types of relationships (Table 1):

Table 1: Examples of students' expressions of relationships in words

Explanation	Student sample
Covariation & Correspondence	For covariation, for the time it adds by two while the Distance adds by three. For correspondence you multiply the time and divide it by two which equals the distance.
Correspondence only	The distance goes up by the time in seconds x3 ÷2.
Covariation only	time | Distance +2 | +3 The right is added by 3 The ~~left~~ is added By 2 left

Expressing correspondence using variables: Verbal descriptions of the relationships act as the connecting link between context and symbolic representation (Van de Walle, Karp, & Bay-Williams, 2013). We introduced word and then letter representations as a shortcut asked the students to write equations to represent their relationships in words. Though 5th-grade students are not expected by the Common Core Standards to write symbolic equations for these types of relationships, our experience was that students were capable of expressing the correspondence relationship using words and letters as variables. Some students created equivalent expressions after finding the rule (Table 2).

Table 2: Examples of students' expressions of correspondence using variables

Level	Description	Student Sample
3	Using letters and creating equivalent expressions	d) Describe your relationship u represent? T= time D= distance. T×3÷2=D T÷2×3=D
2	Using letters as variables	t×3÷2=D D = any number of distance t = any number of time
1	Using words as variables	The distance goes up by the time in seconds ×3 ÷2.

Expressing relationships in different representations: Considering it important that students use multiple representations to compare covariation and correspondence relationships, we asked students to represent the function table values as ordered pairs on a dynagraph and a coordinate graph, and describe what they saw in those representations. Dynagraphs and coordinate graphs both model the one-to-one correspondence of the functional relationship, and illustrate linear and increasing/decreasing qualities of such relationships. Most of the students described the covariation relationship in a graph qualitatively or quantitatively. A few of them mentioned the correspondence rule as defining the "line" created in the graph; some described both relationships (Table 3).

Table 3: Examples of students' expressions of relationships in graphs

Level	Description	Student sample
4	Correspondence and Covariation	What do you notice? Describe what the graph shows. I notice that the time is increasing the distance is increasing What do you notice about the change in time and the distance covered? Describe what the graph shows. Time x 3 ÷ 2 = Distance
3	Quantitative Covariation and viewing the relationship as a line	What do you notice about the change in time and the distance covered? Describe what the graph shows. The time increased by two and the distance increased by three. The whole line is growing in a positive direction...
2	Quantitative Covariation	What do you notice? Describe what the graph shows. The Time is going by 2. The Distance goes by 1.
1	Qualitative Covariation	What do you notice? Describe what the graph shows. I notice that the time is increasing the distance is increasing

<u>Comparing representations:</u> Each representation--table, equation, graph, or dynagraph--provides a different way for students to examine and compare the relationships. By the end of the teaching experiment, students were able to describe whether a dynagraph and graph involved arithmetic or geometric patterns and whether they were increasing or decreasing. While comparing the different graphs created throughout the teaching experiment, some of the students' generalizations included "*If it is arithmetic it is going in a angle while if it is geometric if it going in a curve*" (in graphs), or "*If it is decreasing/increasing [arithmetic], the line is going down/up*", or "*Curves involve multiplying or division.*" Although these are not sophisticated

mathematical generalizations, we anticipate that identifying connections between those representations may lead to deeper understanding and flexible reasoning about functions at a later stage (Carraher, Schliemann, Brizuela, & Earnest, 2006; Hackbarth & Wilsman, 2008; Van de Walle et al., 2013).

Discussion and Implications

This study aimed to give additional insight into students' conceptual development of functional relationships based on research on student learning, facilitating the use of distinct but related strategies (covariation and correspondence), models, and representations. It showed that students were able to describe both the covariation and correspondence relationships of two patterns. We also have noticed that the more contextual problems the students were solving, the more they seemed to answer the first question (the one examining prior knowledge) using covariation and/or correspondence strategies.

We believe contextually-based instructional experiences based on describing covariation and correspondence relationships between two patterns can support students in developing robust understanding of functional relationships. As Confrey (2012) argued "the success of the CCSS-M rests on its potential to support alignment, including curriculum, assessment (formative and summative), and professional development, at a level not previously possible" (p. 4-5). Consequently, we believe that by providing instructional opportunities based on those findings, teachers can support students in developing functional relationships through contextual tasks. These experiences may establish a robust foundation for more advanced understandings of functions at a later stage.

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