NCTM Regional Conference Houston: Burst Session: Three Models to Preview Calculus Concepts, Grades 6-12

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Fill in the Vocabulary Knowledge Rating and survey individually. For the numbered columns, check 4 if you could explain the concept to the group; 3 if you know it but….; 2 if you have heard of it; and 1 if you do not know it at all.

Calculus Topic	4	3	2	1	^ ^	I preview these in my classes	The topic should be previewed in MS math
Vocabulary Knowledge Rating					^		X
Derivatives					^		X
Definite Integrals					^		X
End behavior of graphs					^		X
Extrema problems					^		X
Asymptotic behavior					^		X
Differentiable functions					^		X
Graph (function) analysis					^		X
Limits					^		X
Reimann sums					^		X
Mean Value Theorem					^		X
Inflection points					^		X
Average rate of change					^		X
# of checks					^		
Total score							

Previewing Differential Calculus

Distance vs. Time story problem:

John is standing at the classroom door. He decides to move the 4 feet to the water fountain. It takes him 2 seconds to do that. For the next 8 seconds, he drinks and stands at the fountain. He returns to the door in 1 more second. Draw a graph of the situation.

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Could your students draw the graph of this problem?

Stop and think of the questions your text would ask about the graph....

Questions for MS and/or early HS:

1. What are the units of the rate of change of the three segments?
2. What does this tell you about the situation?
3. When is John moving the fastest? Justify your answer without any calculations.
4. What is his speed at 1 second? At 4 seconds? At 10.5 seconds?
5. Is the average speed from 0 to 10 seconds greater or less than the average speed from 0 to 4 seconds? Use the graph to justify your answer. How else could you justify your answer without using value of the rate of change?
6. What is the average speed from 0 to 10 seconds? From 0 to 4 seconds?
7. Is there a time when John's instantaneous speed is equal to his average speed from 0 to 10 seconds? If so, when does that occur?
8. What is his speed at exactly 2 seconds? Justify your answer
9. What Calculus topics did we preview?

Solutions for the Distance vs. Time problems:

1. The units of the rate of change are yards per second.
2. Rate of change tells you the velocity, speed if you don't care about the direction of the movement.
3. He was moving fastest when he returned to the door, since the line is the steepest during the last second.
4. His velocity {speed, if all answers are positive} was 2 ft/sec.)@ 1 sec.; 0 ft./sec @ 4 sec.; and -4 ft./sec. at 10.5 seconds.
5. The average speed from 0 to 10 seconds is less than the average from 0 to 4 seconds since the slope of the line to 10 seconds is less steep.
6. The average speed from the start to 10 seconds is $\frac{4}{10}$ or $\frac{2}{5}$ ft./sec.; from 0 to 4, seconds, average speed is 1 ft./sec..
7. There is no place on the graph where the slope of the graph equals the slope of the segment from 0 to 10 seconds.
8. The speed from 0 to anything short of 2 seconds is 2ft./sec.. The speed any tiny amount after 32 seconds is 0 ft./sec.. So, according to the model, you can not say what his speed is at exactly 2 seconds. This previews the concept of non-differentiable functions. (Functions that have angles (or cusps) cannot be differentiated at the vertex.) You could point out the model is inaccurate, that you *cannot* go from 2 ft./sec. to a stop instantaneously. If you wanted to discuss a different modeling function, you could make the graph curve at 2 and 10 seconds instead of having an angle where the segments meet. That would let you argue that the person might be slowing down before 2 seconds, but did not come to a complete stop until after 2 seconds, so it would be possible to find the speed at exactly 2 seconds if you had an accurate graph or the function equation.
9. This previews: average rate of change; derivatives, differentiable and non-differentiable functions; graph analysis; limits; and delta epsilon proofs; and it sets up the Mean Value Theorem preview.

Non-linear Problem:

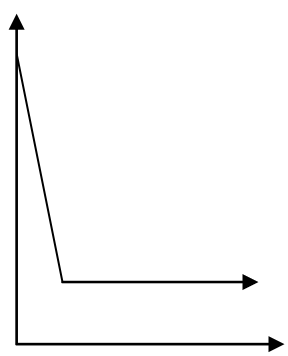
You boil water to make hot tea, but decide it is so warm outside that you want iced tea instead. You fill a cup with ice cubes and pour in the hot tea.

Sketch the graph of the tea temperature vs. time for 3 hour, starting when you pour the hot tea into the cup of ice.

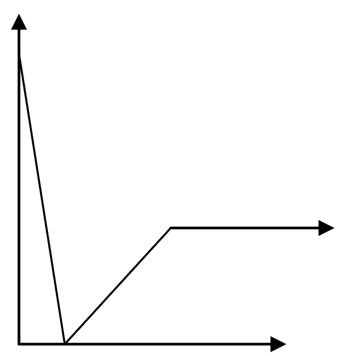
Can your students draw the graph?

What questions would you ask your students?

Graph issues: Here are two segment graphs, but students from late middle school on should be encouraged to draw smooth curve graphs, Some things to watch for in either segment or smooth curve graphs:



or



The left one never warms up after all the ice melts. Some students do not have the graph level off. You probably should point out that the temperature change would be curved, not line segments. Ask the person who drew the graph on the right what their minimum temperature is. If they say it is 0 or near 0, ask how it got below freezing if they used degrees F, and how it got that cold if they used degrees C. You may have graphs that are curved, but do the same as the segment graphs above.

1. What are the units of the rate of change of the problem?
    a. Degrees per minute.
2. Where is the temperature changing the fastest? Why?
    a. Where the graph is the "steepest", which will vary with each person.

3. What does the Y-intercept represent?
    a. The initial temperature -- when you first starting taking the temperature. This value will be different depending on a number of factors, including whether you used F or C for the degrees, and how long after you put the tea in the ice and started taking the temperatures.
4. If you did not label it, label your Y-intercept.
    a. What is your value?
        i. Answers will vary, possibly widely.
    b. What can you tell us if your Y-intercept is 190 and someone else's is 98?
        i. Most likely, the 190 is degrees F, while the 98 is probably someone who used degrees C. Also, the 98 degrees probably meant the water was very close to boiling, or if they used F, they waited a long time before putting the thermometer in the tea.
5. What is the maximum temperature?
    a. Answers will vary, possibly widely.
6. Can you use your graph to predict the temperature 10 seconds before you started your graph? What do you predict it was? What about 10 minutes prior?
    a. You might be able to predict 10 seconds before, but probably not too accurately, since the ice would dramatically impact the temperature change. Predictions will vary. For 10 minutes, you might get answers from room temperature up to boiling! Be sure to ask why they chose the temperature they chose.
7. Where does the minimum temperature occur? Label the point.
    a. Answers will vary, possibly widely.
8. Why is there a minimum temperature?
    a. After all the ice melts, the tea will warm up to room temperature.
9. When is the temperature falling? Rising?
    a. From the start to the low temperature, it is falling. It remains relatively steady for a short time after the ice melts, since water is a heat reservoir. Then, it warms up until it reaches room temperature. You can

decide whether or not you want to discuss that there are slight fluctuations in “room temperature” or not.

10. Where is the rate of change increasing? Decreasing?
    a. The rate of change is actually increasing most of the time the temperature is dropping. As more and more ice melts, the temperature drop becomes less severe. The rate of change is also increasing from when the tea begins to warm up until it gets to the inflection point of the logistic curve for the rising temperature part of the graph.   After the inflection the rate of change is decreasing until it hits room temperature.
11. Can you find the point(s) where the *rate of change* changes from increasing to decreasing and/or decreasing to increasing? (This is called an inflection point.)
    a. There is at least one, where the temperature starts to rise, but then levels off to the ambient temperature. If the graph is curved like a parabola as it shows cooling, there will be a second point somewhere before the low temperature. However, I expect that the drop in temperature is more likely another steeper logistic curve, which would have an inflection point somewhere on the curve. If the curve for the dropping temperatures is exponential, there would be no inflection point. The temperature change for the first 10 to 20 seconds, or so, is all over the place, as the hot tea and the melted ice would be mixing for a while, so it might be best to focus on the graph after 30 seconds.
12. When is the rate of change zero? What does that tell you about the tea?
    a. The rate of change is zero at the lowest point (or at each relative maximum and minimum if you point out that the tea in my ceramic cup actually changed temperature very minimally for about 10 minutes, going up and down very slightly).
13. What would happen to your graph if you extended it another three hours?
    a. The temperature would get to room temperature if you stayed in the house, but to the ambient temperature f the tea was outside. Room temperature would be pretty steady. Ambient outside temperature could fluctuate more than room temperature.

14. Draw the line segment that shows the average temperature change from the start to 1 hour. From the start to 2 hours. Which has the smaller temperature change? If I had asked that question before you drew the graph, would you have given the same answer??
    a. The smaller temperature change is most likely going to be the one from the start to two hours, since the tea has probably warmed up from what it was after 1 hour.
15. Can you find a place where the instantaneous rate of change for the graph equals the average rate of change from the start to two hours?
    a. Draw the line segment from start to the temperature at 2 hours, then find the place where the tangent line to the curve is parallel to the two hour change line.
16. What Calculus concepts from the VKR have we previewed?
    a. This problem previews: average rate of change; derivatives; differentiable function; the Mean value Theorem (# 15); end behavior of graphs; asymptotic behavior; inflection points (#11);
17. How would MS parents in your school react if their daughter said " In science we studied trees, in art we started to draw our gym shoes, in math we did some calculus, and in English we talked about…." When her parents asked, "So, what did you do at school today dear?"
    a. I don't think she would have gotten much beyond the word "calculus"…. But if your 6th or 7th graders (or algebra 1 students) could draw a decent graph, you could preview the Calculus topics listed above in your class!

The boiling temperature of water at various altitudes:

The common answer to the question “What is the boiling temperature of water?” is 100° C or 212° F, but that is only part of the story. The boiling temperature of water is effected by both altitude above sea level and barometric pressure. In this session, we addressed the issue of the boiling temperature of water at various altitudes.

If you use the “true” boiling temperature of water for your city when you do the hot tea to cold tea problem, be sure that your students understand that any question about boiling water on any standardized test (state exams, SAT, ACT, AP, etc) expects students to use 100° C or 212°F, unless it specifically states otherwise.

Here is a table showing the boiling temperature of water at various altitudes:

Altitude above sea level	°F	° C	Possible city
0 ft.	212	100	Any coastal city
1000 ft.	210.1	98.9	Phoenix, Atlanta
3000 ft.	208.1	97.8	Wall Drug, S. Dak.
5280 ft.	202	94.4	Denver, Colo
10,000 ft.	193.2	89.6	Leadville, Colo.

3, 2, 1, ….. Fire …. Cannon ball problem questions.

Traditional questions:

What is the maximum height? When does it occur? (Ans.: The maximum height is 78.6608+ feet, or nearly 78 feet and 8 inches -- 7.9296+inches. This occurs 1.5544+ seconds after the cannon baller leaves the cannon.)

How high is the person after 2 seconds? (Ans.: At 2 seconds, the person is 75.48 feet or 75 feet, 5¾ inches (5.76 inches).)

When is the cannon baller at a height of 50 feet? (Ans.: The cannon baller is 50 feet above the ground twice, once at 0.216+ seconds and at 2.8928+ seconds.)

How long until the person gets to the net? (Ans.: The person gets to the net after 3.625990+ seconds.)

Non-traditional questions:

What are the units of the rate of change of the graph and what does this tell us about the situation? (Ans.: The units of the rate of change are vertical feet per second, which tells us how fast the cannon baller is rising or falling.)

What is the initial vertical velocity? Is the initial velocity a reasonable velocity? (a research questions for students). (Ans.: The initial velocity is 49.742 ft/sec, which is 33.915 mph. Student could research human cannon ball shots to see if this is reasonable. From videos I have seen, it seems like the amount of time in flight is reasonable.)

What is not finished with the sketch (or my graph)? (Ans.: I do not show the net bouncing a little once the person hits the net. Also, the person never jumps out of the net and goes to the ground.)

When is the person moving the fastest? (Ans.: The person is moving the fastest just before hitting the net, nearly 3.626 seconds after launch.)

What is the biggest velocity? Is that a reasonable velocity? (a research questions for students) (Ans.: The greatest velocity is –66.28 ft/sec, or –45.2 mph.)
When is the person going up at a speed of approximately 20 feet per second? Down at approximately 20 feet per second? (Ans.: At 0.62 seconds, the person is ggoing up at 20.002 ft/sec. After 2.18 seconds, they are falling at –20.01 ft/sec..)

Can you find a place where the downward vertical velocity is equal to the upward vertical velocity when the person exited the cannon? When does that happen? (Ans.: The downward velocity is the opposite of the upward velocity at the start when the person is falling through the 40 foot height. This occurs at 3.108890897 seconds.)

What is the average speed from start to the net? (Ans.: The average velocity is net distance divided by the time it takes. This is –30/3.62599, which equals –8.2736+ ft/sec.

Does the person ever go exactly that vertical velocity? (Ans.: Yes, at 1.813008 seconds, dy/dx of the graph equals –8.274 ft/sec.)

Find some values for the rate of change and create a scatter diagram of them.
What kind of relationship seems to be true? (Ans.: It looks linear.)
Find the line of best fit using Modify on PRIZM (or Transfrm App on TI).
(Ans.: By using Modify on a PRIZM or the TRANSFRM app on the TI-83/84 calculator, you can find the line of best fit is Y = –32X +49.7 (or 49.742)
What do the slope and Y-intercept of the line of best fit tell you about the situation? (Ans.: The slope of the line is its rate of change. Since the points tell us the vertical velocity at certain times, the rate of change units are ft/sec^2, which is the acceleration of the cannon baller. While you may have been a little surprised when we found that the slope was –32, we then said that this is the force of gravity when you use feet. The Y-intercept of 49.742 was the initial *vertical* velocity of the cannon baller. This value is the initial velocity coming out of the cannon times the angle of elevation of the cannon (60×sin56°).

Can your Algebra 1 students find the graph of the derivative of a parabolic graph? We just did!!

Previewing Integral Calculus

Speed vs. Time Problem 1

You are driving on the interstate and you travel at exactly 80 mph for the next three hours.

Sketch a graph of the problem.

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Questions and answers for Speed vs. Time Problem 1:

1. What are the units of the sides of the rectangle?

   Horizontal units are hours.

   Vertical Units are miles per hour.

2. What is the area of the rectangle, including units?

   The area is 240 miles.

3. What does that mean in terms of the problem?

   That is the distance you drove in the 3 hours.

4. What did we just figure out?

   Actually, we just calculated $\int_0^3 80dt$, the area under the

   "curve"!!

If your MS students can graph the original problem, and answer the first three questions, *then they can do calculus!*

Speed vs. Time Story Problem 2

Pat, a student in your 3rd period class, is standing near the green door of the red brick school when her school bus arrives. Pat speeds up at a steady pace until she is moving at 4 ft/sec. This takes 2 seconds. She maintains this pace for 8 seconds before stopping to talk to her friend Chris, who is standing five feet from the bus. Stopping takes 1 second.

Draw a graph of the situation.

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Could your students draw the graph of this problem?

Stop and think of the questions your text would ask about the graph....

Here is a list of the questions you could ask about this problem.... Including a few questions that are not previewing Calculus, but are non-traditional questions.

Questions for MS and/or early HS:

1. How far did Pat walk in the 11 seconds?
2. What was Pat's average speed?
3. How fast was she moving at 1 second? At 7 seconds? At 10¼ seconds?
4. Are those speeds "reasonable"?
5. What are the **units** of the slopes of the three line segments?
6. What does the slope of the line segments tell you about the situation? What are the 3 slopes (include units)?
7. How far was Pat from the bus when the bus first arrived?
8. Was Pat walking, jogging, or running?
9. What information in the problem is not needed?
10. What gender are the two people in the problem. Justify your answer.
11. What Calculus concepts does this problem preview?

Solutions to Speed vs. Time Problem 2 questions:

1. Pat walked 38 feet (the area of the trapezoid).
2. Her average speed was $3\frac{7}{11}$ft./sec. 3.454545.... ft./sec. (38 ft. in 11 secs.),
3. 2 ft/sec @ 1; 4 ft./sec. @ 7; & 3 ft./sec. @ 10¼ sec. (do you want to include the idea of a negative or not in the last answer? Does that depend on the grade or ability level of your students?)
4. Those speeds are from 0 to 4 ft./sec. (see #8 answer).
5. The units of the slopes are ft./sec.^{2} (feet per second per second).
6. The slope of the line segments tell you Pat's *acceleration*. The slopes are 2ft./sec.^{2} for the first segment; 0 ft./ sec.^{2} for the horizontal one; and -4ft./ sec.^{2} for the last segment. Where is acceleration covered in MS science at your school?
7. Pat was 43 feet from the bus when the bus first arrived. (Remember, in the story problem, Chris was 5 feet from the bus door.)
8. Pat's speed of 4 ft./sec. is about 2.5 mph, she was walking.
9. Several things were not needed: Did you need to know the students' names (maybe), but did it matter that Pat is in *your* class, let alone in your *3rd period class*? The color of the door and the school (as well as it being brick)
10. Pat is a girl ("...until *she* is moving...") and Chris is up to you as long as you justify your choice!
11. This problem previews definite integrals (area under the curve) and derivatives (rate of change of the graph); as well as justifying your answers and reading problems carefully, which are both important for any AP test, not just AP Calculus.

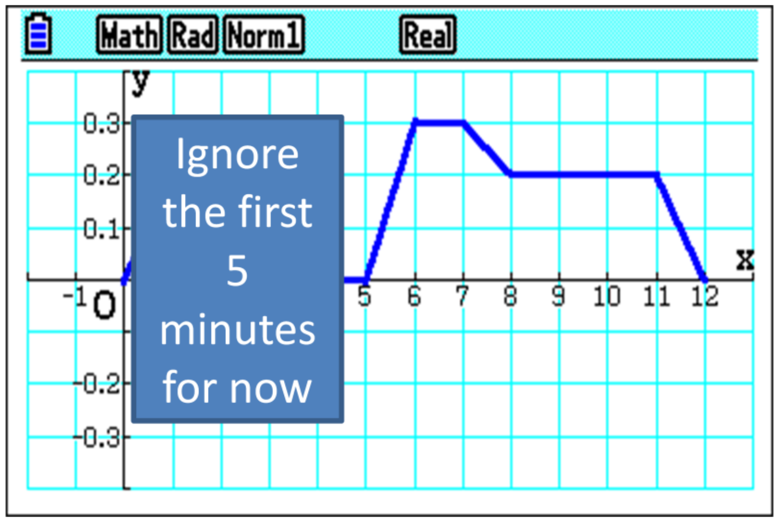
A Speed vs. Time problem

(A minimal adaptation of AP Calc AB/BC 2009 #1)

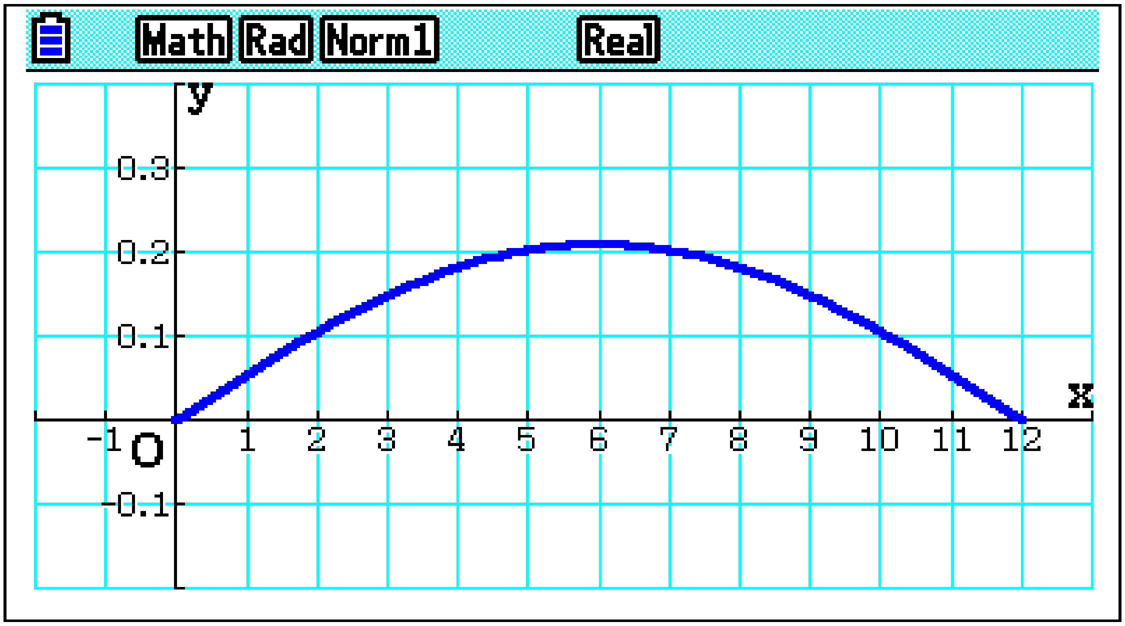
Caren rides her bicycle along a straight road from home to school, starting at home at time t = 0 minutes and arriving at school at timeT = 12 minutes. During the time interval 0 ≤ t ≤ 12 minutes, her velocity v(t), **in miles per minute**, is modeled by the piecewise-linear function whose graph is shown here.

How far does Caren live from her school?

What about Larry? See next graph…….



Larry rides his bike from his home to the same school as well. Here is the graph of Larry's ride to school: (count rectangles)-2

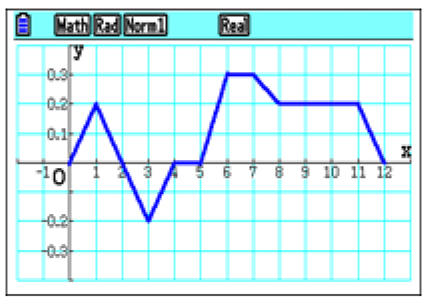


Here is the actual AP Free Response Problem:

AP Test Questions:

a.) Find the acceleration of Caren's bicycle at time t = 7.5 minutes. Indicate units of measure.

b.) Using the correct units, explain the meaning of the integral from 0 to 12 in terms of Caren's trip.
Find the value of the integral.



c.) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

d.) Larry also rides his bicycle along a straight road home to school in 12 minutes. His velocity is modeled by the function ***w*** given by ***w***(t)=(∏/15)(sin(∏*t/12), where ***w***(t) is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

Note: All of the questions in the AP test question can be explored by students starting in 6th grade as long as you avoid the calculus terms and symbols and ask some additional questions.

Excluding some of the vocablulary and the equation for Larry's ride, middle school students should be able to answer all of the AP questions. By giving middle school or Algebra 1 students the graph of Larry's ride on graph paper, they should be able to find a very good estimate for the distance he travelled.

Additional questions and adaptations of the AP problem:

It is probably best not to start with part “a” of the AP question unless students have had significant practice working on these kinds of problems. For students in grades 6 through 11, it might be good to start by asking what one square of graph paper represents in terms of the problem. If you start with part “b”, you may also want to break it into these questions:

1. How far did Caren travel in the first two minutes? In between 2 and 3 minutes?
2. In terms of the problem, what is happening from 4 to 5 minutes?
3. What part of the graph tells you how far Caren lives from school?
4. What is the total distance Caren travelled?

5. Before asking the acceleration question, ask your students what are the units of the rates of change of the segments, and what do the units tell us about the situation.
6. Next, ask your students to explain the meaning of the rate of change represented by the different segments of the graph.

By breaking down the single question “b” into parts, as I did above, your students can answer the more complicated single quesiton.

7. Traditional text questions such as:
    a. How fast is Caren moving at 30 seconds? At 2½ minutes? At 9 minutes?
    b. When is her velocity not changing?
    c. When is her velocity positive? Negative?
    d. What is the fastest she rides?

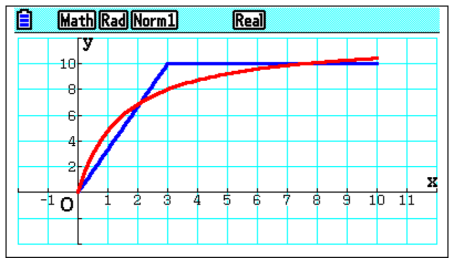
   can also be asked.

A smooth curve and a segment graph.

This is another AP released question.
We will investigate it in depth in my 2nd session.

Here is the graph of the velocity of two runners, in m/sec vs. time in seconds. What does it mean when the two graphs intersect?

Who is ahead the first time they cross?
Justify your answer.



The second time they cross? Justify your answer.

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The two runner problem is investigated thouroughly in the “Released AP Qiestions” session
You can use rate versus time problems from your texts, adapt Calculus problems from Calculus texts or released AP exams and AP review material. Talk to the Calculus teacher in your building or at a high school that your middle school students will attend. Letting teachers in other grades know you are previewing Calculus concepts will hopefully get them to do the same. Also, the more of yo who are doing this, the wider variety of problems you all may think of using.

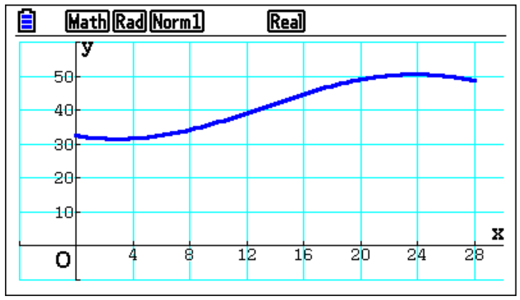
You can also make up some problems like the next one:

A smooth curve problem:

The graph shows the growth *rate* of bacteria in a petri dish(sq.mm./hour) vs. hours.

- *What is the area of the bacteria after 24 hours?*

(this is more involved than you may think.)



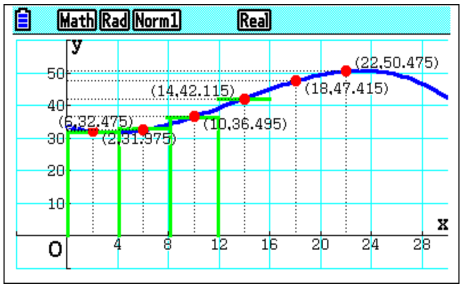
What does the area of each small rectangle on the grid represent?
What does the Y-intercept tell us?
According to the graph what happens initially?
What appears to be happen after the end of one day? Explain.

The area of each small grid rectangle represents a growth of 40 sq.mm. of bacteria.
The Y-intercept tells us the initial growth rate of the bacteria, not the starting area of the bacteria.
Initially, the graph dips slightly, so the bacteria slows their growth, slightly.
After one day, the bacteria growth rate starts to lessen, possibly because they are running out of food in the petri dish or many of them are starting to die.

The next page has one screen shot that allows students to approximate the area under the curve, using Riemann Sum rectangles that have the midpoint of the tops of the rectangles as a point on the graph. There is also a screen shot showing the value of the integral.

A smooth curve problem

- The area of all 6 rectangles equals 963.76 sq.mm.
  4(31.975+32.475+36.495+42.115+47.415+50.475) . But this does not answer the question, "What is the area after 24 hours?"'

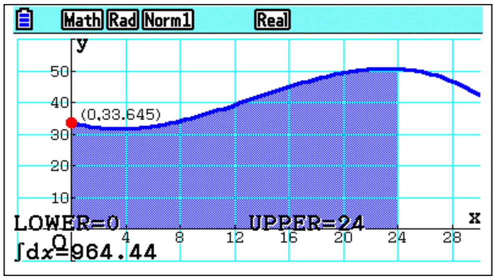


- What else do we need??

Remember: The graph shows the *growth rate* of bacteria in a petri dish(mm/hour) vs. hours.

What is the area after 24 hours?

- The area under the curve tells us how much the bacteria has grown in 24 hours. We need to add the starting area.
- This graph shows the integral value as well as the Y-intercept.



Our estimate of the total growth in area was 963.8 sq.mm., which is only 0.664% below the calculus answer.

The area of the bacteria is approximately 964 sq. mm. mmore than whatever the initial area of the bacteria was when we started. We do not know this area. The Y-intercept of 33.645 (mm^2/day) shown in the above screen shot is the initial *rate of growth*, **not** the initial area of trhe bacteria.

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I hope these problems have piqued your interest. If you do start using the ideas I talked about, make sure the teachers in grades after you and Calculus teachers know you are doing this so they can continue to develop and preview these ideas.

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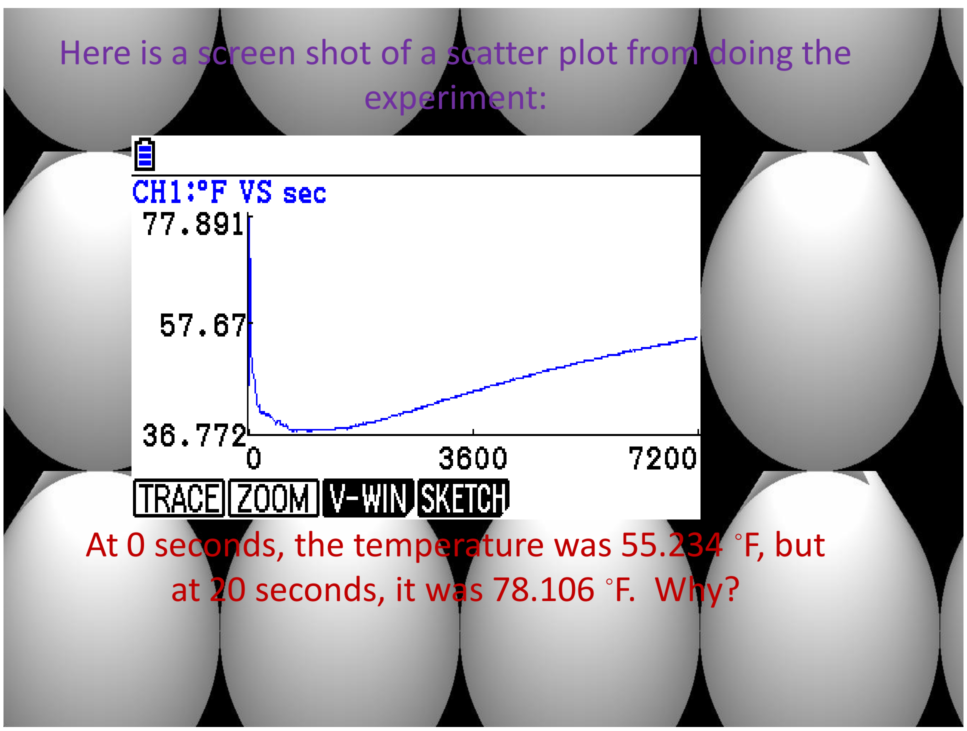
"As teachers your job is to perturb your students, but not so much that they shut down." Prof. Bob Horton, Clemson University.

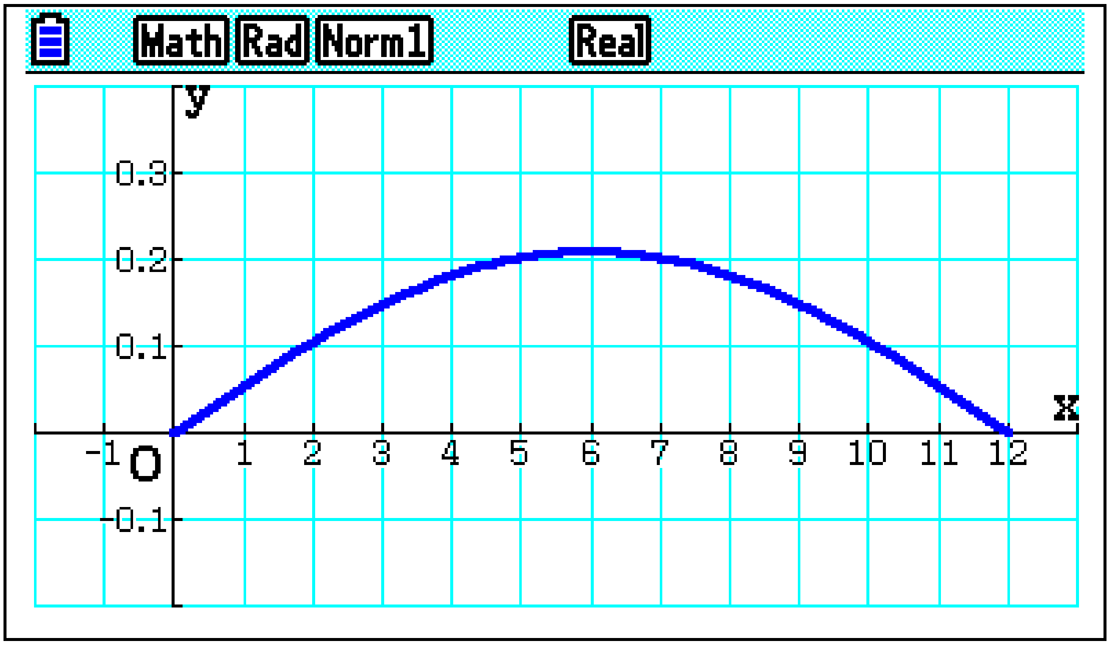
Here is the graph from my experiment with making hot tea into iced tea. I used a heavy ceramic mug filled with ice cubes. A lighter weight cup would likely have warmed to room temperature in the two hours that I collected data.

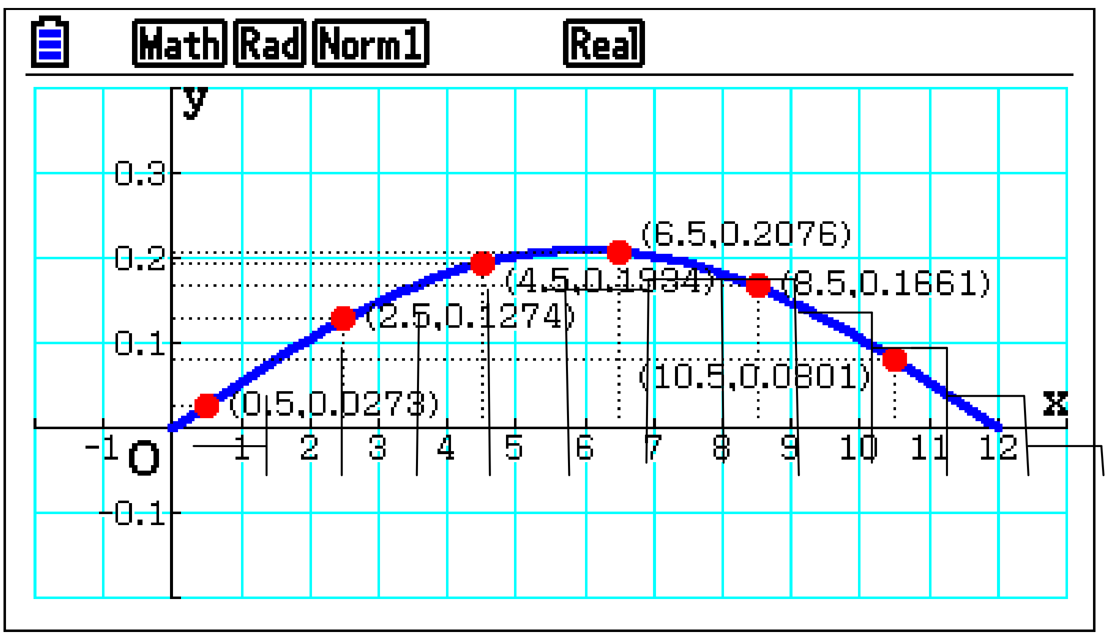


Here is how we found the line of beat fit for the Cannon Ball problem:

For Casio PRIZM:
We traced to several points on the height versus time graph with the Derivative feature On. We entered 8 different dy/dx values and the corresponding times into a table (on paper first, then in the PRIZM statistics menu) and made a scatter plot of the Derivatives (which tell us the vertical speed) versus time. After saving the scatter plot as a Picture, we used it as background for the graph of Y = MX + B. Using the MODIFY feature, we found the Y-intercept, which was the initial vertical velocity, then found the slope of the line of best fit. The slope of the line of best fit is the value of the force of gravity, -32 ft/sec^2 in the American measureing system. Next, we graphed the height vs. time graph and the velocity vs. time graph on the same axes and noticed the relationship between the two graphs.

For TI-83/84 family calculators:
You need to "Calculate" dy/dx for each point by tracing to it then going into calculate (2nd/Trace) and wrote down the time and the value of dy/dx. Make a scatter plot and leave the Stat Plot on. Use the TRANSFRM app to figure out the line of best fit. NOTE: you can not use M for your equation in the app, you need to use Y = AX + B. TI's Nspire can show the tangent line and the derivative as you "trace' just like the PRIZM does, so use that feature to find the values of dy/dx.





X	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5
Y	0.03	0.08	0.13	0.17	0.19	0.21	0.21	0.19	0.17	0.13	0.08	0.03

$\sum y = 1.62$ $\int_0^{12} \frac{\pi}{15} sin \frac{\pi t}{12} = 1.6$ Reimann Sums = 1.25% error

Estimation by counting unit rectangles: 16 rectangles = 1.6 miles.