The Math of Student Loans

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Student Loans

71% of students take out student loans for their undergraduate degree

A typical student in the class of 2013 graduated with $32,000 in student loan debt

http://www.ohe.state.mn.us/mPg.cfm?pageID=1342
http://money.cnn.com/2013/05/17/pf/college/student-debt/

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Student loans are a financial commitment that span decades.

Many students take out loans in their late teens and early twenties, unaware of the long-term impact of the loans.

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“After realizing the extent of their debt, 39% said they would have done things differently.”

- started saving earlier
- thoroughly researched financial aid
- looked for ways to save more
- spent less while at school

http://money.cnn.com/2013/05/17/pf/college/student-debt/

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Because of student loan debt

Young adults have decided:
- against pursing graduate school
- delayed getting married
- delayed having children
- to take a job they ordinarily would not have

Fetterman, M. & Hansen, B. (2006, November 20). Young people struggle to deal with kiss of debt. *USA Today*.

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There is currently a **trillion dollars** out in student loan debt.

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Meet Hypothetical Student!

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Hypothetical Student

- Takes out a $7,000 loan for the first year of college.
- Payments begin after graduation.
- How much is owed when repayment starts?

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Two Types of Student Loans

Subsidized Loans:

- The government pays interest that accrues until repayment begins
- Must demonstrate financial need
- Must fill out FAFSA form
- There are borrowing limits ($3,500 to $5,500)

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Unsubsidized Loans

- The loan is accruing interest while the student is in school.
- Assume it's 6.8% compounded annually.
- Every year the interest is added to the balance of the loan
- This new balance then accrues interest.

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During the first year of the loan

I = PRT

I: interest accrued (what we want to find)

P: principal ($7,000)

R: rate (6.8%)

T: time (1 year)

(CCSS 7.RP.3)

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During the first year of the loan

I = (7,000)(0.068)(1)

I = 476

After the first year, the balance is
$7,000 + $476 = $7,476

Or in one step:

Balance = $7000(1 + 0.068) = $7,476

But there are three more years to go!

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During the second year of the loan

What variables have a different value?

I = PRT

I: interest

P: principal

R: rate

T: time

During the second year of the loan

What variable has a different value?

I = PRT

I: interest

P: principal ($7,476)

R: rate (6.8%)

T: time (1 year)

During the second year of the loan

I = PRT

I = ($7,476)(0.068)(1)

I = $508.37

More interest is being charged during the second year. The interest is accruing interest.

Balance is $7,476 + $508.37 = $7,984.37

Or in one step: ($7476)(1+ 0.068) = $7,984.37

	Interest	Balance
End of Year 1	$476	$7,476 =7000(1.068)
End of Year 2	$508.37	$7,984.37 =**7476**(1.068)
End of Year 3		
End of Year 4		

	Interest	Balance
End of Year 1	$476	$7,476 =7000(1.068)
End of Year 2	$508.37	$7,984.37 =**7476**(1.068) =**(7000)(1.068)**(1.068)
End of Year 3		
End of Year 4		

	Interest	Balance
End of Year 1	$476	$7,476 =7000(1.068)
End of Year 2	$508.37	$7,984.37 =**7476**(1.068) =**(7000)(1.068)**(1.068) =$(7000)(1.068)^2$
End of Year 3		
End of Year 4		

	Interest	Balance
End of Year 1	$476	$7,476 =7000(1.068)
End of Year 2	$508.37	$7,984.37 =7476(1.068) =(7000)(1.068)(1.068) =(7000)(1.068)^2
End of Year 3		=(7000)(1.068)^3
End of Year 4		=(7000)(1.068)^4

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	Interest	Balance
End of Year 1	$476	$7,476 =7000(1.068)
End of Year 2	$508.37	$7,984.37 =7476(1.068) =(7000)(1.068)(1.068) =(7000)(1.068)^2
End of Year 3	$542.94	$8,527.31 =(7000)(1.068)^3
End of Year 4	$579.86	$9,107.16 =(7000)(1.068)^4

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Compound interest

$FV = PV(1 + i)^n$

FV: Future Value

PV: Present Value ($7,000)

i: interest rate per time period (0.068)

n: number of time periods (4)

$FV = 7000(1 + 0.068)^4$

FV = $9,107.16

(CCSS F.LE.1 and F.IF.8b)

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Geometric Sequence:

Each term in the sequence is obtained by multiplying the preceding term by a constant (a common ratio).

Common ratio is 1.068

Sequence:

7000, 7000(1.068), 7000(1.068)^2, 7000(1.068)^{3,}, 7000(1.068)^4

(CCSS F.IF. 1, 2, 3)

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Hypothetical Student’s Loan

Loan amount student took out: $7,000

Amount due four years later: $9,107.16

Wait! There’s more!

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Grace Period

- Payments do not start until six months after graduation.

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Grace Period

- Payments do not start until six months after graduation.
- But interest is accruing during this time!
- Tack on another half a year to that compound interest calculation.

$FV = 7000(1 + 0.068)^{4.5} = \$9411.71$

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College is Four Years

Assume a $7,000 loan taken out every year

Year 1: $FV = 7000(1 + 0.068)^{4.5}$ = **$9,411.71**

Year 2: $FV = 7000(1 + 0.068)^{3.5}$ = **$8,812.47**

Year 3: $FV = 7000(1 + 0.068)^{2.5}$ = **$8,251.37**

Year 4: $FV = 7000(1 + 0.068)^{1.5}$ = **$7,726.00**

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Hypothetical Student's Loans

Loan amount student took out: **$28,000**

Amount due when repayment begins: **$34,201.55**

($9,411.71 + $8,812.47 + $8,251.37 + $7,726)

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Now Repayment Begins

- Hypothetical Student will pay a set amount every month for 15 years.
- This is a "present value annuity": money was provided up front and is paid back in equal payments made at regular time intervals.
- How much will the student pay every month?

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Now Repayment Begins

The student will make 180 payments

(12 payments/year)(15 years) = 180

The amount due is $34,201.55

Why isn't the monthly payment determined this way: $34,201.55/180

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Interest is Accruing

- The loan is accruing interest during repayment
- During repayment assume interest is 6.8% compounded monthly.

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Present Value Annuity

PV = PMT$a_{n|i}$

PV: Present Value (the balance of the loan when repayment begins) $34,201.55

PMT: The amount of each payment

$a_{n|i}$: The present value annuity factor

$$a_{n|i} = \frac{\frac{(1+i)^n - 1}{i}}{(1+i)^n} \qquad a_{180|\frac{0.068}{12}} = 112.6527114$$

(Derivation provided at the end of handout.) 31

Present Value Annuity

PV = PMT$a_{n|i}$

34,201.55 = PMT(112.6527114)

303.60 = PMT

So Hypothetical Student will pay $303.60 a month for fifteen years.

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How Much Interest Will be Paid?

Loan amount student took out: **$28,000**

How much will the student pay in interest?

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How Much Interest Will be Paid?

Loan amount student took out: **$28,000**

Total amount student will repay:

($303.60)(12)(15) = 54,648

Student will repay **$54,648.**

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How Much Interest Will be Paid?

Loan amount student took out: **$28,000**

Total amount student will repay:

($303.60)(12)(15) = 54,648

Student will repay **$54,648.**

Student will pay **$26,648** in interest on her $28,000 loan

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Where Do Monthly Payments Go?

- The student is writing a check every month for $303.60.
- What is happening to that money?
- Amortization table

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Amortization table

Payment number	Payment amount	Interest amount	Principal amount	Remaining balance
1	$303.60			
2	$303.60			
…	…			
180	$303.60			

Where Do Monthly Payments Go?

- Interest is paid off as it accumulates
- It isn't being carried month to month
- Interest for the first month is

  I = PRT

  I = (34201.55)(0.068)(1/12)

  I = $193.81

- The rest of the $303.60 payment goes towards principal

Amortization table

	Payment amount	Interest amount	Principal amount	Remaining balance
1	$303.60	$193.81	$109.79 303.60-193.81	$34,091.76 34201.55 – 109.79
2	$303.60			

Amortization table

	Payment amount	Interest amount	Principal amount	Remaining balance
1	$303.60	$193.81	$109.79 303.60-193.81	$34,091.76 34201.55 – 109.79
2	$303.60	$193.19	$110.41 303.60-193.19	$33,981.35 34091.76 – 110.41

Interest for the second month is

I = PRT

I = (**34091.76**)(0.068)(1/12)

I = $193.19

Amortization table

The payment is the same each time, but:

- amount going toward interest decreases
- amount going toward principal increases

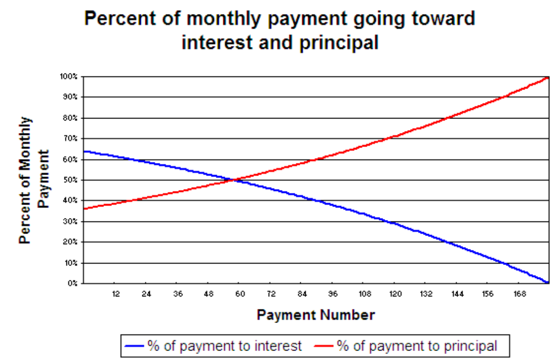
Feel like doing all 180 rows by hand?

This is where a spreadsheet is handy!

Amortization Table

Amortization table for $34,201.55 student loan at 6.8% interest for 15 years

Payment Number	Payment Amount	Interest Amount	Principal Amount	Remaining Balance	
1	$303.60	$ 193.81	$ 109.79	$ 34,091.76	
2	$303.60	$ 193.19	$110.41	$ 33,981.35	
89	$303.60	$ 123.08	$180.52	$ 21,539.54	
90	$303.60	$ 122.06	$181.54	$ 21,358.00	Halfway mark with payments, 37% of loan paid off
91	$303.60	$ 121.03	$182.57	$ 21,175.43	
92	$303.60	$ 119.99	$183.61	$ 20,991.82	
111	$303.60	$ 99.18	$204.42	$ 17,298.74	
112	$303.60	$ 98.03	$205.57	$ 17,093.17	50% of loan paid off
113	$303.60	$ 96.86	$206.74	$ 16,886.43	
179	$303.60	$ 3.41	$300.19	$ 302.40	
180	$303.60	$ 1.71	$301.89	$ 0.52	**loan paid off**



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What if Hypothetical Student paid an extra $100 a month?

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What if Hypothetical Student paid an extra $100 a month?

Making the regular montly payment			Making extra $100 extra towards principal every month		
Payment number	Remaining Balance		Payment Number	Remaining Balance	
1	$ 34,091.76		1	$ 33,991.76	
2	$ 33,981.35		2	$ 33,780.78	
66	$ 25,444.02		66	$ 17,467.49	
67	$ 25,284.60		67	$ 17,162.87	50% paid off
68	$ 25,124.28		68	$ 16,856.52	
89	$ 21,539.54		89	$ 10,006.74	
90	$ 21,358.00	Halfway mark with payments	90	$ 9,659.84	Paid off 72% of loan
91	$ 21,175.43	Paid off 37% of loan	91	$ 9,310.98	
112	$ 17,093.17	50% paid off	112	$ 1,510.52	
113	$ 16,886.43		113	$ 1,115.48	
114	$ 16,678.52		114	$ 718.20	
115	$ 16,469.43		115	$ 318.67	
116	$ 16,259.16		116	$ (83.13)	
117	$ 16,047.69		paid off		
179	$ 302.40				
180	$ 0.52				

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What if Hypothetical Student paid an extra $100 a month?

- Pay off the loan in 9 years 8 months
- Shaved more than 5 years off the payments
- “Only” repay $46,734.47 rather than $54,648
- Save $7,913.53 in interest
- Write “apply extra to principal” on check

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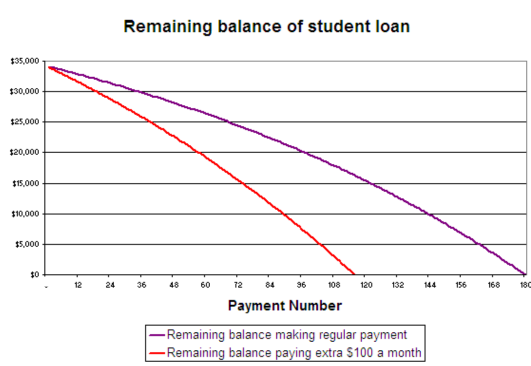

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Thank you!

Questions?
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Links

www.usnews.com/education/best-colleges/the-short-list-college/articles/2012/12/14/10-colleges-where-graduates-have-the-most-debt

money.cnn.com/interactive/pf/college/student-debt-map/?iid=EL

www.citizensbank.com/student-loans/federal-direct.aspx

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Deriving the present value annuity formula

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Future value annuity

- Let's put the student's loan aside
- I deposit $1 at the **end** of each year for four years into an account that earns 5% interest compounded annually.
- This is a "future value" annuity.
- How many years would the first dollar be accruing interest?

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Deposit from Year	Deposit amount	Years earning interest	Future value
1	$1	3 (deposit at end of year)	$1.16 $=1(1.05)^3$
2			
3			
4			

What are the mathematical expressions for the other deposits?

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Deposit from Year	Deposit amount	Years earning interest	Future value
1	$1	3 (deposit at end of year)	$1.16 $=1(1.05)^3$
2	$1	2	$1.10 $=1(1.05)^2$
3	$1	1	$1.05 $=1(1.05)^1$
4	$1	0	$1

How do we find total? Add the future values.

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Future Value Annuity

$FV = (1)(1.05)^3 + (1)(1.05)^2 + (1)(1.05)^1 + 1$

$FV = 1.16 + 1.10 + 1.05 + 1$

$4.31 is what we have saved after four years.

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Future Value Annuity

- Won't always be investing for 4 years
- Won't always be investing at 5% interest.
- We need a flexible formula.

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Future value annuity

The previous specific situation for how $1 behaved:

$FV = (1)(1.05)^3 + (1)(1.05)^2 + (1)(1.05)^1 + 1$

A flexible formula for how $1 behaves:

$s_{n|i} = (1)(1 + i)^{n-1} + (1)(1 + i)^{n-2} + \ldots + (1)(1 + i)^2 + (1)(1+i) + 1$

- $s_{n|i}$ is the future value annuity factor
- i is interest rate, n is number of payments
- Need it in a more compact form

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Original equation:

$s_{n|i} = (1)(1 + i)^{n-1} + (1)(1 + i)^{n-2} + \ldots + (1)(1 + i)^2 + (1)(1+i) + 1$

Multiply both sides by (1 + i):

$(1 + i)s_{n|i} = (1)(1 + i)^n + (1)(1 + i)^{n-1} + \ldots + (1)(1 + i)^2 + (1)(1+i)$

Now subtract original equation

$(1 + i)s_{n|i} = (1)(1 + i)^n + (1)(1 + i)^{n-1} + \ldots + (1)(1 + i)^2 + (1)(1+i)$

$-s_{n|i} = -(1)(1 + i)^{n-1} - (1)(1 + i)^{n-2} - \ldots - (1)(1 + i)^2 - (1)(1+i) - 1$

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Original equation:

$s_{n|i} = (1)(1 + i)^{n-1} + (1)(1 + i)^{n-2} + \ldots + (1)(1 + i)^2 + (1)(1+i) + 1$

Multiply both sides by (1 + i):

$(1 + i)s_{n|i} = (1)(1 + i)^n + (1)(1 + i)^{n-1} + \ldots + (1)(1 + i)^2 + (1)(1+i) + 1$

Now subtract original equation

$(1 + i)s_{n|i} = (1)(1 + i)^n + \cancel{(1)(1 + i)^{n-1}} + \ldots + \cancel{(1)(1 + i)^2} + \cancel{(1)(1+i)}$

$-s_{n|i} = \cancel{-(1)(1 + i)^{n-1}} - \cancel{(1)(1 + i)^{n-2}} - \ldots - \cancel{(1)(1 + i)^2} - \cancel{(1)(1+i)} - 1$

$is_{n|i} = (1 + i)^n - 1$

$s_{n|i} = \frac{(1 + i)^n - 1}{i}$

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Future value annuity

- Why is this so powerful?
- I can calculate the annuity factor no matter how many payments or the interest rate

$$s_{n|i} = \frac{(1 + i)^n - 1}{i}$$

- Then I can use it to scale it up to the dollar amount I'm investing

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Future value annuity

$FV = PMTs_{n|i}$

FV: Future Value (the amount of money saved at the end of the annuity)

PMT: The amount of each payment

$s_{n|i}$: Future value annuity factor

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Present Value Annuity

- But Hypothetical Student has the opposite situation.
- The student borrowed a certain amount of money that needs to be paid off.
- This is a present value annuity
- How can we calculate the student's monthly payment?

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Let's Pretend…

- The student won't make recurring payments to the loan company.
- The student will pay off the entire loan in one lump sum at the end of 15 years.
- To save up for this, each month the student puts money into a bank account.
- At the end of 15 years, the balance of this bank account will pay off the loan.
- This would never happen, but if it did…

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Calculating Monthly Payments

- How much student will owe in 15 years?
- Compound interest problem
- Calculate FV

$$FV = PV(1 + i)^n$$

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Calculating Monthly Payments

- Will pay off this loan in one lump sum
- Student sets up a savings account
- Account pays 6.8% interest compounded monthly
- Student will make deposits every month for 15 years into the account
- What should the monthly deposit be?

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Calculating Monthly Payments

$FV = PMTs_{n|i}$

FV: Future value, amount needed to pay off loan

PMT: Monthly payment (need to find)

$s_{n|i}$: future value annuity factor (can calculate)

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Calculating Monthly Payments

We know:
FV = PMT$s_{n|i}$
FV = PV$(1 + i)^n$

Let's set them equal to each other:
PV$(1 + i)^n$ = PMT$s_{n|i}$

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Present Value Annuity

$$PV(1+i)^n = PMTs_{n|i}$$

$$PV = \frac{PMTs_{n|i}}{(1+i)^n}$$

$$PV = PMT\left(\frac{s_{n|i}}{(1+i)^n}\right) \quad \text{where} \quad a_{n|i} = \frac{s_{n|i}}{(1+i)^n}$$

$$PV = PMTa_{n|i}$$

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Present Value Annuity

PV = PMT$a_{n|i}$

PV: Present Value (the balance of the loan when repayment begins) $34,201.55

PMT: The amount of each payment

$a_{n|i}$: The present value annuity factor

Now we can calculate the monthly payment!

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Present Value Annuity

First calculate $s_{n|i}$

$$s_{n|i} = \frac{(1+i)^n - 1}{i}$$

$$s_{180|\frac{0.068}{12}} = \frac{\left(1+\frac{0.068}{12}\right)^{180} - 1}{\frac{0.068}{12}}$$

$$s_{180|\frac{0.068}{12}} = 311.5097406$$

Then calculate $a_{n|i}$

$$a_{n|i} = \frac{s_{n|i}}{(1+i)^n}$$

$$a_{180|\frac{0.068}{12}} = \frac{311.5097406}{\left(1+\frac{0.068}{12}\right)^{180}}$$

$$a_{180|\frac{0.068}{12}} = 112.6527114$$

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Present Value Annuity

PV = PMT$a_{n|i}$
34,201.55 = PMT(112.6527114)
303.60 = PMT

So Hypothetical Student will pay $303.60 a month for fifteen years.

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