NOT ALL DEFINITIONS ARE CREATED EQUAL

(IN THE ELEMENTARY CLASSROOM)

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Mathematical Definitions: What are they?

In discussing the characteristics of mathematical definitions, Borasi identifies two functions they must fulfill.

A definition of a given mathematical concept should:

1. Allow us to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency (by simply checking whether a potential candidate satisfies all the properties stated in the definition).

2. “Capture” and synthesize the mathematical essence of the concept (all the properties belonging to the concept should be logically derivable from those included in its definition). (Borasi, 1992, pp.17-18)

Why focus on definitions?

- It is now well documented in the literature that teachers' ability to teach mathematics depends on their mathematics content knowledge (e.g., Ma, 1999).
- It is still unclear what teachers need to know about each mathematical topic or activity they teach, and also how this knowledge can be effectively promoted in mathematics teacher preparation programs.
- Ball, Thames, and Phelps (2008) listed "choosing and developing useable definitions" (p. 400) as one of the challenges that are distinctive to the work of teaching mathematics.
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Why focus on definitions?

- Reasoning-and-proving (RP) is at the heart of mathematical sense-making and important for all students' learning as early as the elementary school. Yet, RP typically has a marginal place in elementary classrooms.
- This situation is partly attributed to that many (prospective) elementary teachers have (1) weak mathematical knowledge about RP and (2) counterproductive beliefs concerning its teaching.

-Stylianides, Stylianides, & Shilling-Traina (2013)

Mathematical Definitions: A matter of CHOICE (Morgan, 2006)

- A characteristic of the ways mathematicians use definitions that should also be highlighted at the school level – the **element of choice**. While dictionary definitions describe the ways a word is actually used in practice, mathematical definitions are chosen according to which is best suited for the situation.
- The idea of choice and purposeful formulation of definitions constructs an **active role** for the mathematician him/herself, not simply as a user of correct mathematical vocabulary but as one who chooses between alternative definitions or creates new ones.
- This role is very different from that typically constructed for school student—but why should it be??

A focus on definitions: What we'll do

- Start our work on mathematical definitions:
  - Criteria for evaluating the appropriateness of definitions in school mathematics.
  - Analyzing textbook definitions.

Why do we need to work on definitions?

- In the discipline of mathematics:
  - –Definitions are crucial for mathematical reasoning and communication.
  - –Definitions help make important distinctions.
- In the work of teaching:
  - –Teachers often face the challenge of how to handle the tension between using mathematically precise definitions and definitions that are appropriate for their students.
  - –Teachers need to judge the appropriateness and accuracy of the definitions presented in textbooks.
  - –Teachers need to make sense of and evaluate definitions used by their students.
  - –Teachers need to understand how definitions can be used to reconcile disagreements.
  - –Teachers need to understand the power of definitions in mathematical reasoning.

The challenge of defining even and odd numbers in the elementary school

- Make a list of numbers you are sure are "even" and a list of numbers you are sure that are "odd." Also try to write down your definition for even numbers.

  Compare your lists with the people sitting next to you. Also share your definitions.
- In the **discipline of mathematics**, the set of **even numbers** is, in general, considered to be **{…, -4, -2, 0, 2, 4, …}** and the set of **odd numbers** is considered to be **{…, -5, -3, -1, 1, 3, 5, …}**.
- In the **elementary school**, it is usually the case that even and odd numbers are treated only in the set of **whole numbers** (i.e., the set **{0, 1, 2, 3, …}**).

How do we define even and odd numbers in the elementary school so that the definitions are sensitive both to mathematics and to children?

Gaining some insight into the challenge

- Some teachers say to their students ***"You cannot subtract 5 from 2."***
- Can you think of a different way to tell elementary school students ***"You cannot subtract 5 from 2"*** that would be sensitive to both the mathematics and the children?
- **Another example:** ***"You cannot find the square root of a negative number."***

Two considerations for evaluating the appropriateness of definitions in the elementary school

How might a definition of **even numbers in the elementary school** be formulated so that it is *sensitive* to the following **two considerations**?

- **Mathematics as a discipline**
  - Is the definition accurate mathematically? (e.g., does it use mathematical language in a precise way?)
  - Does it exclude all numbers that are not even and does it include all numbers that are even? (in the discipline, the set of even numbers is considered to be {…, -4, -2, 0, 2, 4, …})
- **Students as learners of mathematics**
  - Does the definition use terms that are known to the students? (assume that elementary students do not yet know about negative numbers)
  - Does the definition prepare well the students for their future learning of mathematics? (students will one day learn about negative numbers)

When you only take into account one of the considerations…

imbalance

Mathematics as a discipline

Students as learners of mathematics

Task for the teacher: To achieve a defensible balance between the two considerations

balance

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Examining textbook definitions

Discuss in your groups the following definitions, having in mind the two considerations mentioned previously.

1. An even number is a number of the form $2k$, where k is an integer.
2. An even number is a whole number that it is a whole number times 2.
3. An even number is a natural number that is divisible by 2.
4. An even number is a number that has 0, 2, 4, 6, or 8 in the ones place.
5. An even number ends in 0, 2, 4, 6, or 8.
6. An even number is a number that is not odd.

7. A whole number is even if it is **a** whole number times 2.

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Equivalent definitions of even numbers (in the set of whole numbers) that may be used in the elementary grades

- A whole number is even if it is 2 times a whole number (or a whole number times 2).
- A whole number is even if it is divisible by 2 (i.e., if there is no remainder when you divide the number by 2).
- A whole number is even if you can divide that number into groups of 2 with none left over.
- A whole number is even if you can divide that number into two equal groups of whole numbers (i.e., without taking any halves).

What level of precision should a teacher seek for the definitions used in his/her class?

- It basically depends on the class and the teacher's goals.
  - A teacher may sometimes make some conscious decisions to let the class proceed with some definitions that he/she considers as being "not that precise."
  - The level of mathematical knowledge of a class may not allow for something "more precise."
  - A teacher may just want to give the students a sense of a concept.
- The **two considerations** provide the teacher with a **tool** to examine definitions critically and *make decisions* that best suite his/her class and goals.
- The application of the **two considerations** is **not** limited to the definitions of even and odd numbers; these considerations can be applied more broadly.

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Video clip from 3rd grade: A Student's novel idea about the parity of 6

- Deborah Ball's third-grade classroom (1989-90)
- Mid-January (ten days into number theory unit).
- Diverse classroom, many English language learners
- Goals in this classroom all year: taking mathematics seriously, attending carefully to other students' ideas, justifying claims, and working together on mathematics

- Resources from *Mathematics Teaching and Learning to Teach*, University of Michigan. (2010)
- http://deepblue.lib.umich.edu/handle/2027.42/65013

Guiding questions for watching the clip

1. Try to identify all the different definitions of even numbers being used in this segment. Who is using each one?
2. What were the definitions used for?
3. What did the teacher have to manage mathematically in terms of definitions in this episode?

Issues about definitions raised by the clip

1. Definitions as resources for reconciling disagreements.
2. Students were using different definitions for even numbers.
3. To make sense of students' thinking the teacher needed to understand the definitions they had in mind.
4. It was important for the teacher to be able to evaluate mathematically these different definitions and see how they corresponded to each other.
5. Students also tried to make sense of each other's definitions.
6. How usable each definition is and in what situations?
7. Shea had in mind an important idea but he lacked the language to express it well (a meaningful description but a confusing name).

An Act of Sense-Making

“The activities in our mathematics classrooms can and must reflect and foster the understandings that we want students to develop with and about mathematics. **That is: if we believe that doing mathematics is an act of sense-making; if we believe that mathematics is often a hands on, empirical activity; if we believe that mathematical communication is important; if we believe that the mathematical community grapples with serious mathematical problems collaboratively, making tentative explanations of these phenomena, and then cycling back through those explanations (including definitions and postulates); if we believe that learning mathematics is empowering, and that there is a mathematical way of thinking that has value and power, then our classroom practices must reflect these beliefs**. Hence we must work to construct learning environments in which student actively engage in the science of mathematical sense-making (Schoenfeld, 1994, pp. 60–61).”

Some true statements about even and odd numbers

+	Odd	Even
Odd	Even	
Even	Odd	Even

x	Odd	Even
Odd	Odd	
Even	Even	Even

Proving a statement about even and odd numbers

The sum of any two odd numbers is an even number.

1 **Individually, try to prove this statement on the set of whole numbers. If you can provide more than one proof, do so.**
(Do **not** be concerned about whether your proof(s) would be understood by elementary school students.)

2 **Do you think you have actually produced a proof? How would a skeptic react to what you have produced?**

Video clip from 3rd grade: Students' idea about the provability of the conjecture "odd + odd = even"

- Deborah Ball's third-grade classroom (1989-90)
- http://deepblue.lib.umich.edu/handle/2027.42/65012
- End-January (two weeks into number theory unit).
- In this episode: Jillian and Shekira argue that it is not possible to prove Bernadette's conjecture:

  An odd number plus an odd number equals an even number.

Guiding questions for watching the clip:

> What was the mathematical issue raised by the students? How might it be addressed?

QuickTime™ and a
DV/DVCPRO - NTSC decompressor
are needed to see this picture.

Proving a statement about even and odd numbers

The sum of any two odd numbers is an even number.

First individually and then in groups of 4, prove this statement on the set of whole numbers. If you can provide more than one proof, do so.
(Try to develop proof(s) that you think would be understood by elementary school students.)

Continue our work on proving a statement about even and odd numbers

The sum of any two odd numbers is an even number.

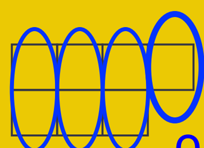
First individually and then in groups of 4, prove this statement on the set of whole numbers. If you can provide more than one proof, do so.
(Try to develop proof(s) that you think would be understood by elementary school students.)

Do you think you have actually produced a proof?
How would a skeptic react to what you have produced?

Bernadette's proof for the conjecture "odd + odd = even"

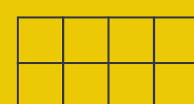
"All odd numbers if you circle them … by twos, there's one left over, so … if you plus another odd number, then the two ones left over will group together, and it will make an even number."

The students in the class frequently used pictures to represent odd and even numbers. For example, for the numbers 7 and 8 they would draw:



7

one left over



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What would be a pictorial proof for the conjecture "odd + odd = even" that would be analogous to Bernadette's proof?

What would be an algebraic proof for the statement “odd + odd = even”?

Bernadette‘s proof

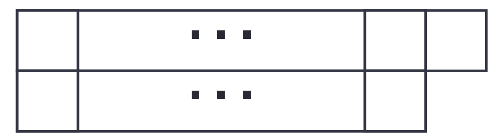
[A]ll odd numbers if you circle them by twos, thereÕs one left over,

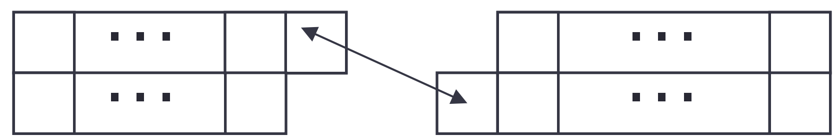
so if you ... plus one, um, or if you plus another odd number, then the two ones left over will group together,

and it will make an even number.

Pictorial proof

[A]ll odd numbers are of the form:





and it will make an even number.

What would be an algebraic proof for the conjecture “odd + odd = even” that would be analogous to Bernadette’s proof and the pictorial proof? Explain the correspondences among the different proofs.

Another possible elementary proof

(Definitions used: A whole number is even if it ends in 0, 2, 4, 6, or 8; A whole number is odd if it ends in 1, 3, 5, 7, or 9.)

- Proof: In order to determine if the sum of two whole numbers (which is also a whole number) is even or odd, we simply have to look at the last digit. The following chart lists all possible final digits upon adding two odd numbers together:

+	**1**	**3**	**5**	**7**	**9**
1	2	4	6	8	10
3	4	6	8	10	12
5	6	8	10	12	14
7	8	10	12	14	16
9	10	12	14	16	18

- The ones digit in these numbers will be the ones digit in the sum of the two odd numbers, and we see from the chart this digit will either be a 0, 2, 4, 6, or 8, which (by definition) means that the sum will be an even number.

Some other important points we discussed

- There are no canonical definitions.
- Most of the decisions about which definitions are appropriate and precise enough depend on the teachers' goals and the level of their students.
- Checking a few examples is not enough to prove general mathematical statements.
- Students can expand their ability to prove general mathematical statements by using definitions.
- Students can produce general arguments without necessarily using algebra -- Teachers need to be able to identify and highlight these general aspects in students' arguments.

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Thank you!

- Please feel free to send any questions/comments/ suggestions/requests for slides to me at shillingtrainaln@longwood.edu